Weakly normal subgroups and classes of finite groups

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Abstract. A subgroup $K$ of a group $G$ is said to be weakly normal in $G$ if $K^g \leq N_G(K)$ implies $g \in N_G(K)$. In this paper we establish certain characterizations of solvable $PST$-groups using some weakly normal subgroups.

Keywords: Weakly normal subgroups, solvable $PST$-groups

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1 Introduction and statements of results.

All groups are finite.

A subgroup $H$ of a group $G$ is said to be weakly normal in $G$ provided that if $g \in G$ and $H^g \leq N_G(H)$, then $g \in N_G(H)$. It is not difficult to see that $H$ is weakly normal in $G$ if and only if whenever $H < \langle H, H^g \rangle$, then $H < \langle H, g \rangle$.

For each group $G$ let $WN(G)$ denote the set of all weakly normal subgroups of $G$. Notice that if $H$ is normal in $G$ or $N_G(H) = H$, then $H \in WN(G)$. Thus $WN(G)$ contains all maximal subgroups of $G$. Moreover, $WN(G)$ contains all the pronormal subgroups of $G$ but there are weakly normal subgroups of $G$ which are not pronormal (see [2, p. 28]). Also the join of two pronormal subgroups of a group need not be pronormal but belongs to $WN(G)$ (see [2, p. 28]).

In [5] the authors introduce the concept of a $\mathcal{H}$-subgroup of a group and prove a number of very interesting results about such subgroups. A subgroup $X$ of a group $G$ is called a $\mathcal{H}$-subgroup provided that $X^g \cap N_G(X) \leq X$ for all $g \in G$. The authors of [5] denote by $\mathcal{H}(G)$ the set of all $\mathcal{H}$-subgroups of $G$. Assume that $X \in \mathcal{H}(G)$ and $X^g \leq N_G(X)$. Then $g \in N_G(X)$ so that $\mathcal{H}(G) \subseteq WN(G)$. In particular, by Proposition 3 of [5], a Sylow subgroup of a normal subgroup of $G$ belongs to $WN(G)$. Moreover, a Hall subgroup of $G$ also is a $WN$-subgroup of $G$.

Example 1. ([3]). Let $G = S_4$ and $H = \langle (1 2 3 4) \rangle$. Then $N_G(H) = \langle (1 2 3 4), (1 3) \rangle$, and if $g = (1 2 3)$, $H^g = \langle 1 4 2 3 \rangle$, whence $H^g \cap N_G(H) = \langle (1 2)(3 4) \rangle \not\leq H$. 

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Thus \( H \) is not an \( H\)-group of \( G \). Note that \( N_G(H) \) has a unique cyclic group of order 4. Consequently, if \( H^x \leq N_G(H) \), \( x \) an element of \( G \), then \( H^x = H \) and \( H \) is weakly normal in \( G \).

However, Ballester-Bolinches and Esteban-Romero [3] proved the following very interesting results.

**Theorem 1.** Let \( H \) be a weakly normal subgroup of the supersolvable group \( G \). Then

(a) If \( H \) is a \( p\)-group, \( p \) a prime, then \( H \) is an \( H\)-subgroup of \( G \).

(b) If every subgroup of \( H \) is weakly normal in \( G \), then \( H \) is an \( H\)-subgroup of \( G \).

A group \( G \) is called a T-group if whenever \( H \) and \( K \) are subgroups of \( G \) such that \( H \triangleleft K \triangleleft G \), then \( H \triangleleft G \). W. Gaschütz [7] introduced the class of finite T-groups and characterized solvable T-groups. He proved that a solvable group \( G \) is a T-group if and only if the nilpotent residual \( L \) of \( G \) is a normal abelian Hall subgroup of \( G \) such that \( G \) acts by conjugation on \( L \) as power automorphisms and \( G/L \) is a Dedekind group.


**Theorem 2.** ([3, 5]). Let \( G \) be a group. The following statements are equivalent:

(a) \( G \) is a solvable T-group.

(b) Every subgroup of \( G \) belongs to \( \mathcal{H}(G) \).

(c) Every subgroup of \( G \) belongs to \( WN(G) \).

**Theorem 3.** ([6]) Let \( G \) be a solvable group. Then \( G \) is a T-group if and only if there exists a subgroup \( L \) of \( G \) such that

(a) \( L \) is a normal Hall subgroup of \( G \).

(b) \( G/L \) is a Dedekind group.

(c) Every subgroup of \( L \) of prime power order belongs to \( \mathcal{H}(G) \).

Throughout this paper \( S_p(G) \) denotes the set of cyclic subgroups of a group \( G \) of prime order or of order 4. Also \( \overline{S_p(G)} \) denotes the set of subgroups of \( G \) of prime power order. Note that \( S_p(G) \subseteq \overline{S_p(G)} \).

**Theorem A.** Let \( G \) be a group. Then
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(a) If $S_p(G) \subseteq WN(G)$, then $G$ is supersolvable.

(b) If $S_p(G') \subseteq WN(G)$, then $G$ is supersolvable.

**Theorem B.** Let $G$ be a group. Then $S_p(G) \leq WN(G)$ if and only if there exist subgroups $L$ and $D$ of $G$ such that

(a) $G = L \rtimes D$, the semidirect product of $L$ by $D$.

(b) $L$ and $D$ are nilpotent Hall subgroups of $G$.

(c) $S_p(L) \subseteq WN(G)$.

(d) $S_p(D) \subseteq WN(D)$.

**Theorem C.** A group $G$ is a solvable $T$-group if and only if $G$ has subgroups $L$ and $D$ such that

(a) $G = L \rtimes D$, the semidirect product of $L$ by $D$.

(b) $L$ and $D$ are nilpotent Hall subgroups of $G$.

(c) $S_p(L) \subseteq WN(G)$.

(d) $S_p(D) \subseteq WN(D)$.

One of the purposes of this paper is to determine if a theorem like Theorem C might be proven for some classes of groups related to solvable $T$-groups. Such classes include solvable $PST$- and $PT$-groups. Let $G$ be a group. A subgroup $H$ of $G$ is said to permute with a subgroup $K$ of $G$ if $HK$ is a subgroup of $G$. $H$ is said to be $S$-permutable if it permutes with all the Sylow subgroups of $G$. Kegel [8] showed that an $S$-permutable subgroup of $G$ is subnormal in $G$. Kegel’s result generalized Ore’s result [11] that a permutable subgroup of $G$ is subnormal in $G$. A group $G$ is called a $PST$- (resp. $PT$-)group if $S$-permutability (resp. permutability) is a transitive relation in $G$, that is, if $H \subseteq K$ are subgroups of $G$ such that $H$ is $S$-permutative (resp. permutable) in $K$ and $K$ is $S$-permutable (resp. permutable) in $G$, then $H$ is $S$-permutable (resp. permutable) in $G$. By Kegel’s (resp. Ore’s) result $G$ is a $PST$- (resp. $PT$-)group if every subnormal subgroup of $G$ is $S$-permutable (resp. permutable) in $G$. $PST$-groups have been studied in detail in [1, 2, 4]. Solvable $PST$- and $PT$-groups have been characterized by Agrawal [1]. He proved the following theorem.

**Theorem 4.** Let $G$ be a group. Then
(a) $G$ is a solvable PST-group if and only if it has a normal abelian Hall subgroup $L$ such that $G/L$ is nilpotent and $G$ acts by conjugation on $L$ as power automorphisms.

(b) $G$ is a solvable PT-group if and only if it is a solvable PST-group with Iwasawa Sylow subgroups.

G. Zacher [13] proved part (b) of Theorem 4 in 1964. An Iwasawa group is one in which every subgroup is permutable.

Theorem D. A group $G$ is a solvable PST-group if and only if it has subgroups $L$ and $D$ such that

(a) $G = L \rtimes D$.

(b) $L$ and $D$ are nilpotent Hall subgroups of $G$.

(c) $\overline{S}_p(L) \subseteq WN(G)$.

Theorem E. Let $L$ be a normal Hall subgroup of $G$ such that

(a) $G/L$ is a PST-group.

(b) Every subnormal subgroup of $L$ is weakly normal in $G$.

Then $G$ is a PST-group.

Theorem F. A group $G$ is a solvable PST-group if and only if it has a normal nilpotent Hall subgroup $L$ such that $G/L$ is a solvable PST-group and $\overline{S}_p(L) \subseteq WN(G)$.

2 Preliminary results.

By Lemma 2.1 of [10] and Lemma 3 of [3] we have

Lemma 1. Let $H$, $K$ and $N$ be subgroups of $G$. Then

(a) If $H \leq K$ and $H \in WN(G)$, then $H \in WN(K)$.

(b) Let $N \leq H$. Then $H \in WN(G)$ if and only if $H/N \in WN(G/N)$.

(c) If $H \trianglelefteq K$ and $H \in WN(G)$, then $H \trianglelefteq K$.

Lemma 2. Let $G$ be a group.

(a) Let $M$ and $L$ be subgroups of $G$ such that $ML = LM$, $(|L|, |M|) = 1$ and $G = MN_G(L)$. Then $L \in WN(G)$. 
(b) Let $N \trianglelefteq G$ and $P$ a $p$-subgroup of $G$ such that $(p, |N|) = 1$. If $P \in \text{WN}(G)$, then $PN \in \text{WN}(G)$ and $PN/N \in \text{WN}(G/N)$.

Proof. The exact same proofs of Lemmas 5 and 6 of [6] establish parts (a) and (b) of Lemma 2. QED

The next lemma contains results established by Li in [9].

Lemma 3. Let $G$ be a group, $p$ a prime and $P$ a Sylow $p$-subgroup of $G$.

(a) If $p > 2$ and every minimal subgroup of $P$ lies in the center of $N_G(P)$, then $G$ is $p$-nilpotent.

(b) If $p = 2$ and every cyclic subgroup of $P$ of order 2 or 4 is normal in $N_G(P)$, then $G$ is 2-nilpotent.

(c) If $G$ possesses a normal 2-complement $N$ and if every minimal subgroup of any Sylow subgroup $R$ of $N$ is normal in $N_G(R)$, then $G$ is supersolvable.

3 Proofs of the Theorems.

Proof of Theorem A. (a) Assume that $S_p(G) \subseteq \text{WN}(G)$. Let $P$ be a Sylow $p$-subgroup of $G$, $p$ a prime and let $U \leq P$ be a subgroup of order $p$ or 4 if $p = 2$. Then $U \in S_p(G) \subseteq \text{WN}(G)$ and $U \triangleleft N_G(P)$. By part (c) of Lemma 1 $U \triangleleft N_G(P)$. Hence for every prime $p$ the subgroups of $S_p(P)$ are normal in $N_G(P)$. Thus $G$ is supersolvable by parts (b) and (c) of Lemma 3. Thus (a) is true. We also note (a) follows from Theorem 3.1 of [10].

Assume that $S_p(G') \subseteq \text{WN}(G)$. We may assume that $G' \neq 1$. Note that $S_p(G') \subseteq \overline{S_p(G')} \subseteq \text{WN}(G')$ by part (a) of Lemma 1 and hence $G'$ is supersolvable. Clearly $G$ is solvable. Let $M$ be a minimal normal subgroup of $G$ contained in $G'$. $M$ is an elementary abelian $q$-group for some prime $q$. Let $\langle x \rangle$ be a subgroup of $M$ of order $q$. Then $\langle x \rangle \in \text{WN}(G)$ and by part (c) of Lemma 1 $\langle x \rangle \triangleleft G$. Consider $G/\langle x \rangle$ and note $G'/\langle x \rangle = (G/\langle x \rangle)'$. By part (b) of Lemma 1 and part (b) of Lemma 2 $\overline{S_p(G'/\langle x \rangle)} \leq \text{WN}(G/\langle x \rangle)$. Hence, by induction, $G/\langle x \rangle$ is supersolvable and so $G$ is also supersolvable. Thus (b) is also true. QED

Proof of Theorem B. Assume that $S_p(G) \subseteq \text{WN}(G)$. By part (a) of Theorem A $G$ is supersolvable. Thus $S_p(G) \subseteq \text{WN}(G) \subseteq \mathfrak{H}(G)$ by part (a) of Theorem 1. By Theorem 10 of [6] there are subgroups $L$ and $D$ of $G$ which satisfy $L$ and $D$ are Hall nilpotent subgroups of $G$ such that $L \triangleleft G$ and $G = L \times D$. Also $S_p(L)$ consists of normal subgroups of $G$ and $S_p(D)$ consists of normal subgroups of $D$. Therefore, (a)–(d) holds.
Conversely, assume that $G$ has Hall subgroups $L$ and $D$ which satisfy (a)--(d). By part (c) of Lemma 1 the subgroups in $S_p(L)$ are all normal in $G$ and the subgroups in $S_p(D)$ are normal in $D$. Thus, by Theorem 10 of [6], $S_p(G) \subseteq \mathcal{H}(G)$. But $\mathcal{H}(G) \subseteq WN(G)$.

This completes the proof of Theorem B.

Proof of Theorem C. Assume $G$ satisfies (a)--(d) and let $X$ be a $q$-subgroup of $L$, $q$ a prime. By (b) $L$ is nilpotent and so $X \vartriangleleft G$. By (c) $X \in WN(G)$ and hence, by part (c) of Lemma 1, $X \vartriangleleft G$. Thus $L$ is a Dedekind group. Now let us assume $X$ has order $q$. Then $L/X$ is a Hall nilpotent subgroup of $G/X$ and $\overline{S_p}(L/X) \subseteq WN(G/X)$ by part (b) of Lemma 1. Also $DX/X$ is a nilpotent Hall subgroup of $G/X$ and it is isomorphic to $D$. Thus $\overline{S_p}(DX/X) \subseteq WN(DX/X)$. This means that $G/X$ satisfies (a)--(d) with respect to $L/X$ and $DX/X$. By induction on the order of $G$, $G/X$ is supersolvable and so $G$ is supersolvable. This means $L$ is abelian and $G$ acts on $L$ as power automorphisms. By Gaschütz’s Theorem [7] it is enough to show $D$ is a Dedekind group. By part (a) of Theorem 1 $\overline{S_p}(L) \subseteq \mathcal{H}(G)$ and $\overline{S_p}(D) \subseteq \mathcal{H}(D)$. Let $r$ be a prime divisor of the order of $D$ and let $R$ be the Sylow $r$-subgroup of $D$. Note that $\overline{S_p}(R) = \mathcal{H}(R)$. Let $Y$ be a subgroup of $R$. Then $Y \vartriangleleft R$ by part (c) of Lemma 1. Thus $D$ is a Dedekind group.

Conversely, assume that $G$ is a solvable $T$-group. Then, by Gaschütz’s Theorem [7], the nilpotent residual $L$ of $G$ is a normal abelian Hall subgroup of $G$ on which $G$ acts as power automorphisms and $G/L$ is a Dedekind group. Let $D$ be a system normalizer of $G$. By the Gaschütz, Shenkman and Carter Theorem, Theorem 9.2.7 of [12, p. 264], $G = L \rtimes D$. Thus (a) and (b) hold. Since all the subgroups of $L$ are normal in $G$ it follows that $\overline{S_p}(L) \subseteq WN(G)$. Likewise, since $D$ is a Dedekind group, $\overline{S_p}(D) \subseteq WN(D)$. This completes the proof.

Proof of Theorem D. Assume that $G$ is a solvable $PST$-group and let $L$ be the nilpotent residual of $G$. By part (a) of Theorem 4 $L$ is an abelian normal Hall subgroup of $G$ on which $G$ acts by conjugation as power automorphisms. Let $D$ be a system normalizer of $G$. By Theorem 9.2.7 of [12, p. 264], $G = L \rtimes D$. Note that $D$ is a nilpotent Hall subgroup of $G$. Let $X$ be a subgroup of $L$. Then $X \vartriangleleft G$ and so $\overline{S_p}(L) \subseteq WN(G)$. Therefore, (a), (b) and (c) are true.

Conversely, assume (a), (b) and (c) hold for the group $G$. Then $G$ has nilpotent Hall subgroups $L$ and $D$ such that $L \vartriangleleft G$ and $G = L \rtimes D$. Clearly $G$ is solvable. Note that $G/L$ is nilpotent and hence is a $PST$-group. Let $X$ be a subgroup of $L$ of prime power order. Then $X \in \overline{S_p}(L) \subseteq WN(G)$ and $X \vartriangleleft G$. By part (c) of Lemma 1 $X \vartriangleleft G$. Therefore, by Theorem 2.4 of [1] $G$ is a solvable $PST$-group.
**Proof of Theorem E.** Let $L$ be a normal Hall subgroup of $G$ such that $G/L$ is a PST-group and every subnormal subgroup of $L$ belongs to $WN(G)$. Let $X$ be a subnormal subgroup of $L$. Then $X$ is subnormal in $G$ and hence $X$ is normal in $G$ by part (c) of Lemma 1. Thus $G$ is a PST-group by Theorem 2.4 of [1].

**QED**

**Proof of Theorem F.** This follows from Theorem 2.3 of [1] and Theorem E.

**QED**

**References**


