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(λ, μ)-statistical convergence of double sequences in n-normed spaces

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Abstract. In this paper, we introduce the concept of $(\lambda,\ \mu)$-statistical convergence in n-normed spaces, where $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two non-decreasing sequences of positive real numbers, each tending to ∞ and such that $\lambda_{r+1} \leq \lambda_r + 1,\ \lambda_1 = 1;\ \mu_{s+1} \leq \mu_s + 1,\ \mu_1 = 1$. Some inclusion relations between the sets of statistically convergent and $(\lambda,\ \mu)$-statistically convergent double sequences are established. We find its relation to statistical convergence, $(C, 1, 1)$-summability and strong (V, λ, μ)-summability in n-normed spaces.

Keywords: Statistical convergence; λ-statistical convergence; double sequence; n-norm.

MSC 2000 classification: primary 40A05; 40B50, secondary 46A19; 46A45.

1 Introduction

The concept of statistical convergence plays a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [38] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view. For example, statistical convergence has been investigated in summability theory by (Fridy [7], Connor [3], Šalát [25]), topological groups (Çakalli [1]) and (Çakalli and Savas [30]) topological spaces (Maio and Kočinac [18]), measure theory (Miller [19]), locally convex spaces (Maddox[17]) and many

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others. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions [2].

The notion of statistical convergence depends on the (natural or asymptotic) density of subsets of $\mathbf{N}$. A subset E of $\mathbf{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

Note that if $K \subset \mathbf{N}$ is finite set, then $\delta(K) = 0$, and for any set $K \subset \mathbf{N}, \delta(K^C) = 1 - \delta(K)$.

Definition 1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbf{N} \mid |x_k - \ell| \geq \varepsilon\}) = 0.$$

In other words we can write the sequence (x_k) *statistical converges* to ℓ if

$$\lim_{r\to\infty} \frac{1}{r} |\{k \leq r \mid |x_k - \ell| \geq \varepsilon\}| = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and S denote the set of all statistically convergent sequences.

Mursaleen and Osama [22] extended the above idea from single to double sequences of scalars and established relations between statistical convergence and strongly Cesàro summable double sequences. Besides this, Mursaleen [21] presented a generalization of statistical convergence with the help of λ-summability methods and called it λ-statistical convergence. Later on, Savaş [37] presented λ- statistical convergence of fuzzy numbers. Also asymptotically λ -statistical equivalent sequences of fuzzy numbers and almost λ -statistical convergence were studied by Savaş [35] and [36] respectively. Furthermore, double λ- statistical convergence was studied and examined by Savaş ([31], [33], [35]) and Savaş and Patterson ([32], [34]).

Let $\lambda = (\lambda_r)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{r+1} \leq \lambda_r + 1, \lambda_1 = 1.$$

The collection of such sequence λ will be denoted by Δ.

The generalized de la Vallée Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k\in I_r} x_k,$$

where $I_r = [r - \lambda_r + 1, r]$.

Definition 2. [15] A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number ℓ if

$$t_r(x) \to \ell, \quad \text{as} \quad r \to \infty.$$

If $\lambda_r = r$, then (V, λ)-summability reduces to $(C, 1)$-summability. We write

$$[C,1] = \left\{ x = (x_k) \mid \exists\, \ell \in \mathbf{R}, \lim_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} |x_k - \ell| = 0 \right\}$$

and

$$[V,\lambda] = \left\{ x = (x_k) \mid \exists\, \ell \in \mathbf{R}, \lim_{r\to\infty} \frac{1}{\lambda_r} \sum_{k\in I_r} |x_k - \ell| = 0 \right\}$$

for the sets of sequences $x = (x_k)$ which are *strongly Cesàro summable* (see [6]) and *strongly* (V, λ)*-summable* to ℓ, i.e. $x_k \to \ell[C, 1]$ and $x_k \to \ell[V, \lambda]$, respectively.

Let $K \subseteq \mathbf{N}$ be a set of positive integers. Then

$$\delta_\lambda(K) = \lim_r \frac{1}{\lambda_r} |\{r - \lambda_r + 1 \le k \le r \mid k \in K\}|$$

is said to be λ-*density* of K provided the limit exists.

Definition 3. [21]. A sequence $x = (x_k)$ is said to be λ-statistically convergent or S_λ-convergent to ℓ if for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{\lambda_r} |\{k \in I_r \mid |x_k - \ell| \ge \varepsilon\}| = 0.$$

In this case we write $S_\lambda - \lim x = \ell$ or $x_k \to \ell(S_\lambda)$ and

$$S_\lambda = \{x = (x_k) \mid \exists\, \ell \in \mathbf{R}, S_\lambda - \lim x = \ell\}.$$

It is clear that if $\lambda_r = r$, for all r then S_λ reduces to S and since $\left(\frac{\lambda_r}{r}\right) \le 1, \delta(K) \le \delta_\lambda(K)$ for every $K \subseteq \mathbf{N}$.

The concept of 2-normed space was initially introduced by Gähler [9], in the mid 1960's, while that of n-normed spaces can be found in Misiak [20]. Since then, many others authors have studied this concept and obtained various results (see, for instance, Gunawan[11], Gähler [8], Gunawan and Mashadi ([10], [12]), Lewandowska [16], Dutta [4])and Savas ([26], [27], [28] and [29]).

2 Definitions and Preliminaries

Let n be a non negative integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $||.,\dots,.||$ from X^n into $\mathbf{R}$ satisfying the following conditions:

(1) $||x_1, x_2, \dots, x_n|| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,

(2) $||x_1, x_2, \dots, x_n||$ is invariant under permutation,

(3) $||\alpha x_1, x_2, \dots, x_n|| = |\alpha| ||x_1, x_2, \dots, x_n||$, for any $\alpha \in \mathbf{R}$,

(4) $||x + \overline{x}, x_2, \dots, x_n|| \leq ||x, x_2, \dots, x_n|| + ||\overline{x}, x_2, \dots, x_n||$ is called an n-norm on X and the pair $(X, ||.,\dots,.||)$ is called an n-normed space.

A trivial example of an n-normed space is $X = \mathbf{R}^n$, equipped with the Euclidean n-norm $||x_1, x_2, \dots, x_n||_E =$ the volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \dots, x_n$ which may be given expicitly by the formula

$$||x_1, x_2, \dots, x_n||_E = |\det(x_{ij})| = |(\det(< x_i, x_j >))|$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbf{R}^n$ for each $i = 1, 2, 3, \dots, n$.

Let $(X, ||.,\dots,.||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ be a linearly independent set in X. Then the function $||.,\dots,.||_\infty$ from X^{n-1} into $\mathbf{R}$ is defined by

$$||x_1, x_2, \dots, x_{n-1}||_\infty = \max_{1 \leq i \leq n} \{||x_1, x_2, \dots, x_{n-1}, a_i||\}$$

defines as $(n-1)$-norm on X with respect to $\{a_1, a_2, \dots, a_n\}$ and this is known as the derived $(n-1)$-norm (for details see [10]).

The standard n-norm on X a real inner product space of dimension $d \geq n$ is as follows:

$$||x_1, x_2, \dots, x_n||_S = [\det(< x_i, x_j >)]^{\frac{1}{2}},$$

where $<,>$ denote the inner product on X. If we take $X = \mathbf{R}^n$ then this n-norm is exactly the same as the Euclidean n-norm $||x_1, x_2, \dots, x_n||_E$ mentioned earlier. For $n = 1$ this n-norm is the usual norm $||x_1|| = \sqrt{< x_1, x_1 >}$(for further details see [10]).

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense [23]. A double sequence $x = (x_{k,l})$ has a *Pringsheim* limit L (denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "$P - convergent$".

Let $K \subset \mathbf{N} \times \mathbf{N}$ and let $K(m, n)$ denote the number of (i, j) in K such that $i \leq m$ and $j \leq n$, (see [22]). Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n\to\infty} \frac{K(m,n)}{mn}$. In case, the sequence $(\frac{K(m,n)}{mn})$ has a limit in Pringsheim's sense, then we say that K has a double natural density and is defined by $P - \lim_{m,n\to\infty} \frac{K(m,n)}{mn} = \delta_2(K)$.

Example 1. Let $K = \{(i^2, j^2) \mid i, j \in \mathbf{N}\}$. Then

$$\delta_2(K) = P - \lim_{m,n\to\infty} \frac{K(m,n)}{mn} \leq P - \lim_{m,n\to\infty} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set K has double natural density zero, while the set $\{(i, 3j) \mid i, j \in \mathbf{N}\}$ has double natural density $\frac{1}{3}$.

Definition 4. A double sequence $(x_{k,l})$ in an n-normed space $(X, ||., \ldots, .||)$ is said to be *statistically-convergent* to some $\ell \in X$ with respect to the n-norm if for each $\varepsilon > 0$ such that the set

$$\{(k, l) \in \mathbf{N} \times \mathbf{N} \mid ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| \geq \varepsilon\}$$

has double natural density zero, for all $z_1, z_2, \ldots, z_{n-1} \in X$.

In other words the double sequence $(x_{k,l})$ *statistical converges* to ℓ in n-normed space X if

$$P - \lim_{r,s\to\infty} \frac{1}{rs} |\{k \leq r,\ l \leq s \mid ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| \geq \varepsilon\}| = 0,$$

for all $z_1, z_2, \ldots, z_{n-1} \in X$.

Let $S_2^{nN}(X)$ denote the set of all statistically convergent sequences in n-normed space X.

Recently Gürdal and Pehlivan [13] studied statistical convergence in 2-normed spaces. And also B.S. Reddy [24] extended this idea to n-normed spaces and studied some properties. In [14], Hazarika and Savaş studied λ-statistical convergence in n-normed spaces.

In the present paper we study $(\lambda,\ \mu)$-statistical convergence of double sequences in n-normed spaces. We show that some properties of $(\lambda,\ \mu)$-statistical convergence of real numbers also hold for sequences in n-normed spaces. We find some relations related to statistically convergent, $(\lambda,\ \mu)$-statistically convergent double sequences, $(C$,1,1)-summability and strong $(V,\ \lambda,\ \mu)$-summability in n-normed spaces.

3 (λ, μ)-statistical convergence in n-normed space X

In this section we define $(\lambda,\ \mu)$-statistically convergent double sequence in n-normed linear space X. Also we obtained some basic properties of this notion in n-normed spaces.

Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two non-decreasing sequences of positive real numbers, each tending to ∞ and such that $\lambda_{r+1} \leq \lambda_r + 1,\ \lambda_1 = 1;\ \mu_{s+1} \leq \mu_s + 1,\ \mu_1 = 1$. Let $I_r = [r - \lambda_r + 1, r]$, $I_s = [s - \mu_s + 1, s]$ and $I_{r,s} = I_r \times I_s$.

For any set $K \subseteq \mathbf{N} \times \mathbf{N}$, the number,

$$\delta_{(\lambda,\mu)}(K) = P - \lim_{r,s\to\infty,\infty} \frac{1}{\lambda_r \mu_s} |\{(i,j) \in I_r \times I_s : (i,j) \in k\}|;$$

is called (λ, μ)-*density* of the set K, provided the limit exists, (see, Savas [34].

Throughout we shall denote $\lambda_{r,s} = \lambda_r \mu_s$ and the collection of such sequences λ will be denoted by Δ_2.

Definition 5. A double sequence $x = (x_{k,l})$ in an n-normed space $(X, ||.,\dots,.||)$ is said to be (λ,μ)-*statistically convergent* or $S_{\lambda,\mu}$-*convergent* to $\ell \in X$ with respect to the n-norm if for every $\varepsilon > 0$

$$P - \lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} |\{(k,l) \in I_{r,s} \mid ||x_{k,l} - \ell, z_1, z_2, \dots, z_{n-1}|| \geq \varepsilon\}| = 0,$$

for all $z_1, z_2, \dots, z_{n-1} \in X$.

i.e., the set $K(\varepsilon) = \{(k,l) \in I_r \times I_s \mid ||x_{k,l} - \ell, z_1, z_2, ..., z_{n-1}|| \geq \varepsilon\}$ has (λ,μ)-density zero.

In this case we write $S^{nN}_{\lambda,\mu} - \lim x = \ell$ or $x_{k,l} \to \ell(S^{nN}_{\lambda,\mu})$ and

$$S^{nN}_{\lambda,\mu}(X) = \{x = (x_{k,l}) \mid \exists\ \ell \in \mathbf{R}, S^{nN}_{\lambda,\mu} - \lim x = \ell\}.$$

Let $S^{nN}_{\lambda,\mu}(X)$ denote the set of all (λ,μ)-statistically convergent of double sequences in n-normed space X.

If $\lambda_r = r$ and $\mu_s = s$ for all r, s then the space $S^{nN}_{\lambda,\mu}(X)$ is reduced to the space $S^{nN}_2(X)$ and since $\delta_2(K) \leq \delta_{\lambda,\mu}(K)$, we have $S^{nN}_{\lambda,\mu}(X) \subset S^{nN}_2(X)$.

We define the generalized double de la Vallée Poussin mean by

$$t_{r,s}(x) = \frac{1}{\lambda_r \mu_s} \sum_{k\in I_r} \sum_{l \in I_s} x_{k,l}.$$

Definition 6. A double sequence $x = (x_{k,l})$ in an n-normed space $(X, ||.,\dots,.||)$ is said to be (V,λ,μ)-*summable* to $\ell \in X$ with respect to the n-norm if

$$P - \lim_{r,s} t_{r,s}(||x - \ell, z_1, z_2, \dots, z_{n-1}||) = 0,$$

for all $z_1, z_2, \ldots, z_{n-1} \in X$. If $\lambda_{r,s} = rs$, then (V, λ, μ)-summability reduces to $(C, 1, 1)$-summability with respect to the n-norm. We write

$$[C,1,1]^{nN}(X) = \Big\{x = (x_{k,l}) \mid \exists\, \ell \in \mathbf{R}, P - \lim_{r,s\to\infty} \frac{1}{rs} \sum_{k,l=1,1}^{r,s} ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| = 0\Big\}$$

and

$$[V,\lambda,\mu]^{nN}(X) = \Big\{x = (x_{k,l}) \mid \exists\, \ell \in \mathbf{R}, P - \lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} \sum_{(k,l)\in I_{r,s}} ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| = 0\Big\}$$

for the sets of X-valued double sequences $x = (x_{k,l})$ which are *strongly Cesàro summable* and *strongly (V, λ, μ)-summable* to ℓ with respect to the n-norm , i.e. $x_{k,l} \to \ell([C, 1, 1]^{nN}(X))$ and $x_{k,l} \to \ell([V, \lambda, \mu]^{nN}(X))$, respectively.

Theorem 1. *Let X be an n-normed space and $(\lambda_{r,s}) \in \Delta_2$. If $(x_{k,l})$ is a sequence in X such that $S_{\lambda,\mu}^{nN} - \lim x_{k,l} = \ell$ exists, then it is unique.*

Proof. Suppose that there exist elements ℓ_1, ℓ_2 $(\ell_1 \neq \ell_2)$ in X such that

$$S_{\lambda,\mu}^{nN} - \lim_{k,l\to\infty} x_{k,l} = \ell_1; S_{\lambda,\mu}^{nN} - \lim_{k,l\to\infty} x_{k,l} = \ell_2.$$

Since $\ell_1 \neq \ell_2$, then $\ell_1 - \ell_2 \neq 0$, so there exist $z_1, z_2, \ldots, z_{n-1} \in X$ such that $\ell_1 - \ell_2$ and $z_1, z_2, \ldots, z_{n-1}$ are linearly independent. Therefore

$$||\ell_1 - \ell_2, z_1, z_2, \ldots, z_{n-1}|| = 2\varepsilon > 0.$$

Since $S_{\lambda,\mu}^{nN} - \lim_{k,l\to\infty} x_{k,l} = \ell_1$ and $S_{\lambda,\mu}^{nN} - \lim_{k,l\to\infty} x_{k,l} = \ell_2$ it follows that

$$P - \lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} |\{(k,l) \in I_{r,s} \mid ||x_{k,l} - \ell_1, z_1, z_2, \ldots, z_{n-1}|| \geq \varepsilon\}| = 0$$

and

$$P - \lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} |\{(k,l) \in I_{r,s} \mid ||x_{k,l} - \ell_2, z_1, z_2, \ldots, z_{n-1}|| \geq \varepsilon\}| = 0.$$

There are $k, l \in I_{r,s}$ such that

$$||x_{k,l} - \ell_1, z_1, z_2, \ldots, z_{n-1}|| < \varepsilon \text{ and } ||x_{k,l} - \ell_2, z_1, z_2, \ldots, z_{n-1}|| < \varepsilon.$$

Further, for these k, l we have

$$||\ell_1 - \ell_2, z_1, z_2, \ldots, z_{n-1}||$$
$$\leq ||x_{k,l} - \ell_1, z_1, z_2, \ldots, z_{n-1}|| + ||x_{k,l} - \ell_2, z_1, z_2, \ldots, z_{n-1}|| < 2\varepsilon$$

which is a contradiction. This completes the proof.

The next theorem gives the algebraic characterization of (λ, μ)-statistical convergence on n-normed spaces.

Theorem 2. *Let X be an n-normed space, $(\lambda_{r,s}) \in \Delta_2$, $x = (x_{k,l})$ and $y = (y_{k,l})$ be two sequences in X.*
(a) If $S^{nN}_{\lambda,\mu} - \lim_{k,l\to\infty} x_{k,l} = \ell$ and $c(\neq 0) \in \mathbf{R}$, then $S^{nN}_{\lambda,\mu} - \lim_{k,l\to\infty} cx_{k,l} = c\ell$. (b) If $S^{nN}_{\lambda,\mu} - \lim_{k,l\to\infty} x_{k,l} = \ell_1$ and $S^{nN}_{\lambda,\mu} - \lim_{k,l\to\infty} y_{k,l} = \ell_2$, then $S^{nN}_{\lambda,\mu} - \lim_{k,l\to\infty}(x_{k,l} + y_{k,l}) = \ell_1 + \ell_2$.

Proof of the theorem is straightforward, thus omitted.

Theorem 3. *$S^{nN}_{\lambda,\mu}(X) \cap \ell^2_\infty(X)$ is a closed subset of $\ell^2_\infty(X)$, if X an n-Banach space.*

Proof. Suppose that $(x^i)_{i\in\mathbf{N}} = (x^i_{k,l})_{k,l\in\mathbf{N}}$ is a convergent sequence in $S^{nN}_{\lambda,\mu}(X) \cap \ell^2_\infty(X)$ converging to $x \in \ell^2_\infty(X)$. We need to prove that $x \in S^{nN}_{\lambda,\mu}(X) \cap \ell^2_\infty(X)$. Assume that $x^i \to \ell_i(S^{nN}_{\lambda,\mu}(X))$, for each $i \in \mathbf{N}$. Take a positive decreasing convergent sequence $(\varepsilon_i)_{i\in\mathbf{N}}$, where $\varepsilon_i = \frac{\varepsilon}{2^i}$, for a given $\varepsilon > 0$. Clearly $(\varepsilon_i)_{i\in\mathbf{N}}$ converges to 0. Choose a positive integer i such that $||x - x^i, z_1, z_2, \ldots, z_{n-1}||_\infty < \frac{\varepsilon_i}{4}$, for all $z_1, z_2, \ldots, z_{n-1} \in X$. Then we have

$$P - \frac{1}{\lambda_{r,s}}|\{(k,l) \in I_{r,s} \mid ||x^i_{k,l} - \ell_i, z_1, z_2, \ldots, z_{n-1}|| \geq \frac{\varepsilon_i}{4}\}| = 0$$

and

$$P - \frac{1}{\lambda_{r,s}}|\{(k,l) \in I_{r,s} \mid ||x^{i+1}_{k,l} - \ell_{i+1}, z_1, z_2, \ldots, z_{n-1}|| \geq \frac{\varepsilon_{i+1}}{4}\}| = 0.$$

Now

$$\frac{1}{\lambda_{r,s}}|\{(k,l) \in I_{r,s} \mid ||x^i_{k,l} - \ell_i, z_1, z_2, \ldots, z_{n-1}|| \geq \frac{\varepsilon_i}{4} \vee ||x^{i+1}_{k,l} - \ell_{i+1}, z_1, z_2, \ldots, z_{n-1}|| \geq \frac{\varepsilon_{i+1}}{4}\}| < 1$$

and for all $k, l \in \mathbf{N}$

$$\{(k,l) \in I_{r,s} \mid ||x^i_{k,l} - \ell_i, z_1, z_2, ..., z_{n-1}|| \geq \frac{\varepsilon_i}{4}\} \cap \{(k,l) \in I_{r,s} \mid ||x^{i+1}_{k,l} - \ell_{i+1}, z_1, z_2, \ldots, z_{n-1}|| \geq \frac{\varepsilon_{i+1}}{4}\}$$

is infinite. Hence there must exist $k,l\in I_{r,s}$ for which we have simultanously,

$$||x^i_{k,l}-\ell_i,z_1,z_2,\dots,z_{n-1}||<\frac{\varepsilon_i}{4}\ \text{and}\ ||x^{i+1}_{k,l}-\ell_{i+1},z_1,z_2,\dots,z_{n-1}||<\frac{\varepsilon_{i+1}}{4}.$$

Then it follows that

$$\begin{aligned}&||\ell_i-\ell_{i+1},z_1,z_2,\dots,z_{n-1}||\\ &\le ||\ell_i-x^i_{k,l},z_1,z_2,\dots,z_{n-1}||+||x^i_{k,l}-x^{i+1}_k,z_1,z_2,\dots,z_{n-1}||\\ &+||x^{i+1}_{k,l}-\ell_{i+1},z_1,z_2,\dots,z_{n-1}||\\ &\le ||x^i_{k,l}-\ell_i,z_1,z_2,\dots,z_{n-1}||+||x^{i+1}_{k,l}-\ell_{i+1},z_1,z_2,\dots,z_{n-1}||\\ &+||x-x^i,z_1,z_2,\dots,z_{n-1}||_\infty+||x-x^{i+1},z_1,z_2,\dots,z_{n-1}||_\infty\\ &\le\frac{\varepsilon_i}{4}+\frac{\varepsilon_{i+1}}{4}+\frac{\varepsilon_i}{4}+\frac{\varepsilon_{i+1}}{4}\le\varepsilon_i.\end{aligned}$$

This implies that (ℓ_i) is a Cauchy sequence in X and thus there is number $\ell\in X$ such that $\ell_i\to\ell$ as $i\to\infty$. We need to prove that $x\to\ell(S^{nN}_{\lambda,\mu}(X))$.

For any $\varepsilon>0$, choose $i\in\mathbf{N}$ such that $\varepsilon_i<\frac{\varepsilon}{4}$,

$$||x-x^i,z_1,z_2,\dots,z_{n-1}||_\infty<\frac{\varepsilon}{4}\ \text{and}\ ||\ell_i-\ell,z_1,z_2,\dots,z_{n-1}||<\frac{\varepsilon}{4}.$$

Then

$$\begin{aligned}&\frac{1}{\lambda_{r,s}}|\{(k,l)\in I_{r,s}\mid ||x_{k,l}-\ell,z_1,z_2,\dots,z_{n-1}||\ge\varepsilon\}|\\ &\le\frac{1}{\lambda_{r,s}}|\{(k,l)\in I_{r,s}\mid ||x^i_{k,l}-\ell_i,z_1,z_2,\dots,z_{n-1}||\\ &+||x_{k,l}-x^i_{k,l},z_1,z_2,\dots,z_{n-1}||_\infty+||\ell_i-\ell,z_1,z_2,\dots,z_{n-1}||\ge\varepsilon\}|\\ &\le\frac{1}{\lambda_{r,s}}|\{(k,l)\in I_{r,s}\mid ||x^i_{k,l}-\ell_i,z_1,z_2,\dots,z_{n-1}||+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\ge\varepsilon\}|\\ &\le\frac{1}{\lambda_{r,s}}|\{(k,l)\in I_{r,s}\mid ||x^i_{k,l}-\ell_i,z_1,z_2,\dots,z_{n-1}||\ge\frac{\varepsilon}{2}\}|\to0\ \text{as}\ \ r,s\to\infty\end{aligned}$$

in Pringsheim sense. This gives that $x\to\ell(S^{nN}_{\lambda,\mu}(X))$, which completes the proof.

Theorem 4. *Let X be an n-normed space and let $(\lambda_{r,s})\in\Delta_2$. Then*

(i) $x_{k,l}\to\ell([V,\lambda,\mu]^{nN}(X))\Rightarrow x_{k,l}\to\ell(S^{nN}_{\lambda,\mu}(X))$,

(ii) $[V,\lambda,\mu]^{nN}(X)$ is a proper subset of $S^{nN}_{\lambda,\mu}(X)$,

(iii) $x\in\ell^2_\infty(X)$ and $x_{k,l}\to\ell(S^{nN}_{\lambda,\mu}(X))$ then $x_{k,l}\to\ell([V,\lambda,\mu]^{nN}(X))$ and hence $x_{k,l}\to\ell([C,1,1]^{nN}(X))$, provided $x=(x_{k,l})$ is not eventually constant,

(iv)$S^{nN}_{\lambda,\mu}(X)\cap\ell^2_\infty(X)=[V,\lambda,\mu]^{nN}(X)\cap\ell^2_\infty(X)$.

Proof. (i) If $\varepsilon > 0$ and $x_k \to \ell([V,\lambda,\mu]^{nN}(X))$, we can write

$$\sum_{(k,l)\in I_{r,s}} ||x_{k,l}-\ell, z_1, z_2, \ldots, z_{n-1}||$$

$$\geq \sum_{(k,l)\in I_{r,s}, ||x_{k,l}-\ell, z_1, z_2,\ldots,z_{n-1}||\geq\varepsilon} ||x_{k,l}-\ell, z_1, z_2, \ldots, z_{n-1}||$$

$$\geq \varepsilon |\{(k,l)\in I_{r,s} \mid ||x_{k,l}-\ell, z_1, z_2, \ldots, z_{n-1}|| \geq \varepsilon\}|$$

and so

$$\frac{1}{\varepsilon\lambda_{r,s}} \sum_{(k,l)\in I_{r,s}} ||x_{k,l}-\ell, z_1, z_2, \ldots, z_{n-1}|| \geq$$

$$\frac{1}{\lambda_{r,s}} |\{(k,l)\in I_{r,s} \mid ||x_{k,l}-\ell, z_1, z_2, \ldots, z_{n-1}|| \geq \varepsilon\}|.$$

This proves the result.

(ii) In order to establish that the inclusion $[V,\lambda,\mu]^{nN}(X) \subset S^{nN}_{\lambda,\mu}(X)$ is proper. We define a sequence $x = (x_{k,l})$ by

$$x_{k,l} = \begin{cases} kl, & \text{if } r-[\sqrt{\lambda_r}]+1 \leq k \leq r \text{ and } s-[\sqrt{\mu_s}]+1 \leq l \leq s; \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \notin \ell^2_\infty(X)$ and for every $\varepsilon \in]0,1[$,

$$\frac{1}{\lambda_{r,s}} \sum_{(k,l)\in I_{r,s}} ||x_{k,l}-0, z_1, z_2, \ldots, z_{n-1}|| \leq \frac{[\sqrt{\lambda_r}]\ [\sqrt{\mu_s}]}{\lambda_r\ \mu_s} \to 0, \quad \text{as} \quad r,s\to\infty,$$

i.e. $x_{k,l} \to 0(S^{nN}_{\lambda,\mu}(X))$. On the other hand,

$$\frac{1}{\lambda_{r,s}} |\ \{(k,l)\in I_{r,s} : ||x_{k,l}-0, z_1, z_2, \ldots, z_{n-1}|| \geq \varepsilon\}\ | \to \infty, \quad \text{as} \quad r,s\to\infty,$$

in Pringsheim sense, i.e. $x_{k,l}$ does not converge to 0 in $[V,\lambda,\mu]^{nN}(X)$.

(iii) Suppose that $x_{k,l} \to \ell(S^{nN}_{\lambda,\mu}(X))$ and $x \in \ell^2_\infty(X)$. Then there exists a $M > 0$ such that $||x_{k,l}-\ell, z_1, z_2, \ldots z_{n-1}|| \leq M$ for all $k,l \in \mathbf{N}$. Given $\varepsilon > 0$, we have

$$\frac{1}{\lambda_{r,s}} \sum_{(k,l)\in I_{r,s}} ||x_{k,l}-\ell, z_1, z_2, \ldots, z_{n-1}|| =$$

$$\frac{1}{\lambda_{r,s}} \sum_{(k,l)\in I_{r,s}, ||x_{k,l}-\ell, z_1, z_2,\ldots,z_{n-1}||\geq\frac{\varepsilon}{2}} ||x_{k,l}-\ell, z_1, z_2, \ldots, z_{n-1}||$$

$$+\frac{1}{\lambda_{r,s}}\sum_{(k,l)\in I_{r,s},||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||<\frac{\varepsilon}{2}}||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||$$

$$\leq\frac{M}{\lambda_{r,s}}|\{(k,l)\in I_{r,s}\mid ||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||\geq\frac{\varepsilon}{2}\}|+\frac{\varepsilon}{2},$$

This shows that $x_{k,l}\to\ell([V,\lambda,\mu]^{nN}(X))$.

Again , we have

$$\frac{1}{rs}\sum_{k,l=1}^{r,s}||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||$$

$$\leq\frac{1}{rs}\sum_{k,l=1,1}^{r-\lambda_r,s-\mu_s}||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||+\frac{1}{rs}\sum_{(k,l)\in I_{r,s}}||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||$$

$$\leq\frac{1}{\lambda_{r,s}}\sum_{k,l=1,1}^{r-\lambda_r,s-\mu_s}||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||+\frac{1}{\lambda_{r,s}}\sum_{(k,l)\in I_{r,s}}||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||$$

$$\leq\frac{2}{\lambda_{r,s}}\sum_{(k,l)\in I_{r,s}}||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||.$$

Hence $x_{k,l}\to\ell([C,1,1]^{nN}(X))$, because $x_{k,l}\to\ell([V,\lambda,\mu]^{nN}(X))$.

(iv) This is an immediate consequence of (i), (ii) and (iii).

If we let $\lambda_{r,s}=rs$ in Theorem 4, then we have the following corollary.

Corollary 1. *Let X be an n-normed space. Then*
(i) $x_{k,l}\to\ell([C,1,1]^{nN}(X))\Rightarrow x_{k,l}\to\ell(S_2^{nN}(X))$,
(ii) $[C,1,1]^{nN}(X)$ is a proper subset of $S_2^{nN}(X)$,
(iii) $x\in\ell_\infty^2(X)$ and $x_{k,l}\to\ell(S_2^{nN}(X))$ then $x_{k,l}\to\ell([C,1,1]^{nN}(X))$,
(iv)$S_2^{nN}(X)\cap\ell_\infty^2(X)=[C,1,1]^{nN}(X)\cap\ell_\infty^2(X)$.

Theorem 5. *Let X be an n-normed space and let $(\lambda_{r,s})\in\Delta_2$. Then $S_2^{nN}(X)\subset S_{\lambda,\mu}^{nN}(X)$ if and only if $\liminf_{r,s}\frac{\lambda_{r,s}}{rs}>0$.*

Proof. Suppose first that $\liminf_{r,s}\frac{\lambda_{r,s}}{rs}>0$. Then for given $\varepsilon>0$, we have

$$\frac{1}{rs}|\{k\leq r,l\leq s\mid ||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||\geq\varepsilon\}|$$

$$\geq\frac{1}{rs}|\{(k,l)\in I_{r,s}\mid ||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||\geq\varepsilon\}|$$

$$\geq\frac{\lambda_{r,s}}{rs}.\frac{1}{\lambda_{r,s}}|\{(k,l)\in I_{r,s}\mid ||x_{k,l}-\ell,z_1,z_2,\ldots,z_{n-1}||\geq\varepsilon\}|.$$

It follows that $x_{k,l} \to \ell(S_2^{nN}(X)) \Rightarrow x_{k,l} \to \ell(S_{\lambda,\mu}^{nN}(X))$. Hence $S_2^{nN}(X) \subset S_{\lambda,\mu}^{nN}(X)$.

Conversely, suppose that $\liminf_{r,s} \frac{\lambda_{r,s}}{rs} = 0$. Then we can select a subsequence $(r(i), s(j))_{i,j=1,1}^{\infty,\infty}$ such that

$$\frac{\lambda_{r(i),s(j)}}{r(i)s(j)} < \frac{1}{ij}.$$

We define a sequence $x = (x_{k,l})$ as follows:

$$x_{k,l} = \begin{cases} 1, & \text{if } k,l \in I_{r(i),s(j)}, i,j = 1,2,3,\ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Then x is statistically convergent, so $x \in S_2^{nN}(X)$. But $x \notin [V,\lambda,\mu]^{nN}(X)$. Theorem 4(iii) implies that $x \notin S_{\lambda,\mu}^{nN}(X)$. This completes the proof.

Theorem 6. *Let X be an n-normed space and if $(\lambda_{r,s}) \in \Delta_2$ such that $\lim_{r,s} \frac{\lambda_{r,s}}{r,s} = 1$, then $S_{\lambda,\mu}^{nN}(X) = S_2^{nN}(X)$.*

Proof. Since $\lim_{r,s} \frac{\lambda_{r,s}}{r,s} = 1$, then for $\varepsilon > 0$, we observe that

$$\begin{aligned} &\frac{1}{rs}|\{k \le r, l \le s \mid ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| \ge \varepsilon\}| \\ &\le \frac{1}{rs}|\{k \le r - \lambda_r, l \le s - \mu_s \mid ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| \ge \varepsilon\}| \\ &\quad + \frac{1}{rs}|\{(k,l) \in I_{r,s} \mid ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| \ge \varepsilon\}| \\ &\le \frac{(r-\lambda_r)(s-\mu_s)}{rs} + \frac{1}{rs}|\{(k,l) \in I_{r,s} \mid ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| \ge \varepsilon\}| \\ &= \frac{(r-\lambda_r)(s-\mu_s)}{rs} + \frac{\lambda_{rs}}{rs}\frac{1}{\lambda_{rs}}|\{(k,l) \in I_{r,s} \mid ||x_{k,l} - \ell, z_1, z_2, \ldots, z_{n-1}|| \ge \varepsilon\}|. \end{aligned}$$

This implies that $(x_{k,l})$ statistically convergent, if $(x_{k,l})$ is (λ,μ)-statistically convergent. Thus $S_{\lambda,\mu}^{nN}(X) \subset S_2^{nN}(X)$.

Since $\lim_{r,s} \frac{\lambda_{r,s}}{r,s} = 1$, implies that $\liminf_{r,s} \frac{\lambda_{r,s}}{rs} > 0$, then from Theorem 5, we have $S_2^{nN}(X) \subset S_{\lambda,\mu}^{nN}(X)$. Hence $S_{\lambda,\mu}^{nN}(X) = S_2^{nN}(X)$.

Remark: We do not know whether the condition $\lim_{r,s} \frac{\lambda_{r,s}}{r,s} = 1$ in the Theorem 6 is necessary and leave it as an open problem.

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