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# Note di Matematica
# Some Results Concerning to Polar Derivative of Polynomials

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**Abstract.** Let $P(z)$ be a polynomial of degree $n$ and for a complex number $\alpha$, let $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to $\alpha$.

In this paper we establish some $L_r$ inequalities for the polar derivative of a polynomial with restricted zeros. Our results not only generalizes some known polynomial inequalities, but also a variety of interesting results can be deduced from these by a fairly uniform procedure.

**Keywords:** Polynomials, Polar derivative, Inequalities, Zeros.

**MSC 2000 classification:** 30A10, 30C10, 30E10, 30C15.

## Introduction

Let $P(z)$ be a polynomial of degree $n$ and $P'(z)$ its derivative. It was shown by Turan [16] that if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1}$$

More generally, if $P(z)$ has all its zeros in $|z| \leq k \leq 1$, it was proved by Malik [11] that (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{2}$$

whereas Govil [8] proved that if $P(z)$ has all its zeros in $|z| \leq k$, where $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{3}$$

As an improvement of (3), Govil [9] proved that if $P(z)$ has all its zeros in $|z| \le k$, where $k \ge 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{4}$$

Again for the class of polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j},\ 1 \le \mu \le n$ of degree $n$ having all its zeros in $|z| \le k$, $k \le 1$, Aziz and Shah [5] proved

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |P(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |P(z)| \right\}. \tag{5}$$

For $\mu = 1$, inequality (5) reduces to an inequality due to Govil [9].

We define $D_\alpha P(z)$, the polar derivative of the polynomial $P(z)$ of degree $n$ with respect to a complex number $\alpha$ by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z). \tag{6}$$

It is easy to see that the polynomial $D_\alpha P(z)$ is of degree atmost $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z). \tag{7}$$

Shah [14] extended (1) to the polar derivative of $P(z)$ and proved:

**Theorem 1.** *If all the zeros of $n$th degree polynomial $P(z)$ lie in $|z| \le 1$, then*

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n}{2}(|\alpha| - 1) \max_{|z|=1} |P(z)|,\ |\alpha| \ge 1. \tag{8}$$

*The result is best possible and equality in (8) holds for* $P(z) = \left(\frac{z-1}{2}\right)^n$.

Aziz and Rather [3] generalized (8) which also extends (2) to the polar derivative of a polynomial. In fact, they proved.

**Theorem 2.** *If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \le k$, where $k \le 1$, then for every real or complex number $\alpha$ with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n \left( \frac{|\alpha| - k}{1+k} \right) \max_{|z|=1} |P(z)| \tag{9}$$

*The result is best possible and equality in (9) holds for $P(z) = (z-k)^n$ with $\alpha \ge k$.*

Further, as a generalization of (3) to the polar derivative of a polynomial, Aziz and Rather [3] proved the following:

**Theorem 3.** *If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \geq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - k}{1 + k^n} \right) \max_{|z|=1} |P(z)|. \tag{10}$$

In the same paper, Aziz and Rather [3] proved the following improvement of inequality (8):

**Theorem 4.** *If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{2} \left\{ (|\alpha| - 1) \max_{|z|=1} |P(z)| + (|\alpha| + 1) \min_{|z|=1} |P(z)| \right\} \tag{11}$$

*The result is best possible and equality in (11) holds for $P(z) = (z-1)^n$ with $\alpha \geq 1$.*

On the other hand Malik [12] obtained an $L_r$ analogue of (1) by proving that if $P(z)$ has all its zeros in $|z| \leq 1$, then for each $r > 0$,

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |P'(z)|. \tag{12}$$

As an extension of (12) and a generalization of (2), Aziz [1] proved that if $P(z)$ has all its zeros in $|z| \leq k \leq 1$, then for each $r > 0$,

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |P'(z)|. \tag{13}$$

If we let $r \to \infty$ in (12) and (13) and make use of the well known fact from analysis (see example [13, p.73] or [15, p.91]) that

$$\left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \to \max_{0 \leq \theta < 2\pi} |P(e^{i\theta})| \ as \ \ r \to \infty, \tag{14}$$

we get inequalities (1) and (2) respectively.

Recently, Dewan et. al. [6] obtained the following result for the polar derivative of polynomials which generalizes inequalities (9) and (13). In fact they proved:

**Theorem 5.** *If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq k$ and for each $r > 0$,*

$$n(|\alpha| - k)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + ke^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1} |D_\alpha P(z)|. \tag{15}$$

In the limiting case, when $r \to \infty$, the above inequality is sharp and equality holds for the polynomial $P(z) = (z - k)^n$ with $\alpha \geq k$.

If we divide both the sides of inequality (15) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (13). If we let $r \to \infty$, in (15), we get inequality (9).

# 1 Main Results

In this paper, we will obtain some $L_r$ inequalities for the polar derivative of a polynomial which generalize the inequalities (8), (9) and (15) in particular. More precisely, we prove the following theorem:

**Theorem 6.** *If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j},\ 1 \leq \mu \leq n$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq k$ and for each $r > 0, p > 1, q > 1$ with $1/p + 1/q = 1$, we have*

$$n(|\alpha| - S_\mu)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + S_\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{16}$$

*where*

$$S_\mu = \left\{\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}\right\} \tag{17}$$

*and*

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \leq k^\mu. \tag{18}$$

**Remark 1.** Since $S_\mu \leq k^\mu \leq k,\ 1 \leq \mu \leq n$ it follows from above theorem that

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}$$
$$\leq \left\{\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{19}$$

For $\mu = 1$, we get a generalization of inequality (15) in the sense that $\max_{|z|=1}|D_\alpha P(z)|$ on the right hand side of (15) is replaced by a factor involving the integral mean of $|D_\alpha P(z)|$ on $|z| = 1$.

Instead of proving inequality (16), we prove the following more general result:

**Theorem 7.** *If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$ is a polynomial of degree $n$ having all its zeros in $|z| \le k$, where $k \le 1$ and $m := \min_{|z|=k}|P(z)|$, then for every real or complex number $\alpha$, $\beta$ with $|\alpha| \ge k$, $|\beta| < 1$ and for each $r > 0, p > 1, q > 1$ with $1/p + 1/q = 1$, we have*

$$n(|\alpha| - A_\mu)\left\{\int_0^{2\pi} |P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}|^r d\theta\right\}^{\frac{1}{r}}$$
$$\le \left\{\int_0^{2\pi} |1 + A_\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta}) - \frac{mn\alpha\beta e^{i(n-1)\theta}}{k^n}|^{qr} d\theta\right\}^{\frac{1}{qr}} \quad (20)$$

*where*

$$A_\mu = \left\{\frac{n\left|a_n - \frac{m\beta}{k^n}\right|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n\left|a_n - \frac{m\beta}{k^n}\right|k^{\mu-1} + \mu|a_{n-\mu}|}\right\}. \quad (21)$$

Finally, we present the $L_r$ analogue of inequalities (8) and (9) by using the location of zeros of $P(z)$. We prove:

**Theorem 8.** *If $P(z) = \prod_{j=0}^{n}(z - z_j)$ is a polynomial of degree $n$ such that $|z_j| \le k_j \le 1$, $1 \le j \le n$, then for every real or complex number $\alpha$ with $|\alpha| \ge t_0$ and for each $r > 0, p > 1, q > 1$ with $1/p + 1/q = 1$, we have*

$$n(|\alpha| - t_0)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} |1 + t_0 e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \quad (22)$$

*where*

$$t_0 = 1 - \frac{n}{\sum_{j=1}^{n}\left(\frac{1}{1-k_j}\right)}. \quad (23)$$

**Remark 2.** If we let $q \to \infty$, $r \to \infty$ so that $p \to 1$, in inequality (22), we get

$$\max_{|z|=1}|D_\alpha P(z)| \ge n\left(\frac{|\alpha| - t_0}{1 + t_0}\right)\max_{|z|=1}|P(z)|, \quad (24)$$

where $t_0$ is defined by (23). The result is best possible and equality in (24) holds for $P(z) = (z - k)^n$ where $\alpha \ge k$.

**Remark 3.** It can be easily seen that inequality (24) includes as special cases inequality (8) when $k_j = 1$ for $1 \le j \le n$ and inequality (9) when $k_j = k$ for $1 \le j \le n$.

**Remark 4.** Dividing the two sides of (24) by $|\alpha|$, letting $|\alpha| \to \infty$ and noting (7), we get

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+t_0} \max_{|z|=1} |P(z)|. \tag{25}$$

By putting the value of $t_0$ in (25) and after simplification, we get

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \left\{ 1 + \frac{1}{1 + \frac{2}{n} \sum_{j=1}^{n} \frac{k_j}{1-k_j}} \right\} \max_{|z|=1} |P(z)| \tag{26}$$

The above inequality was proved by Aziz and Ahmad [2].

## 2 Lemmas

For the proofs of these Theorems we need the following Lemmas.

**Lemma 1.** *If $P(z)$ has all its zeros in $|z| \le k$ where $k \le 1$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then*

$$|Q^{'}(z)| \le S_\mu |P^{'}(z)| \quad for \quad |z| = 1, \tag{27}$$

*where $S_\mu$ is defined by (18).*

The above lemma is due to Aziz and Rather [4].

**Lemma 2.** *If $P(z) = \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree $n$ with $|z_j| \ge k_j \ge 1, 1 \le j \le n$, then for $|z| = 1$,*

$$\left| \frac{q'(z)}{P'(z)} \right| \ge 1 + \frac{n}{\sum_{j=1}^{n} \left( \frac{1}{k_j - 1} \right)} \tag{28}$$

*where $q(z) = z^n \overline{P(1/\bar{z})}$.*

This Lemma is due to Gardner and Govil [7].

## 3 Proof of the Theorems

**PROOF OF THEOREM** If $P(z)$ has a zero on $|z| = k$, then $\min_{|z|=k} |P(z)| = 0$ and the result follows from inequality (16) in this case. Hence, we suppose that all the zeros of $P(z)$ lie in $|z| < k$ where $k \le 1$, so that $m > 0$. Now $m \le |P(z)|$ for $|z| = k$, therefore, if $\beta$ is any real or complex number such that $|\beta| < 1$, then $\left|\frac{m\beta z^n}{k^n}\right| < |P(z)|$ for $|z| = k$. Since all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouche's theorem, that all the zeros of $F(z) = P(z) - \frac{m\beta z^n}{k^n}$ also lie in $|z| < k$. If $G(z) = z^n\overline{F(\frac{1}{\bar{z}})} = Q(z) - \frac{m\bar{\beta}}{k^n}$ then it can be easily verified that for $|z| = 1$,

$$|F^{'}(z)| = |nG(z) - zG^{'}(z)|. \tag{29}$$

As $F(z)$ has all its zeros in $|z| < k \leq 1$, the inequality (27) in conjunction with inequality (29), gives,

$$|G^{'}(z)| \leq A_\mu |nG(z) - zG^{'}(z)| \quad for \quad |z| = 1, \tag{30}$$

where $A_\mu$ is defined in (21).

Now for every real or complex number $\alpha$ with $|\alpha| \geq k$, we have

$$|D_\alpha F(z)| = |nF(z) + (\alpha - z)F'(z)| \tag{31}$$

$$\geq |\alpha||F'(z)| - |nF(z) - zF'(z)|' \tag{32}$$

which gives by interchanging the roles of $F(z)$ and $G(z)$ in (29) for $|z| = 1$ that

$$|D_\alpha F(z)| \geq |\alpha||F'(z)| - |G'(z)| \tag{33}$$

$$\geq (|\alpha| - A_\mu)|F'(z)| \tag{34}$$

$$\left|D_\alpha P(z) - \frac{nm\alpha\beta z^n}{k^n}\right| \geq (|\alpha| - A_\mu)\left|P^{'}(z) - \frac{mn\beta z^{n-1}}{k^n}\right| \tag{35}$$

Again since $F(z)$ has all its zeros in $|z| \leq k \leq 1$, therefore by Gauss-Lucas theorem all the zeros of the polynomial $F^{'}(z) = P^{'}(z) - \frac{mn\beta z^{n-1}}{k^n}$ lie in $|z| < k \leq 1$. Therefore the polynomial $z^{n-1}\overline{F'(\frac{1}{\bar{z}})} = nG(z) - zG^{'}(z)$ has all its zeros in $|z| > \frac{1}{k} \geq 1$. Hence it follows that the function

$$W(z) = \frac{zG^{'}(z)}{A_\mu\{nG(z) - zG'(z)\}} \tag{36}$$

is analytic for $|z| \leq 1$, $|W(z)| \leq 1$ for $|z| = 1$ and $W(0) = 0$. Thus the function $1 + A_\mu W(z)$ is subordinate to the function $1 + A_\mu z$ for $|z| \leq 1$. By a well known property of subordination [10, P.422], we have for each $r > 0$,

$$\int_0^{2\pi} |1 + A_\mu W(e^{i\theta})|^q d\theta \leq \int_0^{2\pi} |1 + A_\mu e^{i\theta}|^q d\theta. \tag{37}$$

Now by (36), we have

$$|1+A_\mu W(z)| = \left|\frac{nG(z)}{nG(z)-zG'(z)}\right| \tag{38}$$

$$= \frac{n\Big|G(z)\Big|}{\Big|P'(z)-\frac{mn\beta z^{n-1}}{k^n}\Big|} \tag{39}$$

$$= \frac{n\Big|F(z)\Big|}{\Big|P'(z)-\frac{mn\beta z^{n-1}}{k^n}\Big|} \tag{40}$$

$$= \frac{n\Big|P(z)-\frac{m\beta z^{n}}{k^n}\Big|}{\Big|P'(z)-\frac{mn\beta z^{n-1}}{k^n}\Big|}. \tag{41}$$

From (35),(37) and (41), we deduce that for each $r>0$,

$$n^r(|\alpha|-A_\mu)^r\left\{\int_0^{2\pi}|P(e^{i\theta})-\frac{m\beta e^{in\theta}}{k^n}|^r d\theta\right\}$$
$$\leq \left\{\int_0^{2\pi}|1+A_\mu e^{i\theta}|^r\int_0^{2\pi}|D_\alpha P(e^{i\theta})-\frac{mn\alpha\beta e^{i(n-1)\theta}}{k^n}|^r d\theta\right\}. \tag{42}$$

Now applying Holder's inequality for $p>1,\ q>1$ with $1/p+1/q=1$, we get

$$n(|\alpha|-A_\mu)\left\{\int_0^{2\pi}|P(e^{i\theta})-\frac{m\beta e^{in\theta}}{k^n}|^r d\theta\right\}^{\frac{1}{r}}$$
$$\leq \left\{\int_0^{2\pi}|1+A_\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})-\frac{mn\alpha\beta e^{i(n-1)\theta}}{k^n}|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{43}$$

and this completes the proof of inequality (20).

**PROOF OF THEOREM** Let $q(z)=z^n\overline{P(1/\bar{z})}$, then for $|z|=1$, we have

$$|q'(z)| = |nP(z)-zP'(z)| \tag{44}$$

Also we have $|z_j| \leq k_j \leq 1, 1 \leq j \leq n$, therefore, $\frac{1}{|z_j|} \geq \frac{1}{k_j} \geq 1, 1 \leq j \leq n$, and hence by inequality (28), for $|z| = 1$,

$$\left|\frac{P'(z)}{q'(z)}\right| \geq 1 + \frac{n}{\sum\limits_{j=1}^{n}\left(\frac{k_j}{1-k_j}\right)} \tag{45}$$

$$= \frac{\sum\limits_{j=1}^{n}\left(\frac{k_j}{1-k_j}+1\right)}{\sum\limits_{j=1}^{n}\left(\frac{k_j}{1-k_j}\right)} \tag{46}$$

$$= \frac{\sum\limits_{j=1}^{n}\left(\frac{1}{1-k_j}\right)}{\sum\limits_{j=1}^{n}\left(\frac{k_j}{1-k_j}\right)} \tag{47}$$

which gives for $|z| = 1$,

$$\left|\frac{q'(z)}{P'(z)}\right| \leq \frac{\sum\limits_{j=1}^{n}\left(\frac{k_j}{1-k_j}\right)}{\sum\limits_{j=1}^{n}\left(\frac{1}{1-k_j}\right)} \tag{48}$$

$$= \frac{\sum\limits_{j=1}^{n}\left(\frac{1}{1-k_j}-1\right)}{\sum\limits_{j=1}^{n}\left(\frac{1}{1-k_j}\right)} \tag{49}$$

$$= 1 - \frac{n}{\sum\limits_{j=1}^{n}\left(\frac{1}{1-k_j}\right)} = t_0. \tag{50}$$

Hence for $|z| = 1$,

$$|q'(z)| \leq t_0|P'(z)|. \tag{51}$$

Now for every real or complex number $\alpha$ with $|\alpha| \geq t_0$, we have

$$|D_\alpha P(z)| = |nP(z) + (\alpha - z)P'(z)| \tag{52}$$

$$\geq |\alpha|\ |P'(z)| - |nP(z) - zP'(z)| \tag{53}$$

which implies by (44) and (51) for $|z| = 1$

$$|D_\alpha P(z)| \geq |\alpha|\ |P'(z)| - t_0|P'(z)| \tag{54}$$

$$= (|\alpha| - t_0)|P'(z)| \tag{55}$$

Again since $P(z)$ has all its zeros in $|z| \leq k \leq 1$, therefore by Gauss-Lucas theorem all the zeros of the polynomial $P^{'}(z)$ lie in $|z| < k \leq 1$. Therefore the polynomial $z^{n-1}\overline{P'(\frac{1}{\bar{z}})} = nQ(z) - zQ^{'}(z)$ has all its zeros in $|z| > \frac{1}{k} \geq 1$. Hence it follows that the function

$$W(z) = \frac{zQ^{'}(z)}{t_0\{nQ(z) - zQ^{'}(z)\}} \tag{56}$$

is analytic for $|z| \leq 1,\ |W(z)| \leq 1$ for $|z| = 1$ and $W(0) = 0$. Thus the function $1 + t_0W(z)$ is subordinate to the function $1 + t_0z$ for $|z| \leq 1$. By a well known property of subordination [10, P.422], we have for each $r > 0$,

$$\int_0^{2\pi} |1 + t_0W(e^{i\theta})|^q d\theta \leq \int_0^{2\pi} |1 + t_0e^{i\theta}|^q d\theta. \tag{57}$$

Now,

$$n|Q(z)| = |1 + t_0W(z)||P'(z)|. \tag{58}$$

Since $|P(z)| = |Q(z)|$ for $|z| = 1$, therefore from (58) we get

$$|P'(z)| = \frac{n|Q(z)|}{|1 + t_0W(z)|} \quad for \quad |z| = 1. \tag{59}$$

From (55),(57) and (59), we deduce that for each $r > 0$,

$$n^r(|\alpha| - t_0)^r\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\} \leq \left\{\int_0^{2\pi} |1 + t_0e^{i\theta}|^r \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^r d\theta\right\}. \tag{60}$$

Now applying Holder's inequality for $p > 1,\ q > 1$ with $1/p + 1/q = 1$, we get

$$n(|\alpha| - t_0)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + t_0e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{61}$$

and this completes the proof of inequality (22).

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