Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **32** (2012) no. 2, 83–88. doi:10.1285/i15900932v32n2p83

Remarks on monochromatic configurations for finite colorings of the plane

Sukumar Das Adhikari
Harish-Chandra Research Institute, Jhusi, Allahabad 211 019, INDIA
adhikari@hri.res.in

Purusottam Rath
Chennai Mathematical Institute, Siruseri, Padur PO-603103, Chennai, INDIA
rath@cmi.ac.in

Received: 23.12.2011; accepted: 4.5.2012.

Abstract. Gurevich had conjectured that for any finite coloring of the Euclidean plane, there always exists a triangle of unit area with monochromatic vertices. Graham ([5], [6]) gave the first proof of this conjecture; a much shorter proof has been obtained recently by Dumitrescu and Jiang [4]. A similar result in the case of a trapezium, claimed by the present authors in [3] does not hold due to an error and a weaker result is recovered for quadrilaterals in this paper. We also take up the original question of triangles.

Keywords: Gurevich's conjecture, van der Waerden's theorem, Monochromatic configurations.

MSC 2000 classification: primary 05D10

1 Introduction

According to a conjecture of Gurevich, for any finite coloring of the Euclidean plane E^2, there always exists a triangle of unit area with monochromatic vertices. Graham [5] settled this particular question by proving the following stronger version:

Theorem 1. *(Graham). For a positive integer r, there is a positive integer $T(r)$ such that given any r-coloring of the set of integer lattice points in E^2, there is always a right angled triangle of area $T(r)$ such that all its vertices have the same color.*

One observes that by suitable scaling, Gurevich's conjecture follows immediately from the above result. In fact, as can be observed, for any $\alpha > 0$ and any pair of intersecting lines, for any finite coloring of the plane, Graham's proof provides a triangle of area α with monochromatic vertices having two sides parallel to the given lines. Furthermore, the proof of Graham uses van der

Waerden's theorem (stated in Section 2, Part III) which allows one to specify a bounded region where the monochromatic right angled triangle can be realized. Later, a shorter proof of the above theorem exploiting the main idea of Graham was furnished by the first author [2]. Recently, Gurevich's conjecture has been re-established with a different technique by Dumitrescu and Jiang [4].

As remarked by Graham [5], it is relatively straightforward to extend his arguments to derive analogous results in E^n. However, it is not clear as to what extent these results can be generalized to other configurations which are not simplexes. In the case of the Euclidean plane, it is not difficult to see that Theorem 1 can not hold for squares. Graham (see concluding remarks in [5] and the notes at the end of Chapter 7 of [6]) raises this question for parallelograms.

In this context, as a first attempt towards this, the simpler question of deriving similar results for trapeziums was considered by the present authors in [3] and the following result was claimed:

Let E^2 denote the Euclidean plane endowed with rectangular Cartesian co-ordinates. For a positive integer r, there is a positive integer T_r such that given any r-coloring of the set of integer lattice points in E^2, there is always a trapezium of area T_r such that all of its four vertices are lattice points of the same color and three of its sides are parallel to the co-ordinate axes.

It has recently come to our notice that our proof [3] of the above result has an error. More precisely, in our proof, T_r depends on a parameter d which, in turn, depends on the particular r-coloring involved. However, for arriving at the result, a T_r should be supplied which works for all r-coloring. Thus, whether the above result is true or not remains unknown. In the present note, we recover a weaker result for quadrilaterals.

We also describe some alternate approaches to the original question of triangles and furnish yet another proof of Gurevich's conjecture. As one can see from our proof, we exploit the idea of coloring of lines introduced by Dumitrescu and Jiang [4], albeit in different ways.

2 Triangles revisited

In this section, we take up the question of Gurevich in three steps; beginning with the naive approach, then with an added ingredient of density considerations and finally invoking van der Waerden's theorem which anyway constitutes the essential ingredient of all the existing proofs.

I. *The naive approach (Pigeon-hole principle).* We first begin with the naive approach, that is of simply applying the pigeon-hole principle. Suppose we are given an r-coloring of the lattice points in the Euclidean plane. Consider the square with diagonal vertices $(0,0)$ and (r,r). It consists of $r+1$ horizontal lines with each line consisting of $r+1$ lattice points. Thus there exists a color (say C) which occurs at least twice in two different lines and hence we are assured of a triangle having an area $\frac{1}{2}ad$ where $0 < a, d \leq r$ and having all its vertices of color C. However, there are too many of such possibilities; namely, the number of different areas is of the order $r^{2-\epsilon}$. This follows immediately from a theorem in additive number theory (see [8], Theorem 1.17) which states the following: if A is a subset of positive integers with cardinality r, with the notations $2A = \{a + a' | a, a' \in A\}$, $A^2 = \{aa' | a, a' \in A\}$, if $|2A| \leq 3r - 4$, then for any $\epsilon > 0$,

$$|A^2| \gg_\epsilon r^{2-\epsilon}.$$

II. *First refinement (Density consideration).* We now refine the above idea as follows. Given an r-coloring of the Euclidean plane as before, we now consider the $r+1$ horizontal lines $y = i$ with $0 \leq i \leq r$. Since only r colors are used, in each such line, one of the given colors necessarily occurs with upper natural density at least $1/r$ among the lattice points on that line. Again, since there are $r+1$ such lines, there exists a color (say C) which has upper natural density $\geq 1/r$ in two distinct lines. Let $y = a$ and $y = b$ be those lines with $b > a$. Let $d = b - a$ and $D := r!/d$. Now, since the color C occurs with upper natural density $\geq 1/r$ among the lattice points on the line $y = a$, looking at all residue classes mod D, at least one of the classes must have color C with upper natural density at least $1/r$. Thus, there exists an i with $0 \leq i \leq D - 1$ and an integer $n \geq 0$ such that C occurs at least twice among the following points

$$(nD + i, a), \cdots, ((n + r)D + i, a).$$

Since C also occurs in line $y = b$, we are ensured of a triangle having all its vertices of color C and of area

$$\frac{1}{2} \cdot d \cdot \frac{r!}{d} \cdot j = \frac{1}{2} j r!,$$

where $1 \leq j \leq r$. Thus the number of possibilities are now reduced to the order of r.

III. *Application of van der Waerden's theorem.* We begin with recalling van der Waerden's theorem which asserts the following ([9], one may also look into [1], [6] or [7]):

Given positive integers k and r, there exists a positive integer $W = W(k,r)$ such that for any r-coloring of $\{1, 2, \cdots, W\}$, there is a monochromatic arithmetic progression of length k.

Let an r-coloring of the lattice points of the Euclidean plane be given and S be the set of colors. As before, in every horizontal line at least one color occurs with upper natural density at least $1/r$ among the lattice points on that line. We now color each such line in the following way; we color (or label) each horizontal line by the non-empty subset T of S where T consists of all colors in S which occur with upper natural density at least $1/r$ among the lattice points on that line with the original coloring. Thus we are considering a $2^r - 1$ coloring of the set of horizontal lines. By van der Waerden's theorem quoted above, we get a collection of monochromatic horizontal lines

$$y = jb + a, \quad 0 \leq j \leq r!, \qquad (*)$$

where

$$(r!)b + a \leq W(r! + 1, 2^r - 1).$$

Arguing as in II above, since a color C occurs with upper natural density $\geq 1/r$ in all the lines in $(*)$, taking $D := (W(r! + 1, 2^r - 1))!/b$ and considering the line $y = a$, looking at all classes mod D, at least one of the classes must have color C with upper natural density at least $1/r$. Thus, there exists an i with $0 \leq i \leq D - 1$ and an integer $n \geq 0$ such that C occurs at least twice among the following points

$$(nD + i, a), \cdots, ((n + r)D + i, a),$$

say the points $((n + s)D + i, a), ((n + t)D + i, a)$, with $t > s$. Let $t - s = l \leq r$. Since C also occurs in line $y = db + a$, with $d = \frac{r!}{l}$, we are ensured of a triangle having all its vertices of color C and of area

$$\frac{r!}{2l} \cdot b \cdot l \cdot D = \frac{r!}{2l} \cdot b \cdot l \cdot (W(r! + 1, 2^r - 1))!/b = \frac{r!(W(r! + 1, 2^r - 1))!}{2}.$$

Since the last expression depends only on r, this proves that for a positive integer r, there is a positive integer $T(r)$ such that for any given r-coloring of the set of integer lattice points in E^2, there is always a triangle of area $T(r)$ such that all its vertices have the same color, once again establishing Gurevich's conjecture (by suitable scaling).

3 Further configurations

In this section, we consider the analog of Gurevich's question for non-simplicial configurations in the plane.

We begin by noting that the arguments given in our work [3] can be modified to deduce the following:

Theorem 2. *Given a $\delta > 0$ and a positive integer r, there exists an integer N_r depending only on r such that for any r coloring of the plane, one can find a monochromatic trapezium with three sides parallel to the coordinate axes and of area $d\delta$ where $d \leq N_r$.*

We now carry out similar investigations for quadrilaterals along the lines sketched in the previous section. We note that the naive approach for triangles as described in part I of the previous section carries through for trapeziums; more precisely:

Given a $\delta > 0$ and a positive integer r, there exists an integer $N_r = O(r^2)$ such that for any r coloring of the plane, one can find a monochromatic trapezium (not necessarily right angled) of area $d\delta$ where $d \leq N_r$.

Let us now consider the density approach for trapeziums. Given an r-coloring of the Euclidean plane as before, we consider the $r+1$ horizontal lines $y = i$ with $0 \leq i \leq r$. Since only r colors are used, in each such line, one of the given colors necessarily occurs with upper natural density at least $1/r$. Again, since there are $r+1$ such lines, there exists a color (say C) which has upper natural density $\geq 1/r$ in two distinct lines. Let $y = a$ and $y = b$ be those lines with $b > a$. Let $d = b - a$ and $D := r!/d$. Now since the color C occurs with upper natural density $\geq 1/r$ in the line $y = a$, looking at all classes mod D, at least one of the classes must have color C with upper natural density at least $1/r$. Thus, there exists an i with $0 \leq i \leq D-1$ and an integer $n \geq 0$ such that C occurs at least twice among the following points

$$(nD + i, a), \cdots, ((n+r)D + i, a).$$

Since C also occurs in line $y = b$ with upper natural density at least $1/r$, it occurs at least twice among the points

$$(mD + j, b), \cdots, ((m+r)D + j, b)$$

for some integers $j, m \geq 0$ with $j \leq D-1$.

Thus we have the following:

Theorem 3. *For a positive integer r, given any r-coloring, we are ensured of a trapezium with two sides parallel to the x-axis and having an area $\frac{1}{2}(i+j)r!$ where $2 \leq i+j \leq 2r$. As in the case of triangles in Part II of the previous section, the number of possible areas is of the order of r.*

Finally, we consider the analog of Gurevich's question for quadrilaterals by applying van der Waerden's theorem. As done in the case of triangles in Part III of the previous section, we locate a collection of $2r!+1$ horizontal lines

$$y = jb + a, \quad 0 \leq j \leq 2r!,$$

where $(2r!)b + a \leq W(2r! + 1, 2^r - 1)$. Now, looking at the central line and arguing as done in the case of triangles for the $r!$ lines lying below and above the central line, we immediately deduce the following theorem:

Theorem 4. *Let E^2 denote the Euclidean plane endowed with rectangular Cartesian co-ordinates. For a positive integer r, there is a positive integer T_r such that given any r-coloring of the set of integer lattice points in E^2, there is always a quadrilateral of area T_r such that all of its four vertices are lattice points of the same color. Further, one of its diagonals divides it into two triangles of equal area.*

Remark. We remark that Theorem 4 can also be obtained by using the line coloring as in the paper of Dumitrescu and Jiang [4]. However, for the result on trapeziums in Theorem 3, a different way of line coloring seems to be necessary. One wonders whether the statement in Theorem 3 is the best possible.

Acknowledgment. We would like to thank the referee for a very careful reading of the earlier manuscript, pointing out some typographical errors and giving numerous suggestions to improve the presentation.

References

[1] S. D. Adhikari, *Aspects of combinatorics and combinatorial number theory*, Narosa, New Delhi, 2002.

[2] S. D. Adhikari, *On a theorem of Graham*, Expo. Math., **19**, no. 4, 365 – 367, 2001.

[3] S. D. Adhikari, and P. Rath, *Monochromatic configurations for finite colourings of the plane*, Note di Matematica, **22**, no 1, 59–63, 2003/04.

[4] A. Dumitrescu and M. Jiang, *Monochromatic simplices of any volume*, Discrete Math. **310**, no 4, 956–960, 2010.

[5] R. L. Graham, *On partitions of E^n*, J. Comb. Th. (A), **28**, 89–97, 1980.

[6] R. L. Graham, *Rudiments of Ramsey Theory*, American Math. Soc., 1981.

[7] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, John Wiley & Sons, 1980.

[8] M. Nathanson, *Additive number theory, Inverse problems and the geometry of sumsets*, Graduate Texts in Mathematics, 165.

[9] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wisk. **15**, 212–216, 1927.