A Note on Groups with Just-Infinite Automorphism Groups

Francesco de Giovanni
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy) degiovan@unina.it

Diana Imperatore
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy) diana.imperatore@unina.it

Received: 6.9.2012; accepted: 5.10.2012.

Abstract. An infinite group is said to be just-infinite if all its proper homomorphic images are finite. We investigate the structure of groups whose full automorphism group is just-infinite.

Keywords: just-infinite group; automorphism group

MSC 2000 classification: 20E36

1 Introduction

Let $D_\infty$ be the infinite dihedral group. Although $D_\infty$ admits non-inner automorphisms, it is well known that $D_\infty$ is isomorphic to its full automorphism group $\text{Aut}(D_\infty)$. Moreover, it has been proved in [3] that there are no other groups with infinite dihedral automorphism group. In other words, the equation

$$\text{Aut}(X) \simeq D_\infty$$

admits, up to isomorphisms, only the trivial solution $X = D_\infty$. It is usually hard to understand which groups $Q$ can occur as full automorphism groups of some other group, i.e. when the equation

$$\text{Aut}(X) \simeq Q$$

admits at least one solution. For instance, it was proved by D.J.S. Robinson [8] that no infinite Černikov group can be realized as full automorphism group of a group, while a classical result of R. Baer [1] showed that for any finite group $Q$ the above equation has no solution within the universe of infinite periodic groups.
The aim of this paper is to obtain informations about groups whose automorphism groups are just-infinite. Here a group $G$ is said to be just-infinite if it is infinite but all its proper homomorphic images are finite. It follows from Zorn’s Lemma that any finitely generated infinite group has a just-infinite homomorphic image, and so just-infinite groups play a relevant role in many problems of the theory of infinite groups (see for instance [4]). The first examples of just-infinite groups are of course the infinite cyclic group and the infinite dihedral group; the first of these cannot occur as the full automorphism group of any group, and we discussed above the dihedral case. We will prove that if $G$ is any group admitting an ascending normal series whose factors are either central or finite, then the automorphism group $\text{Aut}(G)$ cannot be just-infinite. It follows in particular that in any group with just-infinite automorphism group, the centre and the hypercentre coincide. Some examples of groups with just-infinite automorphism groups will also be constructed.

Most of our notation is standard and can for instance be found in [7].

2 Results and examples

The structure of just-infinite groups has been described by J.S. Wilson [11]; of course, all infinite simple groups are just-infinite, while any soluble-by-finite just-infinite group is a finite extension of a free abelian group of finite rank. In the same paper, the class $\mathcal{D}_2$, consisting of all infinite groups in which every non-trivial subnormal subgroup has finite index, is considered; obviously, all $\mathcal{D}_2$-groups are just-infinite, but it is clear that any $\mathcal{D}_2$-group containing an abelian non-trivial subnormal subgroup is either cyclic or dihedral. It follows easily from this remark that the result proved in [3] can be extended to the next statement (recall here that a group is called generalized subsoluble if it has an ascending subnormal series whose factors are either abelian or finite).

**Theorem 1.** Let $G$ be a generalized subsoluble group whose automorphism group $\text{Aut}(G)$ is a $\mathcal{D}_2$-group. Then $G \simeq D_\infty$.

The following example shows that a similar result cannot be proved for groups with just-infinite automorphism groups, even restricting the attention to the case of polycyclic groups.

Let $A = \langle a \rangle \times \langle b \rangle$ be a free abelian group of rank 2, and let $x$ and $y$ be the automorphisms of $A$ defined by the positions

$$a^x = b, \quad b^x = a, \quad a^y = b, \quad b^y = a^{-1}b.$$
Then $\langle x, y \rangle$ is a dihedral subgroup of order 12 of $GL(2, \mathbb{Z})$, and the semidirect product

$$G = \langle x, y \rangle \rtimes A$$

is a polycyclic group. Moreover, $A$ is self-centralizing and has no cyclic non-trivial $G$-invariant subgroups, so that $G$ is just-infinite. On the other hand, the group $G$ is complete, i.e. it has trivial centre and $\text{Aut}(G) = \text{Inn}(G)$, and hence $\text{Aut}(G) \simeq G$ (see [9]).

**Lemma 1.** Let $G$ be an abelian group. If all proper homomorphic images of the full automorphism group $\text{Aut}(G)$ of $G$ are finite, then $\text{Aut}(G)$ is finite.

**Proof.** Assume for a contradiction that $\text{Aut}(G)$ is just-infinite. As the inversion map $\tau$ of $G$ belongs to the centre $Z(\text{Aut}(G))$, we have that $\langle \tau \rangle$ is a finite normal subgroup of $\text{Aut}(G)$, so that $\tau$ is the identity and $G$ is an infinite abelian group of exponent 2. Let $\Gamma$ be the set of all automorphisms $\alpha$ of $G$ acting trivially on a subgroup of finite index of $G$. Then $\Gamma$ is a non-trivial normal subgroup of $\text{Aut}(G)$ and the index $|\text{Aut}(G) : \Gamma|$ is infinite. This contradiction proves the lemma. \[QED\]

**Lemma 2.** Let $G$ be a just-infinite group, and let $N$ be a normal subgroup of $G$. Then $N$ has no finite non-trivial normal subgroups.

**Proof.** Let $X$ be any finite normal subgroup of $N$. Since $G/N$ is finite, the conjugacy class of $X$ in $G$ is finite, and so it follows from the well known Dietzmann’s lemma that the normal closure $X^G$ is a finite normal subgroup of $G$. Therefore $X = \{1\}$ and the lemma is proved. \[QED\]

We can now prove our main result on groups with just-infinite automorphism group.

**Theorem 2.** Let $G$ be a group admitting an ascending normal series whose factors are either central or finite. Then the automorphism group $\text{Aut}(G)$ is not just-infinite.

**Proof.** Assume for a contradiction that the group $\text{Aut}(G)$ is just-infinite, and let

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < \ldots < G_\tau = G$$

be an ascending normal series whose infinite factors are central. As the inner automorphism group $\text{Inn}(G)$ is a non-trivial normal subgroup of $\text{Aut}G$, it follows from Lemma 2 that $\text{Inn}(G)$ has no finite non-trivial normal subgroups. Then the centre $Z(\text{Inn}(G))$ is non-trivial, and so it has finite index in $\text{Aut}(G)$. In particular, the index $|G : Z_2(G)|$ is finite, and hence the term $\gamma_3(G)$ of the
lower central series of $G$ is finite (see [7] Part 1, p.113). Thus $\gamma_3(G)$ is contained in $Z(G)$, and $G$ is nilpotent, so that $\text{Inn}(G)$ lies in the Fitting subgroup of $\text{Aut}(G)$. Therefore

$$G/Z(G) \simeq \text{Inn}(G)$$

is a free abelian group of finite rank $r$ (see [11], Theorem 2). It is well known that the homomorphism group

$$\text{Hom}(G/Z(G), Z(G))$$

is isomorphic to an abelian normal subgroup of $\text{Aut}(G)$, so that it is torsion-free and hence also $Z(G)$ must be torsion-free; moreover, $\text{Hom}(G/Z(G), Z(G))$ is isomorphic to the direct product of $r$ copies of $Z(G)$. On the other hand, the groups $\text{Inn}(G)$ and $\text{Hom}(G/Z(G), Z(G))$ are isomorphic, and hence $Z(G)$ is infinite cyclic and $G$ is torsion-free.

Put $C = Z(G)$ and $Q = G/C$, and let $x$ be any element of $G \setminus C$. The mapping

$$\varphi : g \mapsto [g, x]$$

is a non-trivial homomorphism of $G$ into $C$, whose kernel coincides with the centralizer $C_G(x)$, so that $G/C_G(x)$ is infinite cyclic and

$$G = \langle y \rangle \rtimes C_G(x)$$

for some element $y$ of infinite order. Let $m$ be a non-negative integer such that $(yx)^m$ belongs to $C_G(x)$. As $(yx)^m = y^m x^m z$ for some $z \in C_G(x)$, we have that $y^m$ is in $C_G(x)$, and so $m = 0$. Therefore $\langle yx \rangle \cap C_G(x) = \{1\}$, and hence

$$G = \langle yx \rangle \rtimes C_G(x).$$

It follows that an automorphism $\alpha$ of $G$ can be defined by setting

$$y\alpha = yx \quad \text{and} \quad c\alpha = c$$

for all $c \in C_G(x)$. Then $yo^n = yx^n$ for each positive integer $n$, so that $\alpha$ has infinite order and $\alpha^n$ cannot be an inner automorphism of $G$. This is of course a contradiction, since $\text{Inn}(G)$ has finite index in $\text{Aut}(G)$. $\Box$

The above theorem shows in particular that hypercentral groups cannot have just-infinite automorphism groups. We leave here as an open question whether there exists a locally nilpotent group whose automorphism group is just-infinite. As a consequence of Theorem 2, we can observe that the upper central series of any group with just-infinite automorphism group stops at the centre.
Corollary 1. Let $G$ be a group whose automorphism group $\text{Aut}(G)$ is just-infinite. Then $Z(G) = Z_2(G)$.

Proof. As $\text{Aut}(G)$ is just-infinite, it follows from Theorem 2 that the index $|G : Z_2(G)|$ must be infinite, and hence $Z(G) = Z_2(G)$ because $Z_2(G)$ is a characteristic subgroup of $G$.

As infinite simple groups are just-infinite, we have that complete infinite simple groups are trivial examples of groups with just-infinite automorphism groups. Among such groups we find for instance the universal locally finite groups of cardinality $2^{\aleph_0}$ (see [5]); recall here that a locally finite group $U$ is said to be universal if it contains a copy of any finite group and any two finite isomorphic subgroups of $U$ are conjugate.

Our last result shows how to find examples of non-simple $D_2$-groups occurring as full automorphism groups; here the main ingredient is an infinite simple group with finite non-trivial outer automorphism group. Groups of this kind have for instance been constructed by R.J. Thompson in his study of homeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ (see [2]). Recall here that the outer automorphism group of a group $G$ is the factor group $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$.

Lemma 3. Let $G$ be a group containing a simple normal subgroup $N$ such that $G/N$ is finite and $C_G(N) = \{1\}$. Then every non-trivial subnormal subgroup of $G$ has finite index.

Proof. Let $S$ be any non-trivial subnormal subgroup of $G$. Then $[N, S]$ cannot be trivial, and hence $S \cap N \neq \{1\}$ (see for instance [10], 13.3.1). Thus $S$ contains $N$, and so the index $|G : S|$ is finite.

Theorem 3. Let $G$ be an infinite simple group whose outer automorphism group $\text{Out}(G)$ is finite. Then $\text{Aut}(G)$ is an infinite complete group whose non-trivial subnormal subgroups have finite index.

Proof. The automorphism group $\text{Aut}(G)$ is complete by a well known result of Burnside (see for instance [10], 13.5.9), and $\text{Inn}(G) \simeq G$ is a simple normal subgroup of $\text{Aut}(G)$ of finite index. Moreover, as $Z(G) = \{1\}$, also the centralizer $C_{\text{Aut}(G)}(\text{Inn}(G))$ is trivial, and hence it follows from Lemma 3 that any non-trivial subnormal subgroup of $\text{Aut}(G)$ has finite index.

References


