**Semistable genus 5 general type \( \mathbb{P}^1 \)-curves have at least 7 singular fibres**

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**Abstract.** We prove that if \( f : X \to \mathbb{P}^1 \) is a non-isotrivial, semistable, genus 5 fibration defined on a general type surface \( X \) then the number \( s \) of singular fibres is at least 7.

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**MSC 2000 classification:** primary 14D06, secondary 14J26

**Introduction**

We work on the field of complex numbers. Let \( f : X \to \mathbb{P}^1 \) be a non-isotrivial semistable genus \( g \) fibration defined on the general type surface \( X \) (this is the semistable curve alluded to in the title).

A classical issue (since Parshin’s paper [5]), is determining a lower bound for the number \( s \) of singular fibers of \( f \). The state of the art is as follows:

- If \( g \geq 1 \) then \( s \geq 4 \) ([2]),
- If \( g \geq 2 \) then \( s \geq 5 \) ([6]),
- If \( g \geq 2 \) and the Kodaira dimension of \( X \) is nonnegative, then \( s \geq 6 \) ([7], see also [4]),
- If \( X \) is of general type and \( 2 \leq g \leq 4 \) then \( s \geq 7 \) ([7]).

Some partial results for the case of fibrations on rational surfaces satisfying \( s \geq 6 \) can be found in [1].

In a preprint previous to the appearance of [7] it was conjectured by Tan and Tu that if \( X \) is of general type, then \( s \geq 7 \). Moreover if \( X \) is of general type, \( s = 6 \) and \( g = 5 \) then the minimal model \( S \) of \( X \) satisfies:

\[
K_S^2 = 1, \quad p_g(S) = 2 \quad \text{and} \quad q(S) = 0.
\]

Moreover, in that case the fibration \( f \) is the pull-back of a pencil on \( S \) with 5 simple base points.
Call a pencil $\Lambda$ transversal if two general elements intersect transversally (in particular a general element is non-singular). In section 2 of this short note we shall prove:

**Theorem 1.** Let $S$ be a minimal surface of general type with $K_S^2 = 1$, $p_g(S) = 2$ and $q(S) = 0$. Then $S$ does not admit a transversal pencil $\Lambda$ of genus 5 curves with 5 base points.

Previous remarks imply:

**Theorem 2.** If $f : X \to \mathbb{P}^1$ is a non-isotrivial semistable fibration of genus 5 curves defined on the general type surface $X$, then the number $s$ of singular fibers is at least 7.

The proof of Theorem 1 is based on a construction by Horikawa ([3]): numerical restrictions in the hypothesis of Theorem 1 mean that $S$ is on the ”Noether’s line” and thus after blowing up a point it can be realized as a double cover of $\mathbb{F}_2$. The author is indebted to Prof. M. Mendes-Lopes who pointed out this fact and suggested its use for proving Theorem 1, and to Prof. C. Ciliberto for indicating a mistake in the first version of this paper.

## 1 Proof of Theorem 1

Start with $S$ minimal of general type, $K_S^2 = 1$, $q = 0$ and $p_g = 2$. Assume that a transversal pencil $\Lambda$ of smooth genus 5 curves and with general curve $F$ and $F^2 = 5$ exists on $S$.

After blowing up the base locus of $|K_S|$ consider the ramified double covering:

$$f_2 : \tilde{S} \to \mathbb{F}_2.$$

The map $f_2$ is described as follows: the bi-canonical map of $S$ determines a double cover on the quadric cone in $\mathbb{P}^3$, $f_2$ is the induced map on $\tilde{S}$ after considering the desingularization $\mathbb{F}_2$ of the cone. The branch locus of $f_2$ is a curve $B$ of class $6\Delta_0 + 10\Gamma$, $\Delta_0$ and $\Gamma$ denoting respectively the class of the $(-2)$-section and the class of the fiber in $\mathbb{F}_2$ ([3], Theorem 2.1).

Denote by $|F|$ the induced pencil in $\tilde{S}$. Note that $F^2 = 4$ or 5 depending on whether the base point of $|K_S|$ is a base point of $|F|$ or not. Let $G$ be the image of $F$ under $f_2$. Note that if we denote $G = a\Delta_0 + b\Gamma$, then we have:

i) $G.B = 6b - 2a$,

ii) $G^2 = 2a(b - a)$,

iii) $2p_G - 2 = G^2 + G.(-2\Delta_0 - 4\Gamma) = 2a(b - a) - 2b$, with $p_G$ denoting the arithmetic genus of $G$.

We distinguish two cases:
Case 1: $f_2$ restricted to $\bar{F}$ is $2 : 1$.

Denote by $f_2 : \bar{F} \to G$ the restriction. If $G \equiv a\Delta_0 + b\Gamma$, then

$$2G^2 = (f_2^*G)^2 = \bar{F}^2 = 4 \text{ or } 5.$$  

Thus, $G^2 = 2 = 2a(b - a)$. This forces $a = 1$ and $b = 2$.

By iii):

$$2\rho_G - 2 = 2 - 2b = -2.$$  

Thus, being $G$ irreducible and of arithmetic genus 0 it must be a non-singular rational curve.

Finally, the degree of the ramification divisor of $f_2$ restricted to $\bar{F}$ can be computed into two different ways, namely, using Riemann-Hurwitz or intersecting $G$ with $B$. Using Riemann-Hurwitz we obtain:

$$8 = 2g_{\bar{F}} - 2 = 4(g_G - 1) + B,$$  

and therefore $B = 12$. On the other hand, by i):

$$B = G.B = 6b - 2a = 10.$$  

This contradiction proves that Case 1 is impossible.

Case 2: $f_2$ restricted to $F$ is $1 : 1$.

In this case we use not only the branch locus $B$ but also the ramification divisor $R$ on $\bar{S}$. Denote by $\pi : \bar{S} \to S$ the blowing up. The divisor $R$ is given by $R = 5D + 6E$, with $E$ the exceptional divisor and $D \equiv \pi^*K_S - E$, $B$ and $R$ are related by $f_2^*B = 2R$ ([3], page 129).

Let $f_2^*G = \bar{F} + \tilde{F}$. Note that since the ramifications of $f_2$ occurring on $\bar{F}$ are given by intersections of $\bar{F}$ and $\tilde{F}$, the equality $R.\bar{F} = R.\tilde{F}$ holds. Thus, we have:

$$2(\bar{F}).2R = (\bar{F} + \tilde{F}).2R = f_2^*G.f_2^*B = 2G.B.$$  

(1)

Assume $\bar{F}^2 = 4$. First, we compute the intersection $\bar{F}.R$. Note that $\bar{F}^2 = 4$ means that the center of the blowing up is a base point of $|F|$. Thus, $\bar{F} \equiv \pi^*F - E$, $\bar{F}.E = 1$ and:

$$\bar{F}.R = (\pi^*F - E).5\pi^*K_S - 5E + 6E = 5\pi^*F.\pi^*K_S + 1 = 16,$$

because $g_F = 5$ and $F^2 = 5$ imply $K_S.F = 3$.

Then, by 1 and i):
\[32 = 6b - 2a, \text{ i.e. } a = 3b - 16.\]

On the other hand,

\[0 \leq G^2 = a(b - a) = (3b - 16)(-2b + 16).\]

It follows that: \(b \geq 16/3\) and \(b \leq 8\). Thus \(b = 6, 7\) or \(8\) and correspondingly \(a = 2, 5\) or \(8\). But, being \(f_2\) restricted to \(\overline{F}\) a degree 1 map, the arithmetic genus \(p_G\) of \(G\) must be at least 5 and thus

\[8 \leq 2(p_G - 1) = 2a(b - a) - 2b.\]

None of the possible combinations of \(a\) and \(b\) listed before satisfy this inequality.

The case \(F^2 = 5\) follows by similar considerations. In this case \(\overline{F} = \pi^*F\), \(\overline{F}.E = 0\) and \(\overline{F}.R = 15\). The computations are quite analogous. This prove the Theorem.

References


