Note di Matematica **20,** n. 2, 2000/2001, 135–164.

Submanifolds of framed metric manifolds and S-manifolds

Mukut Mani Tripathi*
Department of Mathematics and Astronomy, Lucknow University
Lucknow 226 007, India
mmt66@satyam.net.in

Ion Mihai
Faculty of Mathematics
Str. Academiei 14, 70109 Bucharest, Romania

Received: 5 April 2001; accepted: 5 April 2001.

Abstract. Almost semi-invariant submanifolds of framed metric manifolds and S-manifolds are studied, where the ambient manifolds generalize almost Hermitian, Kaehler, almost contact metric and Sasakian manifolds while the submanifolds generalize/imply several known classes of submanifolds including CR, semi-invariant, holomorphic, totally real and slant submanifolds. Integrability conditions for certain natural distributions on almost semi-invariant submanifolds of S-manifolds are investigated. Parallelism of the operators arising naturally in the study is investigated leading to a flowchart connecting them. Totally umbilical, totally geodesic and totally contact geodesic submanifolds are studied. Some relations between almost semi-invariant submanifolds of a framed metric manifold and other submanifolds are investigated. Submanifolds of principal toroidal bundles are also studied.

Keywords: Framed metric manifold, S-manifold and S-space form; CR, semi-invariant, almost semi-invariant, holomorphic, totally real and slant submanifolds; Totally contact geodesic submanifold; Tangent bundle and principal toroidal bundle

MSC 2000 classification: 53C15, 53C40, 53C42

Introduction

Framed metric structures were introduced by Yano ([54]) as a generalization of almost contact metric structures and almost Hermitian structures. Blair ([3]) introduced the concept of an S-manifold equipped with a normal framed metric structure satisfying certain relations. The S-structure is analogous to the Kaehler structure in the almost Hermitian case and to the Sasakian structure in the almost contact case. For geometry of framed metric structures we refer to [54] and related references cited therein.

*The paper was presented at the Conference on Mathematics - 2000, held at the Department of Mathematics and Astronomy, Lucknow University, Lucknow 226 007, India during January 2 & 3, 2001.

Recently in [47] the study of almost semi-invariant submanifolds of an ϵ-framed metric manifold was initiated. In this paper we study almost semi-invariant submanifolds of framed metric manifolds and S-manifolds, where the ambient manifolds generalize almost Hermitian, Kaehler, almost contact metric and Sasakian manifolds while the submanifolds generalize/imply several known classes of submanifolds viz. CR, semi-invariant, holomorphic, totally real, slant submanifolds etc. (see the Table in the section 1).

The paper starts with the definitions of framed metric structure, S-structure and S-space form. Some basic results for submanifolds of framed metric manifolds and S-manifolds are given in section 3. Some properties of almost semi-invariant submanifolds of an S-manifold are presented in section 4. Among others it is proved that for a submanifold of an S-space form $\bar{M}(c)$ with $c \neq r$ to be invariant or anti-invariant it is necessary and sufficient that the tangent bundle of the submanifold is invariant under the action of $\bar{R}(X, Y)$ for all vector fields X and Y on the submanifold (Theorem 2). Integrability conditions for certain natural distributions on almost semi-invariant submanifolds of S-manifolds are the subject matter of section 5. Using Lemma 3 it is shown that the anti-invariant distributions $\mathcal{D}^0$ and $\mathcal{D}^0 \oplus \mathcal{E}$ are integrable. In section 6, we investigate certain parallel operators and distributions on submanifolds of S-manifolds. Totally umbilical and totally geodesic submanifolds have been studied in section 7. It is also proved that the invariant submanifolds of S-manifolds (thus of Kaehler and Sasakian manifolds) are minimal (Theorem 17). Section 8 deals with totally contact geodesic submanifolds. The main result of this section is that the second fundamental form of a totally contact geodesic submanifold of an S-manifold is r-contact parallel; and moreover, if the ambient manifold is also an S-space form with $c \neq -3r$, then the totally contact geodesic submanifold is invariant or anti-invariant. In the last section, we investigate some relations between almost semi-invariant submanifolds of a framed metric manifold and other submanifolds. This section is mainly devoted to the study of submanifolds of principal toroidal bundles.

1 Preliminaries

1. 1 Framed metric manifolds

Let $\bar{M}$ be a $(2n + r)$-dimensional *framed metric manifold* [54] (or *almost r-contact metric manifold* [49]) with a framed metric structure $(J, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \ldots, r\}$, that is, J is a (1,1) tensor field defining an f-structure of rank $2n$; $\xi_1, \ldots, \xi_r$ are r vector fields; $\eta^1, \ldots, \eta^r$ are r 1-forms and g is a Riemannian metric on $\bar{M}$ such that

$$J^2 = -I + \eta^\alpha \otimes \xi_\alpha, \qquad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad J(\xi_\alpha) = 0, \qquad \eta^\alpha \circ J = 0, \tag{1}$$

$$g(JX, JY) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\,\eta^\alpha(Y), \tag{2}$$

$$\Omega(X, Y) \equiv g(X, JY) = -\Omega(Y, X), \quad g(X, \xi_\alpha) = \eta^\alpha(X) \tag{3}$$

for all $X, Y \in T\bar{M}$ and $\alpha, \beta \in \{1, \ldots, r\}$ [54].

A framed metric structure is called *normal* [54] if

$$[J, J] + 2d\eta^\alpha \otimes \xi_\alpha = 0, \tag{4}$$

and an *S-structure* [3] if it is normal and

$$\Omega = d\eta^\alpha, \qquad \alpha \in \{1, \ldots, r\}. \tag{5}$$

When $r = 1$, a framed metric structure is an almost contact metric structure, while an S-structure is a Sasakian structure. When $r = 0$, a framed metric structure is an almost Hermitian structure, a normal framed metric structure is a Hermitian structure (integrable almost Hermitian structure) while an S-structure is a Kaehler structure.

If a framed metric structure on $\bar{M}$ is an S-structure then it is known [3] that

$$\left(\bar{\nabla}_X J\right) Y = \sum_\alpha \left(g(JX, JY)\,\xi_\alpha + \eta^\alpha(Y)\,J^2 X\right), \tag{6}$$

$$\bar{\nabla}\xi_\alpha = -J, \qquad \alpha \in \{1, \ldots, r\}. \tag{7}$$

The converse may also be proved. In case of Sasakian structure (that is $r = 1$), (6) implies (7). In Kaehler case (that is $r = 0$), we get $\bar{\nabla} J = 0$.

It is known [11] that in an S-manifold of constant J-sectional curvature c

$$\begin{aligned} \bar{R}(X, Y) Z = &\sum_{\alpha,\beta} \Big(\eta^\alpha(X)\,\eta^\beta(Z)\,J^2 Y - \eta^\alpha(Y)\,\eta^\beta(Z)\,J^2 X - \\ &-g(JX, JZ)\,\eta^\alpha(Y)\,\xi_\beta + g(JY, JZ)\,\eta^\alpha(X)\,\xi_\beta) \\ &+ \frac{c + 3r}{4}\left(-g(JY, JZ)\,J^2 X + g(JX, JZ)\,J^2 Y\right) \\ &+ \frac{c - r}{4}\left(g(X, JZ)\,JY - g(Y, JZ)\,JX + 2g(X, JY)\,JZ\right) \end{aligned} \tag{8}$$

for all $X, Y, Z \in T\bar{M}$, where $\bar{R}$ is the curvature tensor of $\bar{M}$. Such an S-manifold is called an *S-space form* $\bar{M}(c)$.

When $r=0$, an S-space form $\bar{M}(c)$ becomes a complex space form and (8) moves to

$$\begin{aligned} 4\bar{R}(X,Y)Z &= c\left(g(Y,Z)X - g(X,Z)Y\right. \\ &\left.+g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\right). \end{aligned} \tag{9}$$

When $r=1$, an S-space form $\bar{M}(c)$ reduces to a Sasakian space form $\bar{M}(c)$ and (8) reduces to

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) \\ &+\frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ), \end{aligned} \tag{10}$$

where $\xi_1 \equiv \xi$ and $\eta^1 \equiv \eta$.

1. 2 Submanifolds of a Riemannian manifold

Let M be a submanifold of a Riemannian manifold $\bar{M}$ with a Riemannian metric g. Then Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{11}$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{12}$$

for all $X,Y \in TM$ and $V \in T^{\perp}M$, where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\bar{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to A by

$$g(h(X,Y),V) = g(A_V X, Y). \tag{13}$$

Moreover, if J is a (1,1) tensor field on $\bar{M}$, for $X,Y \in TM$ and $V \in T^{\perp}M$ we put

$$JX = PX + FX, \qquad PX \in TM,\ FX \in T^{\perp}M, \tag{14}$$

$$JV = tV + fV, \qquad tV \in TM,\ fV \in T^{\perp}M. \tag{15}$$

In this case we have [47]

$$\begin{aligned} (\bar{\nabla}_X J)Y &= ((\nabla_X P)Y - A_{FY}X - th(X,Y)) \\ &+((\nabla_X F)Y + h(X,PY) - fh(X,Y)), \end{aligned} \tag{16}$$

and

$$(\bar{\nabla}_X J)V = ((\nabla_X t)V - A_{fV}X + PA_V X) + ((\nabla_X f)V + h(X, tV) + FA_V X), \tag{17}$$

where

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y, \qquad (\nabla_X F)Y \equiv \nabla^{\perp}_X FY - F\nabla_X Y,$$
$$(\nabla_X t)V \equiv \nabla_X tV - t\nabla^{\perp}_X V, \qquad (\nabla_X f)V \equiv \nabla^{\perp}_X fV - f\nabla^{\perp}_X V.$$

Let $\bar{R}$ (resp. R) be the curvature tensor of $\bar{M}$ (resp. M). Then the equations of Gauss and Codazzi are given by

$$g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)), \tag{18}$$

$$(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{19}$$

respectively, where $(\bar{R}(X,Y)Z)^{\perp}$ is the normal component of $\bar{R}(X,Y)Z$, and

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{20}$$

The submanifold M is said to be *totally geodesic* in $\bar{M}$ if $h = 0$, *minimal* if $\mathbf{H} \equiv \mathrm{trace}\,(h)\,/\dim\,(M) = 0$, and *totally umbilical* if $h\,(X,Y) = g\,(X,Y)\,\mathbf{H}$. A differential distribution $\mathcal{D}$ on M is said to be *$\mathcal{D}'$-parallel* if $\nabla_X Y \in \mathcal{D}$ for all $X \in \mathcal{D}'$ and $Y \in \mathcal{D}$, where $\mathcal{D}'$ is a differentiable distribution on M. $\mathcal{D}$ is said to be *autoparallel* (resp. *parallel*) if it is $\mathcal{D}$-parallel (resp. TM-parallel). The submanifold M is said to be *$(\mathcal{D},\mathcal{D}')$-mixed totally geodesic* if $h\,(\mathcal{D},\mathcal{D}') = 0$, *$\mathcal{D}$-totally geodesic* if $h\,(\mathcal{D},\mathcal{D}) = 0$, and *$\mathcal{D}$-umbilical* if for all $X, Y \in \mathcal{D}$ we have $h\,(X,Y) = g\,(X,Y)\,\mathbf{K}$ for some normal vector field $\mathbf{K} \in T^{\perp}M$.

2 Almost semi-invariant submanifolds of framed metric manifolds

Let M be a submanifold of a framed metric manifold $\bar{M}$. Then the operator P^2_x is symmetric (that is $g\left(P^2X,Y\right) = g\left(X,P^2Y\right)$) on T_xM and therefore its eigenvalues are real and it is diagonalizable. Moreover, its eigenvalues are bounded by -1 and 0.

Now let all ξ_α's $\in TM = \mathcal{E} \oplus \mathcal{E}^{\perp}$, where $\mathcal{E}$ denotes the distribution spanned by $\xi_1 \ldots \xi_r$ and $\mathcal{E}^{\perp}$ is the complementary orthogonal distribution to $\mathcal{E}$ in M. For each $x \in M$ we may set

$$\mathcal{D}^{\lambda}_x = \ker\left(P^2|_{\mathcal{E}^{\perp}} + \lambda^2\,(x)\,I\right)_x$$

where I is the identity transformation and $\lambda(x)$ belongs to the closed interval $[0,1]$ such that $-\lambda^2(x)$ is an eigenvalue of $\left(P^2|_{\mathcal{E}^\perp}\right)_x$. Since $\left(P^2|_{\mathcal{E}^\perp}\right)_x$ is symmetric and diagonalizable, there is some integer q such that $-\lambda_1^2(x), \ldots, -\lambda_q^2(x)$ are distinct eigenvalues of $\left(P^2|_{\mathcal{E}^\perp}\right)_x$ and $\mathcal{E}_x^\perp$ can be decomposed as the direct sum of the mutually orthogonal P-invariant eigenspaces, that is

$$\mathcal{E}_x^\perp = \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_q}.$$

If $\lambda_i(x) > 0$ then $\mathcal{D}_x^{\lambda_i}$ is even-dimensional. Note that $\mathcal{D}_x^1 = \ker(F_x)$ and $\mathcal{D}_x^0 = \ker(P_x)$. Here $\mathcal{D}_x^1$ is the maximal J-invariant while $\mathcal{D}_x^0$ is the maximal anti-J-invariant subspace of $\mathcal{E}_x^\perp$. For more details we refer to [41, 45, 47].

Now, we recall the definition of almost semi-invariant submanifolds of a framed metric manifold [47, 45].

Definition 1. A submanifold M of a framed metric manifold $\bar{M}$ with all ξ_α's $\in TM$, is called an *almost semi-invariant submanifold* of $\bar{M}$ if there are k functions $\lambda_1, \ldots, \lambda_k$ defined on M with values in the open interval $(0,1)$ such that the following two conditions hold:
(i) $-\lambda_1^2(x), \ldots, -\lambda_k^2(x)$ are distinct eigenvalues of $P^2|_{\mathcal{E}^\perp}$at $x \in M$ with

$$T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0 \oplus \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_k} \oplus \mathcal{E}_x,$$

where $\mathcal{D}_x^1 = \ker(F_x)$, $\mathcal{D}_x^0 = \ker(P_x)$ and $\mathcal{D}_x^{\lambda_i} = \ker\left(P^2 + \lambda_i^2(x) I\right)_x$, $i \in \{1, \ldots, k\}$,
(ii) the dimensions of $\mathcal{D}_x^1$, $\mathcal{D}_x^0$, $\mathcal{D}_x^{\lambda_1}, \ldots, \mathcal{D}_x^{\lambda_k}$ are independent of $x \in M$.

If in addition, each λ_i is constant, then M is called an *almost semi-invariant* submanifold*. If $k = 0$ (that is, in (i) $T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0 \oplus \mathcal{E}_x$) then (i) implies (ii).

Condition (ii) in the above definition enables one to define P-invariant mutually orthogonal distributions

$$\mathcal{D}^\lambda = \bigcup_{x \in M} \mathcal{D}_x^\lambda, \qquad \lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\},$$

on M such that

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k} \oplus \mathcal{E}.$$

In view of the study of Nomizu [39], these distributions are differentiable.

For $X \in TM$ we write

$$X = U^1X + U^0X + U^{\lambda_1}X + \cdots + U^{\lambda_k}X + \eta^\alpha(X)\,\xi_\alpha, \tag{21}$$

where U^1, U^0, $U^{\lambda_1}, \ldots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1$, $\mathcal{D}^0$, $\mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}$ respectively.

For an almost semi-invariant submanifold M of $\bar{M}$ we have

$$T^{\perp}M = \bar{\mathcal{D}}^{1} \oplus \bar{\mathcal{D}}^{0} \oplus \bar{\mathcal{D}}^{\lambda_1} \oplus \cdots \oplus \bar{\mathcal{D}}^{\lambda_k},$$

where $\bar{\mathcal{D}}^{1} = \ker(t)$, $\bar{\mathcal{D}}^{0} = \ker(f)$, $F\mathcal{D}^{\lambda} = \bar{\mathcal{D}}^{\lambda}$ and $t\,\bar{\mathcal{D}}^{\lambda} = \mathcal{D}^{\lambda}$, $\lambda \in \{0, \lambda_1, \ldots, \lambda_k\}$.

Here, we give an example of an almost semi-invariant* submanifold of E^{2n+r}.

Example 1. Let $\mathbf{E}^{2n+r} = \mathbf{C}^n \times \mathbf{R}^r$ be the (2n+r)-dimensional Euclidean space endowed with the framed metric structure $(J, \xi_\alpha, \eta^\alpha, g)$ defined by

$$J(x^1, \ldots, x^{2n}, t^1, \ldots, t^r) = (-x^2, x^1, \ldots, -x^{2n}, x^{2n-1}, 0, \ldots, 0),$$

$$\eta^\alpha = dt^\alpha, \quad \xi_\alpha = \frac{\partial}{\partial t^\alpha},$$

with $\alpha \in \{1, \ldots, r\}$ and $1 < h_1 < h_2 < h_1 + h_2 < n$. The product $M_1 \times M_2 \times M_3 \times \mathbf{R}^r$, where M_1 is a complex submanifold of C^{h_1}, M_2 is a totally real submanifold of C^{h_2} and M_3 is a proper slant submanifold of $C^{n-h_1+h_2}$, is an *almost semi-invariant* submanifold of* E^{2n+r} (for definitions and examples of slant submanifolds we refer to B.-Y. Chen's book [17]).

In 1996, Tripathi and Singh defined and initiated the study of almost semi-invariant submanifolds of an ϵ-framed metric manifold [47]. Here in the special case of framed metric structures the ambient manifolds generalize almost Hermitian and almost contact metric manifolds while the submanifolds generalize/imply several known classes of submanifolds viz. CR, semi-invariant, holomorphic, totally real, slant submanifolds etc. Some of the particular submanifolds have been even defined after 1996. Denoting framed metric by FM, almost contact metric by ACM and almost Hermitian by AH, we present the following Table.

r	k	$\mathcal{D}^1$	$\mathcal{D}^0$	**Manifold**	**Submanifold**
				FM	almost semi-invariant [47, 45]
	0			FM	CR [36, 13]
	0			FM	semi-invariant [29]
	0	$\neq \{0\}$	$\neq \{0\}$	FM	proper semi-invariant [29]
	0		$\{0\}$	FM	invariant [31]
	0	$\{0\}$		FM	anti-invariant [12]
	0		$J\mathcal{D}^0 = T^\perp M$	FM	generic CR [1]
1				ACM	almost semi-invariant [46]
1	0			ACM	contact CR [53]
1	0			ACM	semi-invariant [1]
1	0	$\neq \{0\}$	$\neq \{0\}$	ACM	proper semi-invariant [1]
1	0		$\{0\}$	ACM	invariant [1]
1	0	$\{0\}$		ACM	anti-invariant [1, 52]
1	1 & λ_1=const.	$\{0\}$	$\{0\}$	ACM	(proper) slant [8]
1	1 & λ_1=const.	$\neq \{0\}$	$\{0\}$	ACM	semi-slant [7]
0				AH	generic [41]
0	0			AH	CR [1, 53]
0	0	$\neq \{0\}$	$\neq \{0\}$	AH	proper CR [1]
0	0		$\{0\}$	AH	invariant [54]
0	0	$\{0\}$		AH	anti-invariant [1, 52]
0	0		$J\mathcal{D}^0 = T^\perp M$	AH	anti-holomorphic [1]
0	0		$J\mathcal{D}^0 = T^\perp M$	AH	generic [53]
0	1 & λ_1=const.	$\{0\}$	$\{0\}$	AH	(proper) slant [17]
0	1 & λ_1=const.	$\neq \{0\}$	$\{0\}$	AH	semi-slant [40]
0	1 & $\lambda_1 = \lambda_1(x)$	$\neq \{0\}$	$\{0\}$	AH	quasi-slant [19]

The class of almost semi-invariant submanifold [46] of an almost contact metric manifold contains the almost semi-invariant, generic, almost *CR* and generalized *CR* submanifolds in the sense of [2], [43], [34] and [35] respectively. The class of generic submanifold [41] of an almost Hermitian manifold also generalizes the generic submanifolds in the sense of [16] (or *f*-submanifold in the sense of [50]) and generalized *CR*-submanifold in the sense of [38]. The slant submanifolds given in [32] are different from the slant submanifolds given in [7].

3 Some basic results

It is easy to verify the following

Lemma 1. *Let M be a submanifold of a framed metric manifold $\bar{M}$ such that all ξ_α's $\in TM$. Then*

$$P(\xi_\alpha) = 0 = F(\xi_\alpha), \tag{22}$$

$$\eta^\alpha \circ P = 0 = \eta^\alpha \circ F, \tag{23}$$

$$P^2 + tF = -I + \eta^\alpha \otimes \xi_\alpha, \tag{24}$$

$$FP + fF = 0, \tag{25}$$

$$tf + Pt = 0, \tag{26}$$

$$f^2 + Ft = -I, \tag{27}$$

$$g(PX, Y) = -g(X, PY), \tag{28}$$

$$g(FX, V) = -g(X, tV), \tag{29}$$

$$g(U, fV) = -g(fU, V). \tag{30}$$

Now let all ξ_α's $\in TM$, and let $TM = \mathcal{E} \oplus \mathcal{E}^\perp$, where $\mathcal{E} = \mathrm{span}\{\xi_1, \ldots, \xi_r\}$ and $\mathcal{E}^\perp$ is the complementary orthogonal distribution to $\mathcal{E}$ in M. Then the Lemma 1 leads to the following

Proposition 1. *If M is a submanifold of a framed metric manifold $\bar{M}$ such that all ξ_α's $\in TM$ then at every $x \in M$*

$$\ker(P_x) = \ker\left(P_x^2\right) = \ker\left(tF + I - \eta^\alpha \otimes \xi_\alpha\right)_x, \tag{31}$$

$$\ker(F_x) = \ker(tF)_x = \ker\left(P^2 + I - \eta^\alpha \otimes \xi_\alpha\right)_x, \tag{32}$$

$$\ker(t_x) = \ker(Ft)_x = \ker\left(f^2 + I\right)_x, \tag{33}$$

$$\ker(f_x) = \ker\left(f^2\right)_x = \ker(Ft + I)_x. \tag{34}$$

Consequently, on $\mathcal{E}_x^\perp$

$$\ker\left(P|_{\mathcal{E}^\perp}\right)_x = \ker\left(P^2|_{\mathcal{E}^\perp}\right)_x = \ker\left(tF|_{\mathcal{E}^\perp} + I\right)_x, \tag{35}$$

$$\ker\left(F|_{\mathcal{E}^\perp}\right)_x = \ker\left(tF|_{\mathcal{E}^\perp}\right)_x = \ker\left(P^2|_{\mathcal{E}^\perp} + I\right)_x. \tag{36}$$

Proof. Equations (31)-(34) follow from (24)-(27) and (28)-(30). Since

$$\eta^\alpha(X) = 0 \quad \text{for} \quad X \in \mathcal{E}^\perp,$$

(35) and (36) are implied by (31) and (32) respectively. $\boxed{QED}$

Proposition 2. *Let M be a submanifold of a framed metric manifold satisfying* (7)*. Then for $X, \xi_\alpha \in TM$, and $V \in T^{\perp}M$ we have*

$$P = -\nabla \xi_\alpha, \tag{37}$$

$$FX = -h\left(X, \xi_\alpha\right) \qquad \left(\Longleftrightarrow A_V \xi_\alpha = tV\right). \tag{38}$$

Consequently,

$$\eta^\alpha\left(A_V X\right) = -g\left(FX, V\right). \tag{39}$$

Proof. Using (11) and (14) we have

$$\nabla_X \xi_\alpha + h\left(X, \xi_\alpha\right) = -PX - FX, \qquad X \in TM.$$

Equating tangential and normal parts in the above equation we get (37) and (38) respectively. Lastly,

$$\eta^\alpha\left(A_V X\right) = g\left(A_V X, \xi_\alpha\right) = g\left(h\left(X, \xi_\alpha\right), V\right) = -g\left(FX, V\right),$$

which is (39). QED

Proposition 3. *Let M be a submanifold of a framed metric manifold satisfying* (7)*. Then for $\xi_\beta \in T^{\perp}M$ we have*

$$A_{\xi_\beta} = P, \tag{40}$$

$$\nabla^{\perp} \xi_\beta = -F. \tag{41}$$

Proof. Using (12) and (14) we have

$$-A_{\xi_\beta} X + \nabla^{\perp}_X \xi_\beta = -PX - FX, \qquad X \in TM,\ \xi_\beta \in T^{\perp}M.$$

Equating tangential and normal parts in the above equation we get equations (40) and (41) respectively. QED

Proposition 4. *Let M be a submanifold of an S-manifold with all ξ_α's $\in TM$. Then for all $X, Y \in TM$ and $V \in T^{\perp}M$ we get*

$$(\nabla_X P)Y - A_{FY}X - th(X,Y) = \sum_\alpha \left(g(JX, JY)\xi_\alpha + \eta^\alpha(Y) J^2 X\right), \tag{42}$$

$$(\nabla_X F)Y + h(X, PY) - fh(X,Y) = 0, \tag{43}$$

$$(\nabla_X t)V - A_{fV}X + PA_V X = 0, \tag{44}$$

$$(\nabla_X f)V + h(X, tV) + FA_V X = 0. \tag{45}$$

Consequently,

$$(\nabla P)\xi_\alpha = P^2, \tag{46}$$

$$(\nabla F)\xi_\alpha = FP, \tag{47}$$

$$\nabla_{\xi_\alpha} P = 0, \tag{48}$$

$$\nabla_{\xi_\alpha} F = 2FP = -2fF, \tag{49}$$

$$\nabla_{\xi_\alpha} t = 2tf = -2Pt, \tag{50}$$

$$\nabla_{\xi_\alpha} f = 0, \tag{51}$$

$$\begin{aligned} P[X,Y] = {} & (\nabla_X PY - \nabla_Y PX + A_{FX}Y - A_{FY}X) \\ & + \sum_\alpha \left(\eta^\alpha(X)J^2Y - \eta^\alpha(Y)J^2X\right), \end{aligned} \tag{52}$$

$$F[X,Y] = \nabla^\perp_X FY - \nabla^\perp_Y FX + h(X,PY) - h(PX,Y), \tag{53}$$

$$(\nabla_X h)(Y,\xi_\alpha) = -(\nabla_X F)Y + h(PX,Y). \tag{54}$$

Proof. Using (6) and (14) in (16) and equating tangential and normal parts we get (42) and (43) respectively. Similarly using (6), (15) and $\eta^\alpha(V) = 0$ in (17) and equating tangential and normal parts we get (44) and (45) respectively.

Putting $Y = \xi_\alpha$ in (42) and using (22), (38), (1), (3), (23) and (24) we get (46). Similarly we get (47) - (51). Equations (52) and (53) follow from (42) and (43) respectively. Lastly, putting $Z = \xi_\alpha$ in (20) and using (37) and (38) we get (54). $\boxed{QED}$

Theorem 1. *If M is a submanifold of a framed metric manifold satisfying (5) such that at least one ξ_β is normal to M then M is anti-invariant.*

Proof. We have

$$g\,(X,JY) = \Omega\,(X,Y) = d\eta^\beta\,(X,Y) = 0, \qquad X,Y \in TM,$$

and hence $P = 0$. $\boxed{QED}$

4 Some properties of almost semi-invariant submanifolds

Proposition 5. *If M is an almost semi-invariant or semi-invariant submanifold of an S-manifold $\bar{M}$, then*

$$A_{FX}Y - A_{FY}X = \sum_\alpha \left(\eta^\alpha(Y)J^2X - \eta^\alpha(X)J^2Y\right), \qquad X,Y \in \mathcal{D}^0 \oplus \mathcal{E}. \tag{55}$$

Proof. For $X,Y\in\mathcal{D}^0\oplus\mathcal{E},\ Z\in TM$ in view of (13), (29), (42) and (1) we have

$$\begin{aligned}
g(A_{FX}Y,Z) &= g(h(Y,Z),FX)=-g(th(Y,Z),X)\\
&= -g((\nabla_Z PY - P\nabla_Z Y - A_{FY}Z\\
&\quad -\sum_\alpha (g(JZ,JY)\xi_\alpha+\eta^\alpha(Y)J^2Z)),X)\\
&= -g(\nabla_Z Y,PX)+g(A_{FY}Z,X)\\
&\quad -g\left(\sum_\alpha \left(g(JZ,JY)\xi_\alpha+\eta^\alpha(Y)J^2Z\right),X\right)\\
&= g(A_{FY}X,Z)+\sum_\alpha \eta^\alpha(Y)g(J^2Z,X)\\
&\quad +g\left(\sum_\alpha \xi_\alpha,X\right)g(-J^2Y,Z)\\
&= g\left(A_{FY}X+\sum_\alpha\left(\eta^\alpha(Y)J^2X-\eta^\alpha(X)J^2Y\right),Z\right),
\end{aligned}$$

which implies (55). $\boxed{QED}$

This proposition leads to the following

Corollary 1. *For an almost semi-invariant or semi-invariant submanifold of an S-manifold $\bar{M}$ it follows that*

$$A_{FX}Y-A_{FY}X=0 \qquad X,Y\in\mathcal{D}^0. \tag{56}$$

Proposition 6. *If M is an almost semi-invariant submanifold of an S-manifold, then for all $X\in\mathcal{D}^1\oplus\mathcal{E},\ Y\in TM,\ V\in\bar{\mathcal{D}}^1$*

$$g(Jh(X,Y),V)=g(h(PX,Y),V). \tag{57}$$

Proof. For $X\in\mathcal{D}^1\oplus\mathcal{E}=\ker(F),\ Y\in TM,\ V\in\bar{\mathcal{D}}^1=\ker(t)$ in view of (43) and (29), we obtain

$$\begin{aligned}
g(fh(X,Y),V) &= g(\nabla^\perp_Y FX-F\nabla_Y X+h(Y,PX),V)\\
&= g(\nabla_Y X,tV)+g(h(PX,Y),V)\\
&= g(h(PX,Y),V),
\end{aligned}$$

which provides (57). $\boxed{QED}$

The above proposition immediately implies the following.

Corollary 2. *For an almost semi-invariant submanifold M of an S-manifold we get*

$$(h(PX,Y) - h(X,PY)) \perp \bar{\mathcal{D}}^1, \qquad X, Y \in \mathcal{D}^1 \oplus \mathcal{E}. \tag{58}$$

In particular, if M is a semi-invariant submanifold then

$$f\,(h(PX,Y) - h(X,PY)) = 0.$$

Next, we prove the following proposition.

Proposition 7. *Let M be an almost semi-invariant submanifold of an S-manifold. Then direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}\}$ is $\mathcal{E}$-parallel.*

Proof. First we note that

$$g(\nabla_{\xi_\alpha} X, \xi_\beta) = -g(X, \nabla_{\xi_\alpha}\xi_\beta) = g(X, P\xi_\alpha) = 0, \qquad X \in \mathcal{E}^\perp. \tag{59}$$

For $X \in \mathcal{D}^\lambda$ and $Y \in \mathcal{D}^\mu$, $\lambda, \mu \in \{1, 0, \lambda_1, \ldots, \lambda_k\}$ and $\lambda \neq \mu$, we obtain

$$\begin{aligned}
-\mu^2 g(\nabla_{\xi_\alpha} X, Y) =& g(\nabla_{\xi_\alpha} X, -\mu^2 Y) = g(\nabla_{\xi_\alpha} X, P^2 Y)\\
=& -g(P\nabla_{\xi_\alpha} X, PY) = -g(\nabla_{\xi_\alpha} PX, PY)\\
=& g(PX, \nabla_{\xi_\alpha} PY) = g(PX, P\nabla_{\xi_\alpha} Y)\\
=& -g(P^2 X, \nabla_{\xi_\alpha} Y) = \lambda^2 g(X, \nabla_{\xi_\alpha} Y) = -\lambda^2 g(\nabla_{\xi_\alpha} X, Y),
\end{aligned}$$

which together with (59) leads to $\nabla_{\xi_\alpha} X \in \mathcal{D}^\lambda$ and the proof is achieved. $\boxed{QED}$

Remark 1. Since for $X \in \mathcal{D}^\lambda$ we have $\nabla_{\xi_\alpha} X \in \mathcal{D}^\lambda$ hence

$$-\lambda^2 \nabla_{\xi_\alpha} X = P^2 \nabla_{\xi_\alpha} X = \nabla_{\xi_\alpha} P^2 X = -\nabla_{\xi_\alpha}(\lambda^2 X) = -(\nabla_{\xi_\alpha}\lambda^2)X - \lambda^2(\nabla_{\xi_\alpha} X),$$

from which it follows that $\nabla_{\xi_\alpha}(\lambda^2) = 0$.

Corollary 3. *For X belonging to direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}\}$ on an almost semi-invariant submanifold of an S-manifold, we have $[X, \xi_\alpha] \in \mathcal{D}$.*

Proof. For $X \in \mathcal{D}$ we have $[X, \xi_\alpha] = \nabla_X \xi_\alpha - \nabla_{\xi_\alpha} X = -PX - \nabla_{\xi_\alpha} X \in \mathcal{D}$. $\boxed{QED}$

Let M be a submanifold of an S-space form $\bar{M}(c)$ with all ξ_α's $\in TM$. If M is invariant or anti-invariant then it is easy to verify that TM and $T^\perp M$ are invariant under the action of $\bar{R}(X,Y)$ for all $X, Y \in TM$, that is $\bar{R}(X,Y)Z \in$

TM and $\bar{R}(X,Y)V \in T^{\perp}M$ for all $X,Y,Z \in TM$ and $V \in T^{\perp}M$. If $c \neq r$ and TM is invariant under the action of $\bar{R}(X,Y)$ then

$$\begin{aligned}
\bar{R}(X,Y)X &= \sum_{\alpha,\beta}(\eta^{\alpha}(X)\eta^{\beta}(X)J^2Y - \eta^{\alpha}(Y)\eta^{\beta}(X)J^2X \\
&\quad -g(JX,JX)\eta^{\alpha}(Y)\xi_{\beta} + g(JY,JX)\eta^{\alpha}(X)\xi_{\beta}) \\
&\quad +\frac{c+3r}{4}(-g(JY,JX)J^2X + g(JX,JX)J^2Y) \\
&\quad -\frac{3(c-r)}{4}g(JX,Y)JX
\end{aligned}$$

which implies that $g(JX,Y)JX \in TM$, *i. e.* either $JX \in TM$ or $g(JX,Y) = 0$. Since J is linear, M is either invariant or anti-invariant.

Thus we are able to state (see [10] also)

Theorem 2. *Let M be a submanifold of an S-space form $\bar{M}(c)$ with $c \neq r$. Then M is invariant or anti-invariant if and only if $\bar{R}(X,Y)Z \in TM$ for all $X,Y,Z \in TM$.*

The above theorem implies the following

Corollary 4. *If M is an almost semi-invariant submanifold of an S-space form $\bar{M}(c)$ such that $\mathcal{D}^0 \neq \{0\} \neq \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k}$, then $\bar{R}(X,Y)Z \in TM$ for all $X,Y,Z \in TM$ if and only if $c = r$.*

5 Integrability conditions

Lemma 2. *If M is a submanifold of an S-manifold, tangent to all ξ_{α}'s, then*

$$g(\nabla_X Y, \xi_{\alpha}) = \nabla_X \eta^{\alpha}(Y) + g(PX,Y), \qquad X,Y \in TM. \tag{60}$$

Consequently,

$$g([X,Y],\xi_{\alpha}) = \nabla_X \eta^{\alpha}(Y) - \nabla_Y \eta^{\alpha}(X) + 2g(PX,Y), \qquad X,Y \in TM, \tag{61}$$

$$g(\nabla_X Y, \xi_{\alpha}) = g(PX,Y), \qquad X \in TM,\ Y \in \mathcal{E}^{\perp}, \tag{62}$$

$$g([X,Y],\xi_{\alpha}) = 2g(PX,Y), \qquad X,Y \in \mathcal{E}^{\perp}. \tag{63}$$

Proof. For $X,Y \in TM$, using (3), (4), (42), we get

$$g(\nabla_X Y, \xi_{\alpha}) = \nabla_X g(Y,\xi_{\alpha}) - g(Y, \nabla_X \xi_{\alpha}) = \nabla_X \eta^{\alpha}(Y) + g(PX,Y).$$

The rest of the proof is straightforward. $\boxed{QED}$

Theorem 3. *For a submanifold M of an S-manifold $\bar{M}$ with all ξ_α's $\in TM$, $\mathcal{E}^\perp$ is integrable if and only if M is anti-invariant.*

Proof. The proof follows easily from (63). $\boxed{QED}$

Theorem 4. *. Let M be an almost semi-invariant submanifold of an S-manifold. Neither the direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^1, \mathcal{D}^{\lambda_1}, \dots, \mathcal{D}^{\lambda_k}\}$ nor $\mathcal{D} \oplus \mathcal{D}^0$ is integrable.*

Proof. Choosing an $X \in \mathcal{D}$ such that $PX \neq 0$ by (63), we have $0 \neq 2g(PX, PX) = g([X, PX], \xi_\alpha)$. $\boxed{QED}$

Lemma 3. *. Let M be an almost semi-invariant submanifold of a framed metric manifold. Then $\mathcal{D}^0$ and $\mathcal{D}^0 \oplus \mathcal{E}$ are integrable if and only if*

$$d\Omega(X, Y, Z) = 0, \qquad Y, Z \in \mathcal{D}^0,\ X \in TM. \tag{64}$$

Proof. For $X \in TM,\ Y, Z \in \mathcal{D}^0$, we have

$$3d\Omega(X, Y, Z) = -g([Y, Z], JX) = g(P[Y, Z], X),$$

which implies the proof. $\boxed{QED}$

This Lemma leads to the following two obstructions for almost semi-invariant submanifolds.

Theorem 5. *Let $\bar{M}$ be a framed metric manifold with $d\Omega = \Omega \wedge \omega$ for some 1-form ω on $\bar{M}$. Then in order that M is an almost semi-invariant or semi-invariant submanifold it is necessary that $\mathcal{D}^0 \oplus \mathcal{E}$ is integrable.*

Proof. Let $X \in TM$ and $Y, Z \in \mathcal{D}^0 \oplus \mathcal{E}$. Then $\Omega(X, Y) = 0 = \Omega(Y, Z)$. Consequently, $0 = \Omega \wedge \omega(X, Y, Z) = d\Omega(X, Y, Z)$. Hence, applying Lemma 3, it follows that $\mathcal{D}^0 \oplus \mathcal{E}$ is integrable. $\boxed{QED}$

Theorem 6. *If M is a semi-invariant or almost semi-invariant submanifold of an S-manifold then $\mathcal{D}^0$ and $\mathcal{D}^0 \oplus \mathcal{E}$ are integrable.*

Proof. Since in an S-manifold, $d\Omega = 0$, the proof is obvious. Alternatively, the proof follows from (55), (56) and (52). $\boxed{QED}$

In the following theorem, necessary and sufficient conditions for $\mathcal{D}^1 \oplus \mathcal{E}$ to be integrable have been obtained.

Theorem 7. *If M is an almost semi-invariant submanifold of an S-manifold, then $\mathcal{D}^1 \oplus \mathcal{E}$ is integrable if and only if*
(a) $h(X, PY) = h(PX, Y), \qquad X, Y \in \mathcal{D}^1 \oplus \mathcal{E}$, *or equivalently*
(b) $g(h(X, PY), FZ) = g(h(PX, Y), FZ), \quad X, Y \in \mathcal{D}^1 \oplus \mathcal{E},\ Z \in TM.$

Proof. In view of $\ker(F) = \mathcal{D}^1 \oplus \mathcal{E}$ and (53), $\mathcal{D}^1 \oplus \mathcal{E}$ is integrable if and only if (a) holds. Since $F(TM) = \bar{\mathcal{D}}^0 \oplus \bar{\mathcal{D}}^{\lambda_1} \oplus \cdots \oplus \bar{\mathcal{D}}^{\lambda_k}$, in view of Corollary 2, (a) $\Leftrightarrow$ (b). $\boxed{QED}$

Theorem 8. *On an almost semi-invariant submanifold of an S-manifold, $\mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{E}$ is integrable if and only if the following assertions hold.*
(a) $\nabla_X PY - \nabla_Y PX \in \mathcal{D}^1, \qquad X, Y \in \mathcal{D}^1,$
(b) $\nabla_X PY + A_{FX} Y \in \mathcal{D}^1, \qquad X \in \mathcal{D}^0,\ Y \in \mathcal{D}^1.$
In particular, if M is a semi-invariant submanifold, then (a) *and* (b) *hold.*

Proof. Since $\mathcal{D}^0 \oplus \mathcal{E}$ is integrable, $[X, Y] \in \mathcal{D}^0 \oplus \mathcal{E}$ for $X, Y \in \mathcal{D}^0 \oplus \mathcal{E}$. Since $Z \in \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{E}$ and $PZ \in \mathcal{D}^1$ are equivalent, in view of (52) the proof follows immediately. $\boxed{QED}$

Theorem 9. *For an almost semi-invariant submanifold of an S-manifold the following statements are equivalent:*
(a) $\mathcal{D}^1 \oplus \mathcal{D} \oplus \mathcal{E}$ *is integrable,*
(b) $\left(\nabla^{\perp}_X FY - \nabla^{\perp}_Y FX + h(X, PY) - h(PX, Y)\right) \in \bar{\mathcal{D}}, \quad X, Y \in \mathcal{D}^1 \oplus \mathcal{D} \oplus \mathcal{E},$
where $\mathcal{D}$ is direct sum of a subfamily of $\{\mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}\}$ and $\bar{\mathcal{D}}$ is the direct sum of the corresponding subfamily of $\{\bar{\mathcal{D}}^{\lambda_1}, \ldots, \bar{\mathcal{D}}^{\lambda_k}\}$.

Proof. Taking account of (53) and the equivalence of $Z \in \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_i} \oplus \mathcal{E}$ and $FZ \in \bar{\mathcal{D}}^{\lambda_i}$, the proof is obvious. $\boxed{QED}$

Theorem 10. *For direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}\}$ on an almost semi-invariant submanifold of an S-manifold the following two statements are equivalent:*
(a) $\mathcal{D}^0 \oplus \mathcal{D} \oplus \mathcal{E}$ *is integrable,*
(a) $(\nabla_X PY - A_{FX} Y - A_{FY} X) \in \mathcal{D}, \qquad X \in \mathcal{D}^0,\ Y \in \mathcal{D}.$

Proof. Since $\mathcal{D}^0 \oplus \mathcal{E}$ is integrable we have $[X, Y] \in \mathcal{D}^0 \oplus \mathcal{E}$ for $X, Y \in \mathcal{D}^0 \oplus \mathcal{E}$. For $Y \in \mathcal{D}$ we always have $[\xi_\alpha, Y] \in \mathcal{D}$ by Corollary 7. Thus, using the equivalence of $Z \in \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i} \oplus \mathcal{E}$ and $PZ \in \mathcal{D}^{\lambda_i}$ in (52), we get (a) $\Leftrightarrow$ (b). $\boxed{QED}$

6 Certain parallel distributions and operators

In this section we investigate certain parallel distributions and operators on submanifolds. First we give the following definition.

Definition 2. A submanifold M of a framed metric manifold is said to satisfy the condition (A) if M is an almost semi-invariant * submanifold, and each of the distributions $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}$ and $\mathcal{E}$ is parallel with respect to ∇, and condition (B) if M is an almost semi-invariant* submanifold, and each of the subbundles $\bar{\mathcal{D}}^1, \bar{\mathcal{D}}^0, \bar{\mathcal{D}}^{\lambda_1}, \ldots, \bar{\mathcal{D}}^{\lambda_k}$ is parallel with respect to $\nabla^{\perp}$.

In view of Theorem 6.3 of [47] we have the following

Theorem 11. *For a submanifold M of a framed metric manifold $\bar{M}$ with all ξ_α's $\in TM$ we have the following flow diagram.*

$$\begin{array}{ccccccc} \nabla P = 0 & \Rightarrow & \nabla(P^2) = 0 & \Leftrightarrow & (A) & \Leftarrow & \nabla t = 0 \\ & & & & & & \Updownarrow \\ \nabla f = 0 & \Rightarrow & \nabla(f^2) = 0 & \Leftrightarrow & (B) & \Leftarrow & \nabla F = 0 \end{array}$$

Theorem 12. *A submanifold of an S-manifold, tangent to all ξ_α's, is anti-invariant if and only if $\nabla P = 0$.*

PROOF. By the Theorem 11, anti-invariantness implies $\nabla P = 0$. Conversely, if $\nabla P = 0$ then by the assumption and (46), it follows that for all $x \in M$ and $X \in TM$, $P^2 X = 0$, that is, $T_x M = \ker(P^2)_x$ and therefore M is anti-invariant. $\boxed{QED}$

Theorem 13 ([48]). *For a non-invariant submanifold M of an S-manifold, tangent to all ξ_α's, if $\nabla F = 0$ then M is anti-invariant.*

Remark 2. In particular, for $r = 1$ the Theorem 13 holds for Sasakian manifolds. This result is stronger than the Proposition 3.3 of [55] where it has been proved that every submanifold of a Sasakian manifold with $\xi \in TM$ and $\nabla F = 0$is semi-invariant. Moreover, this result also makes Theorems 3.5 and 3.7 of [55] redundant.

Theorem 14. *For an almost semi-invariant submanifold M of an S-manifold if $\mathcal{D}^1 \oplus \mathcal{E}$ is autoparallel, then*

$$h(X, PY) = h(PX, Y) = fh(X, Y), \qquad X, Y \in \mathcal{D}^1 \oplus \mathcal{E}.$$

In particular, if M is semi-invariant submanifold, then the converse statement also holds.

PROOF. First equality follows from the integrability of $\mathcal{D}^1 \oplus \mathcal{E}$, while the second one follows from $\ker(F) = \ker(tF) = \mathcal{D}^1 \oplus \mathcal{E}$ and (43). $\boxed{QED}$

7 Totally umbilical and totally geodesic submanifolds

We begin this section with the following lemma.

Lemma 4. *Let $\mathcal{D}$ be a distribution on a submanifold M of an S-manifold such that at least one $\xi_\alpha \in \mathcal{D}$. If M is $\mathcal{D}$-umbilical, then M is $\mathcal{D}$-totally geodesic.*

PROOF. If M is $\mathcal{D}$-umbilical then for all $X, Y \in \mathcal{D}$ we have $h(X, Y) = g(X, Y)\mathbf{K}$ for some vector field $\mathbf{K}$ normal to M. But, using (3), (1) and (22) we obtain

$$\mathbf{K} = g(\xi_\alpha, \xi_\alpha)\mathbf{K} = h(\xi_\alpha, \xi_\alpha) = -F\xi_\alpha = 0,$$

which shows that $h(X, Y) = 0$, for all $X, Y \in \mathcal{D}$, that is M is $\mathcal{D}$-totally geodesic. $\boxed{QED}$

Theorem 15. *A totally umbilical submanifold of an S-manifold, tangent to at least one ξ_α, is totally geodesic.*

By (38), we can state the following.

Theorem 16. *A totally geodesic submanifold of an S-manifold, tangent to at least one ξ_α, is invariant.*

From Theorems 15 and 16 we immediately have the following

Corollary 5. *A totally umbilical submanifold of an S-manifold, tangent to at least one ξ_α, is invariant.*

Proposition 8. *Let M be a submanifold of a framed metric manifold $\bar{M}$ and $\mathcal{D}$ be a distribution on M. For $X, Y \in \mathcal{D}$ the following statements are equivalent:*
(a) $h(X, PY) = h(PX, Y)$, (b) $(A_V PX + PA_V X) \perp \mathcal{D}$, $V \in T^\perp M$.
Moreover, if $\bar{M}$ is an S-manifold, then (a) *is equivalent to each of the following equivalent statements:*
(c) $(\nabla_X F)Y - (\nabla_Y F)X = 0$, (d) $F[X, Y] = \nabla^\perp_X FY - \nabla^\perp_Y FX$.

Proof. In view of (13), $(a) \Leftrightarrow (b)$. Using (43) we can prove the equivalence of (a), (c) and (d). $\boxed{QED}$

Definition 3. For a distribution $\mathcal{D}$ on a submanifold M of a framed metric manifold $\bar{M}$ such that all ξ_α's $\in TM$, we say that P is $\mathcal{D}$-commutative if one of the equivalent statements (a) and (b) of the above proposition holds.

Note that P is $\mathcal{D}$-commutative for each distribution $\mathcal{D}$ on M if and only if $PA_V + A_V P = 0$ for all $V \in T^\perp M$. If M is an almost semi-invariant submanifold of an S-manifold, then P is $\mathcal{E}$-commutative, $\mathcal{D}^0$-commutative and $\mathcal{D}^0 \oplus \mathcal{E}$-commutative. Moreover, P is $\mathcal{D}^1 \oplus \mathcal{E}$-commutative if and only if $\mathcal{D}^1 \oplus \mathcal{E}$ is integrable.

For each $\mathcal{D}^\lambda$, $\lambda \in \{0, 1, \lambda_1, \ldots, \lambda_k\}$ on an almost semi-invariant submanifold of a framed metric manifold we choose a local orthonormal basis: $E_1, \ldots, E_{n(\lambda)}$, where $n(\lambda) = \dim(\mathcal{D}^\lambda)$ and put

$$\mathbf{H}_\lambda = \sum_i h(E_i, E_i), \qquad i \in \{1, \ldots, n(\lambda)\}.$$

An almost semi-invariant submanifold of a framed metric manifold with $\mathbf{H}_\lambda = 0$ is called $\mathcal{D}^\lambda$-*minimal*; and it is minimal if

$$\mathbf{H}_\mathcal{E} + \mathbf{H}_0 + \mathbf{H}_1 + \mathbf{H}_{\lambda_1} + \cdots + \mathbf{H}_{\lambda_k} = 0, \text{ where } \mathbf{H}_\mathcal{E} = \sum_\alpha h(\xi_\alpha, \xi_\alpha).$$

Proposition 9. *Let M be an almost semi-invariant submanifold of an S-manifold. If P is $\mathcal{D}^\lambda$-commutative, $\lambda \neq 0$, then M is $\mathcal{D}^\lambda$-minimal and $\mathcal{D}^\lambda \oplus \mathcal{E}$-minimal.*

Proof. We choose a local orthonormal basis for $\mathcal{D}^\lambda$: $E_1, \dots, E_{n(\lambda)/2}, \dots, E_{n(\lambda)}$, where $E_{(n(\lambda)/2)+i} = PE_i/\lambda$, $(1 \leq i \leq n(\lambda)/2)$. Then we have

$$\begin{aligned} h(E_i, E_i) + h(PE_i/\lambda, PE_i/\lambda) &= h(E_i, E_i) + h(P^2E_i, E_i)/\lambda^2 \\ &= h(E_i, E_i) + h(-\lambda^2 E_i, E_i)/\lambda^2 = 0. \end{aligned}$$

Consequently $\mathbf{H}_\lambda = 0$. $\boxed{QED}$

As an application of the above proposition we get the following

Theorem 17 ([31]). *If M is an invariant submanifold of an S-manifold, then M is minimal.*

Proposition 10. *Let M be an almost semi-invariant submanifold of an S-manifold. If $\mathcal{D}^1 \oplus \mathcal{E}$ is integrable and M is $(\mathcal{D}^1, \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i})$-mixed totally geodesic for $1 \leq i \leq k$, then $\mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{E}$ is integrable if and only if $\mathcal{D}^1$ is $\mathcal{D}^0$-parallel.*

Proof. If $\mathcal{D}^1 \oplus \mathcal{E}$ is integrable, then, using Theorem 8, $\mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{E}$ is integrable if and only if

$$\nabla_X PY + A_{FX}Y \in \mathcal{D}^1, \qquad X \in \mathcal{D}^0,\ Y \in \mathcal{D}^1.$$

But, for $X \in \mathcal{D}^0,\ Y \in \mathcal{D}^1,\ Z \in \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i} \oplus \mathcal{E}$, we have

$$g(A_{FX}Y, Z) = -g(h(Y, Z), FX) = 0,$$

because M is $(\mathcal{D}^1, \mathcal{E})$-mixed totally geodesic and it is assumed that M is $(\mathcal{D}^1, \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i})$-mixed totally geodesic, thus $h(Y, Z) = 0$. Therefore $\mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{E}$ is integrable if and only if $\nabla_X PY \in \mathcal{D}^1$ for all $X \in \mathcal{D}^0$, $Y \in \mathcal{D}^1$, that is, $\mathcal{D}^1$ is $\mathcal{D}^0$-parallel. $\boxed{QED}$

Corollary 6. *If $\mathcal{D}^1 \oplus \mathcal{E}$ is integrable and M is $(\mathcal{D}^1, \mathcal{D}^0)$-mixed totally geodesic on a semi-invariant submanifold of an S-manifold, then $\mathcal{D}^1$ is $\mathcal{D}^0$-parallel.*

Finally, we prove the following

Theorem 18. *If P is $\mathcal{D}^\lambda$-commutative for all $\mathcal{D}^\lambda$, $\lambda \in \{1, 0, \lambda_1, \dots, \lambda_k\}$ on an almost semi-invariant submanifold of an S-manifold, then*
(a) *M is minimal if and only if M is $\mathcal{D}^0$-minimal,*
(b) *M is $(\mathcal{D}^\lambda, \mathcal{D}^\mu)$-mixed totally geodesic if $\lambda \neq \mu$,*
(c) *$\mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{E}$ is integrable iff $\mathcal{D}^1$ is $\mathcal{D}^0$-parallel.*

Proof. (a) follows from Proposition 9. By the assumption, we get

$$0 = P^2 A_V X - A_V P^2 X = P^2 A_V X + \lambda^2 A_V X, \qquad X \in \mathcal{D}^\lambda,\ V \in T^\perp M,$$

which implies that $A_V X \in \mathcal{D}\lambda$. If $Y \in \mathcal{D}^\mu,\ \mu \neq \lambda$, then for all $V \in T^\perp M$, we have

$$0 = g(A_V X, Y) = g(h(X, Y), V),$$

which implies (b). Since P is $(\mathcal{D}^1 \oplus \mathcal{E})$-commutative, $\mathcal{D}^1 \oplus \mathcal{E}$ is integrable and (c) follows from Proposition 10. $\boxed{QED}$

8 Totally contact umbilical and totally contact geodesic submanifolds

Let M be a submanifold of a framed metric manifold $\bar{M}$ tangent to all ξ_α's. We give the following definitions, which are analogues to the definitions in Section 2 of [55].

Definition 4. If the second fundamental form h of M satisfies

$$(\nabla_X h)(Y,Z) = r(g(PX,Y)FZ + g(PX,Z)FY), \qquad X,Y,Z \in \mathcal{E}^\perp, \tag{65}$$

then the second fundamental form h of M is said to be r-contact parallel.

Definition 5. M is said to be *totally contact geodesic* if

$$h(J^2X, J^2Y) = 0, \qquad X,Y \in TM, \tag{66}$$

and, respectively, *totally contact umbilical* if

$$h(J^2X, J^2Y) = g(J^2X, J^2Y)\mathbf{K}, \qquad X,Y \in TM, \tag{67}$$

where $\mathbf{K}$ is some vector field perpendicular to M.

Example 2 ([14]). Let

$$\mathbf{R}^{2n+r} = \{(x^1, \ldots, x^n, y^1, \ldots, y^n, z^1, \ldots, z^r)\}$$

be a Euclidean space. It is endowed with a framed metric structure given by

$$\xi_\alpha = 2\frac{\partial}{\partial z^\alpha}, \quad 1 \le \alpha \le r, \quad \eta^\alpha = \frac{1}{2}\left(dz^\alpha - \sum_{k=1}^n y^k dx^k\right),$$

$$JX = \sum_{k=1}^n \left(Y^k \frac{\partial}{\partial x^k} - X^k \frac{\partial}{\partial y^k}\right) + \left(\sum_{k=1}^n Y^k y^k\right)\left(\sum_{\alpha=1}^r \frac{\partial}{\partial z^\alpha}\right)$$
$$\left(\text{where } X = \sum_{k=1}^n \left(X^k \frac{\partial}{\partial x^k} + Y^k \frac{\partial}{\partial y^k}\right) + \sum_{\alpha=1}^r Z^\alpha \frac{\partial}{\partial z^\alpha}\right),$$
$$g = \sum_{\alpha=1}^r \eta^\alpha \otimes \eta^\alpha + \frac{1}{4}\sum_{k=1}^n (dx^k \otimes dx^k + dy^k \otimes dy^k).$$

Then $M = S^{2n-1} \times \mathbf{R}^r$ is a totally contact umbilical semi-invariant submanifold.

In an S-manifold it is easy to see that

$$g(PX,Y) = -(\nabla_X \eta^\alpha)Y = d\eta^\alpha(Y,X), \qquad X,Y \in \mathcal{E}^\perp, \quad \text{and}$$
$$h(J^2X, J^2Y) = h(X,Y) + \sum_\alpha (\eta^\alpha(X)FY + \eta^\alpha(Y)FX),$$

and therefore (65), (66) and (67) become equivalent to the following three equations respectively :

$$(\nabla_X h)(Y,Z) = -\sum_\alpha (((\nabla_X \eta^\alpha)Y)FZ + ((\nabla_X \eta^\alpha)Z)FY), \qquad X,Y,Z \in \mathcal{E}^\perp, \tag{68}$$

$$h(X,Y) = -\sum_\alpha (\eta^\alpha(X)FY + \eta^\alpha(Y)FX), \quad X,Y \in TM, \tag{69}$$

$$h(X,Y) = g(JX,JY)\mathbf{K} - \sum_\alpha (\eta^\alpha(X)FY + \eta^\alpha(Y)FX), \qquad X,Y \in TM. \tag{70}$$

Proposition 11. *If M is totally contact geodesic submanifold of an S-manifold $\bar{M}$ then h is r-contact parallel.*

Proof. For $X,Y,Z \in \mathcal{E}^\perp$, by (69), we get $h(Y,Z)=0$, and

$$h(\nabla_X Y, Z) = -\sum_\alpha (\eta^\alpha(\nabla_X Y)FZ + \eta^\alpha(Z)F(\nabla_X Y)) = \sum_\alpha ((\nabla_X \eta^\alpha)Y)FZ,$$

from which (68) follows immediately. $\boxed{QED}$

Let $X \in TM$ and $Y \in \mathcal{D}^1 \oplus \mathcal{E}$. Then (43) implies that

$$F\nabla_X Y = h(X,PY) - fh(X,Y).$$

If M is totally contact geodesic, we have by (69), $\ker(F) = \mathcal{D}^1 \oplus \mathcal{E}$; by (23) we get

$$F\nabla_X Y = \sum_\alpha \eta^\alpha(Y) fFX, \qquad X \in TM,\ Y \in \mathcal{D}^1 \oplus \mathcal{E}. \tag{71}$$

Thus, we have:

Theorem 19. *If M is totally contact geodesic almost semi-invariant submanifold of a framed metric manifold then $\mathcal{D}^1 \oplus \mathcal{E}$ is autoparallel. In particular, if M is totally contact geodesic semi-invariant submanifold then $\mathcal{D}^1 \oplus \mathcal{E}$ is parallel and the maximal integral submanifold of $\mathcal{D}^1 \oplus \mathcal{E}$ is totally geodesic in M.*

Theorem 20. *If M is a submanifold of an S-space form $\bar{M}(c)$ with $c \neq -3r$ such that the second fundamental form h of M is r-contact parallel, then M is invariant or anti-invariant.*

Proof. Let $X, Y, Z \in \mathcal{E}^{\perp}$. From (65) we obtain

$$\begin{aligned}(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) &= -r\,(g(X,PZ)FY - g(Y,PZ)FX \\ &\quad + 2g(X,PY)FZ)\end{aligned}$$

and on the other hand from (8) we obtain

$$\begin{aligned}(\bar{R}(X,Y)Z)^{\perp} &= \frac{c-r}{4}(g(X,PZ)FY - g(Y,PZ)FX \\ &\quad +2g(X,PY)FZ),\end{aligned}$$

and therefore, in view of (19) it implies that

$$(c+3r)\,(g(X,PZ)FY - g(Y,PZ)FX + 2g(X,PY)FZ) = 0.$$

Putting $Y = Z$ in the above equation we get $g(X,PY)FY = 0$. Thus we have $PY = 0$ or $FY = 0$, that is M is invariant or anti-invariant. $\boxed{QED}$

Proposition 11 and Theorem 20 lead to

Theorem 21. *If M is totally contact geodesic submanifold of an S-space form $\bar{M}(c)$ with $c \neq -3r$, then M is invariant or anti-invariant.*

Theorem 22. *If M is totally contact umbilical almost semi-invariant submanifold of an S-manifold $\bar{M}$, then M is $(\mathcal{D}^{\lambda}, \mathcal{D}^{\mu})$-mixed totally geodesic for $\lambda, \mu \in \{1, 0, \lambda_1, \ldots, \lambda_k\}$ with $\lambda \neq \mu$.*

Proof. For $X \in \mathcal{D}^{\lambda}$, $Y \in \mathcal{D}^{\mu}$ from (70) we obtain

$$\begin{aligned}h(X,Y) &= g(JX,JY)\mathbf{K} - \sum_{\alpha}\left(\eta^{\alpha}(X)FY + \eta^{\alpha}(Y)FX\right), \\ &= g(JX,JY)\mathbf{K} = g(X,Y)\mathbf{K} = 0,\end{aligned}$$

which achieves the proof. $\boxed{QED}$

9 Relations between almost semi-invariant submanifolds of a framed metric manifold and other submanifolds

In this section, we investigate some relations between almost semi-invariant submanifolds of a framed metric manifold and other submanifolds. This section is mainly devoted to the study of submanifolds of principal toroidal bundles. A number of results about submanifolds of principal toroidal bundles are proved in [21] and [37].

Proposition 12. *Let $(\bar{M}, g)$ be a framed metric manifold and $\bar{M} \times \mathbf{R}^r$ be the almost Hermitian manifold with the almost Hermitian structure ϕ given by*

$$\phi X = JX, \quad X \in \mathcal{E}^{\perp}, \quad \phi \xi_\alpha = \frac{\partial}{\partial t^\alpha}, \quad \phi\left(\frac{\partial}{\partial t^\alpha}\right) = -\xi_\alpha, \quad \alpha \in \{1, \ldots, r\}.$$

Let M be a submanifold of $\bar{M}$ tangent to $\mathcal{E}$. Then we have

(1) *M is an almost semi-invariant submanifold of $\bar{M}$ if and only if it is a generic submanifold (in the sense of Ronsse [41]) of $\bar{M} \times \mathbf{R}^r$,*

(2) *M is a generic submanifold (in the sense of Mihai [37]) of $\bar{M}$ if and only if it is a generic submanifold (in the sense of Chen [16]) of $\bar{M} \times \mathbf{R}^r$.*

Next, we obtain relations between some submanifolds of an almost contact metric manifold and some submanifold of a framed metric manifold.

Let $\bar{M}$ be an almost contact metric manifold equipped with structure tensors (ϕ, ξ, η, g). Following the notations of [51] one can prove that the horizontal lift ϕ^h of ϕ with respect to the Riemannian connection of g defines a framed structure on the tangent bundle $T\bar{M}$. One has

$$(\phi^h)^2 = -I + \eta^v \otimes \xi^h + \eta^h \otimes \xi^v, \quad \eta^h(\xi^v) = \eta^v(\xi^h) = 1, \quad \eta^v(\xi^v) = \eta^h(\xi^h) = 0,$$

where ^{h} and ^{v} denote the horizontal and vertical lifts, respectively. The Sasaki metric G on $T\bar{M}$ is adapted to ϕ^h, that is

$$G(\phi^h X, \phi^h Y) = G(X, Y) - \eta^h(X)\, \eta^h(Y) - \eta^v(X)\, \eta^v(Y), \quad X, Y \in T(T\bar{M}).$$

Proposition 13. *Let M be a submanifold of the almost contact metric manifold $\bar{M}$ tangent to ξ. Then we have*

(1) *TM is a generic submanifold of $T\bar{M}$ if and only if M is a generic submanifold of $\bar{M}$,*

(2) *TM is a CR-submanifold of $T\bar{M}$ if and only if M is a CR-submanifold of $\bar{M}$,*

(3) *TM is an almost semi-invariant submanifold of $T\bar{M}$ if and only if M is an almost semi-invariant submanifold of $\bar{M}$.*

The main part of this section is devoted to the study of submanifolds of principal toroidal bundles.

Let $\bar{M}$ be a $(2n+r)$-dimensional S-manifold which is the bundle space of a principal toroidal bundle over a $2n$-dimensional Kaehler manifold $\bar{N}$ with

Kaehler structure $(\tilde{J}, \tilde{g})$; $\bar{\pi} : \bar{M} \to \bar{N}$. Let M be an $(m+r)$-dimensional submanifold of $\bar{M}$, tangent to $\mathcal{E}$ and N an m-dimensional submanifold of $\bar{N}$. We assume that there exists a fibration $\pi : M \to N$ such that the diagram

$$(*) \qquad \begin{array}{ccc} M & \xrightarrow{i'} & \bar{M} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{\pi}} \\ N & \xrightarrow{i} & \bar{N} \end{array}$$

commutes and the immersion i' is a diffeomorphism on the fibres.

Examples.

1. Let N be a submanifold of $\bar{N}$. Then we have the diagram

$$\begin{array}{ccc} \bar{\pi}^{-1}(N) & \xrightarrow{i'} & \bar{M} \\ \downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ N & \xrightarrow{i} & \bar{N} \end{array} .$$

2. Let us consider the commutative diagram

$$\begin{array}{ccc} M^{m+1} & \xrightarrow{i'} & S^{2n+1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ N^m & \xrightarrow{i} & P^n(\mathbf{C}) \end{array} ,$$

where $\tilde{\pi}$ is the Hopf fibration, M^{m+1} and N^m are submanifolds of S^{2n+1} and $P^n(\mathbf{C})$, respectively, and i' is a diffeomorphism on the fibres.

Using the diagonal map, we define a principal bundle by the diagram

$$\begin{array}{ccc} M^{m+r} & \xrightarrow{\triangle'} & M^{m+1} \times \cdots \times M^{m+1} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi \times \cdots \times \pi} \\ N^m & \xrightarrow{\triangle} & N^m \times \cdots \times N^m \end{array} ,$$

where $M^{m+r} = \{(p_1, \ldots, p_r) \in M^{m+1} \times \cdots \times M^{m+1} | \pi(p_1) = \cdots = \pi(p_r)\}$.

Thus we obtain the diagram

$$\begin{array}{ccc} M^{m+r} & \xrightarrow{i'} & H^{2n+r} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \bar{\pi}} \\ N^m & \xrightarrow{i} & P^n(\mathbf{C}) \end{array} ,$$

where $H^{2n+r} = \{(x_1, \ldots, x_r) \in S^{2n+1} \times \ldots \times S^{2n+1} | \tilde{\pi}(x_1) = \cdots = \tilde{\pi}(x_r)\}$.

We can state the following result.

Theorem 23. (i) *M is a CR-submanifold of $\bar{M}$ if and only if N is a CR-submanifold of $\bar{N}$.*

(ii) *M is a generic submanifold (in the sense of Mihai [37]) of $\bar{M}$ if and only if N is a generic submanifold (in the sense of Chen [16]) of $\bar{N}$.*

(iii) *M is an almost semi-invariant submanifold of $\bar{M}$ if and only if N is a generic submanifold (in the sense of Ronsse [41]) of $\bar{N}$.*

Now, we denote by $\bar{\nabla}$ (resp. $\tilde{\bar{\nabla}}$) the Riemannian connections with respect to g (resp. $\tilde{g}$) on $\bar{M}$ (resp. $\bar{N}$); and by ∇ (resp. ∇') the induced Riemannian connections on M (resp. N). Let h (resp. σ) be the second fundamental form of the immersion i' (resp. i) and A' (resp. A) the shape operator of i' (resp. i).

For $X, Y \in TN$, the Gauss formula implies

$$(\nabla'_X Y)^* = -J^2 \nabla_{X^*} Y^*, \quad \sigma(X,Y)^* = h(X^*, Y^*).$$

Analogously, the Weingarten formula leads to

$$(A'_V X)^* = -J^2 A_{V^*} X^*, \quad (D'_X V)^* = D_{X^*} V^*$$

for each $V \in T^{\perp}N$, where we put D (resp. D') for the normal connections induced by $\bar{\nabla}$ (resp. $\tilde{\bar{\nabla}}$).

Then the following result follows.

Proposition 14. *M is a minimal submanifold of the S-manifold $\bar{M}$ if and only if N is minimal in the Kaehler manifold $\bar{N}$.*

On the other hand, we deduce that

$$\begin{aligned}((\nabla_X \sigma)(Y,Z))^* &= (\nabla_{X^*} h)(Y^*, Z^*) + \sum g(Y^*, JX^*) h(\xi_\alpha, Z^*) \\ &\quad + \sum g(Z^*, JX^*) h(\xi_\alpha, Y^*).\end{aligned}$$

If we put $\tilde{J}X = pX + qX$, for each vector field X tangent to N, where pX and qX are the tangential and normal components of $\tilde{J}X$ respectively, the above relation becomes

$$(\nabla_{X^*} h)(Y^*, Z^*) = ((\nabla_X \sigma)(Y,Z))^* + r\{\tilde{g}(Y, pX) qZ + \tilde{g}(Z, pX) qY\}^* .$$

It is easily seen that

$$(\nabla_{X^*} h)(Y^*, \xi_\alpha) = \{D_{X^*} F Y^* - F \nabla_{X^*} Y^* - h(Y^*, PX^*)\},$$

or equivalently

$$(\nabla_{X^*} h)(Y^*, \xi_\alpha) = \{(\nabla_X q) Y - \sigma(Y, pX)\}^* \quad .$$

Thus, we proved the following.

Proposition 15. *Let $\bar{M}$ be an S-manifold and M (resp. N) a submanifold of $\bar{M}$ (resp. $\bar{M}/_{\mathcal{E}}$) such that the diagram (*) is commutative. Then M is a parallel submanifold of $\bar{M}$ if and only if the second fundamental form σ of N satisfies*

$$(\nabla_X \sigma)(Y,Z) = -r\left\{\tilde{g}(Y,pX)qZ + \tilde{g}(Z,pX)qY\right\} \quad \textit{and} \quad (\nabla_X q)Y = \sigma(Y,pX)$$

for any vector fields X, Y, Z tangent to N.

Corollary 7. *Let $\bar{M}$ be an S-manifold. Then M is a parallel submanifold of $\bar{M}$ if and only if N is a parallel submanifold of $\bar{N}$.*

We denote by R and R' the curvature tensor fields on M and N respectively. For $U \in T^{\perp}N$, put $\tilde{J}U = bU + cU$, where bU and cU are the tangential and normal components of $\tilde{J}U$ respectively.

Proposition 16. *The normal connection D of M is flat if and only if the f-structure c in the normal bundle of N is parallel and*

$$R'^{\perp}(X,Y)U = 2r\tilde{g}(X,pY)cU,$$

for all $X, Y \in TN$, $U \in T^{\perp}N$.

Proof. Using Weingarten's formula, we have

$$(D'_{[X,Y]}U)^* = D_{[X^*,Y^*]}U^* + 2rg(Y^*, PX^*)fU^*$$

and

$$g(R^{\perp}(X^*,Y^*)U^*,V^*) = \tilde{g}(R'^{\perp}(X,Y)U,V) + 2r\tilde{g}(Y,pX)G(cU,V).$$

On the other hand

$$\bar{R}(X^*,\xi_\alpha)U^* = -(\bar{\nabla}_{X^*}J)U^* = \sum_{\alpha=1}^{r} g(X^*,U^*)\xi_\alpha.$$

From Ricci's equation, it follows that

$$0 = g(\bar{R}(X^*,\xi_\alpha)U^*,V^*) = g(R^{\perp}(X^*,\xi_\alpha)U^*,V^*) - g([A_{U^*},A_{V^*}]X^*,\xi_\alpha),$$

where $X, Y \in TN$ and $U, V \in T^{\perp}N$.

By a straightforward computation, we obtain

$$\begin{aligned} g([A_{U^*},A_{V^*}]X^*,\xi_\alpha) &= g(A_{V^*}X^*,tU^*) - g(A_{U^*}X^*,tV^*) \\ &= \tilde{g}(A'_V X, bU) - \tilde{g}(A'_U X, bV) \\ &= -\tilde{g}(\tilde{J}A'_V X, U) - \tilde{g}(\sigma(X,bV),U) \\ &= \tilde{g}((\nabla'_X c)V, U), \end{aligned}$$

where bU is the tangential component of $\tilde{J}U$. If $X, Y \in TN$, we have

$$R(\xi_\alpha, X^*, Y^*, \xi_\alpha) = g(PX^*, PY^*)$$

and therefore M can not be flat if it is not anti-invariant. $\boxed{QED}$

Proposition 17. *Let M and N be anti-invariant submanifolds of $\bar{M}$ and $\bar{N}$, respectively. Then M is flat if and only if N is flat. The Ricci tensors S and S' of M and N respectively are related by*

$$S(X^*, Y^*) = S'(X, Y) - 2rg(PX^*, PY^*), \qquad X, Y \in TN,$$

and their scalar curvatures satisfy

$$\rho = \rho' - rm + r\sum_{i=1}^{m} g(FE_i^*, FE_i^*),$$

where $\{E_1, \ldots, E_m\}$ is a local orthonormal basis on N.

Proposition 18. *The scalar curvatures ρ of M and ρ' of N satisfy*

$$\rho' - rm \leq \rho \leq \rho'.$$

Moreover, both M and N are anti-invariant if and only if $\rho = \rho'$ (resp. both M and N are invariant if and only if $\rho = \rho' - rm$).

Proposition 19 ([21]). *If the mean curvature vector $\mathbf{H}$ of M is parallel, then the mean curvature vector $\mathbf{H}'$ of N is parallel too. Moreover, if $f = 0$ (and so $c = 0$), $\mathbf{H}$ is parallel if and only if $\mathbf{H}'$ is parallel.*

Proof. For $X \in TN$, one has

$$(D'_X\mathbf{H}')^* = \frac{m+r}{m} D_{X^*}\mathbf{H}\,.$$

$\boxed{QED}$

Analogously, one can prove the following.

Proposition 20. (i) *M is totally contact umbilical if and only if N is totally umbilical.*

(ii) *M is totally contact geodesic if and only if N is totally geodesic.*

Example. Consider again the S-manifold H as a principal toroidal bundle over $P^n(\mathbf{C})$. A real projective space $P^m(\mathbf{R})$ $(m < n)$ of constant curvature 1 is imbedded in $P^n(\mathbf{C})$ as an anti-invariant and totally geodesic submanifold. The following diagram

$$\begin{array}{ccc} P^m(\mathbf{R}) \times T^{r-1} & \xrightarrow{i'} & H \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \bar{\pi}} \\ P^m(\mathbf{R}) & \xrightarrow{i} & P^n(\mathbf{C}) \end{array},$$

is commutative. Then $P^m(\mathbf{R}) \times T^{r-1}$ is an anti-invariant and totally contact geodesic submanifold of H.

Acknowledgements. The authors are thankful to the referee for comments towards the improvement of the paper.

References

[1] A. Bejancu: Geometry of CR-submanifolds, D. Reidel Publ. Co., 1986.

[2] A. Bejancu, N. Papaghiuc: *Almost semi-invariant submanifolds of a Sasakian manifold*, Bull. Math. Soc. Sci. Math. R.S. Roumanie **28** (76) (1984), 13–30.

[3] D. E. Blair: *Geometry of manifolds with structural group $\mathcal{U}(n) \times \mathcal{O}(s)$*, J. Diff. Geometry **4** (1970), 155–167.

[4] D. E. Blair: *On a generalization of the Hopf fibration*, An. Ştiint. Univ. "Al.I.Cuza" Iaşi **27** (1971), 171–177.

[5] D. E. Blair: Contact manifolds in Riemannian geometry, Lecture Notes in Math. **509**, Springer Verlag, 1976.

[6] D. E. Blair, G. D. Ludden, K. Yano: *Differential geometric structures on principal toroidal bundles*, Trans. Am. Math. Soc. **181** (1973), 175–184.

[7] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, Geometriae Dedicata **72** (1999), 183–199.

[8] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez: *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J. **42** (2000), 125–138.

[9] J. L. Cabrerizo, L. M. Fernandez, M. Fernandez: *The curvature tensor fields on f-manifolds with complemented frames*, An. Ştiint. Univ. "Al.I.Cuza" Iaşi **36** (1990), 151–161.

[10] J. L. Cabrerizo, L. M. Fernandez, M. Fernandez *A classification of certain submanifolds of an S-manifold*, Ann. Polonici. Math. **54** (1991), 117–123.

[11] J. L. Cabrerizo, L. M. Fernandez, M. Fernandez: *The curvature of submanifolds of an S-space form*, Acta Math. Hung. **62** (1993), 373–383.

[12] J. L. Cabrerizo, L. M. Fernandez, M. Fernandez: *On certain anti-invariant submanifolds of S-manifold*, Portugal. Math. **50** (1993), 103–113.

[13] J. L. Cabrerizo, L. M. Fernandez, M. Fernandez *On normal CR-submanifolds of S-manifolds*, Coll. Math. **LXIV** (1993), 203–214.

[14] J. L. Cabrerizo, L. M. Fernandez, M. Fernandez: *A classification of totally f-umbilical submanifolds of an S-manifold*, Soochow. J. Math. **18** (1992), 211–221.

[15] B.-Y. Chen: Geometry of submanifolds, Marcel Dekker NY, 1973.

[16] B.-Y. Chen: Geometry of submanifolds and its applications, Science Univ. Tokyo, 1981.

[17] B.-Y. Chen: Geometry of slant submanifolds, Katholieke Univ. Leuven, 1990.

[18] K. L. Duggal: *On differentiable structures defined by algebraic equations I: Nijenhuis tensor*, Tensor **22** (1971), 238–242.

[19] F. Etayo: *On quasi-slant submanifolds of an almost Hermitian manifold*, Pub. Math. Debrecen **53** (1998), no. 1-2, 217–223.

[20] L. M. Fernandez: *On totally geodesic invariant submanifolds of an S-manifold*, Rev. Acad. Ciencias Zaragoza **43**(1988), 23–30.

[21] L. M. Fernandez: *Submanifolds in toroidal bundles*, Proc. VIth Int. Coll. on Diff. Geom., Santiago, Spain, 1988, Cursos y Congresos, Univ. Santiago de Compostela, Spain, **61** (1989), 85–98.

[22] L. M. Fernandez: *CR-products of S-manifolds*, Portugal. Math. **47** (1990), 167–181.

[23] S. J. Greenfield: *Cauchy-Riemann equations in several variables*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 275–314.

[24] I. Hasegawa: *Totally k-contact umbilical submanifolds in p-th Sasakian manifolds*, J. Hokkaido Univ. Edu. Sect IIA **37** (1987), 109–118.

[25] I. Hasegawa, T. Abe: *Totally k-contact umbilical submanifolds in p-th Sasakian manifolds II*, J. Hokkaido Univ. Edu. Sect IIA **40** (1989), 1–6.

[26] I. Hasegawa, Y. Okuyama, T. Abe: *On p-th Sasakian manifolds*, J. Hokkaido Univ. Edu. Sect IIA **37** n. 1 (1986), 1–16.

[27] M. Kobayashi: *CR submanifolds of a Sasakian manifold*, Tensor **35** (1981), 297–307.

[28] M. Kobayashi: *Contact CR products of Sasakian manifolds*, Tensor **36** (1982), 281–288.

[29] M. Kobayashi: *Semi-invariant submanifolds in an f-manifold with complemented frames*, Tensor **49** (1990), 155–178.

[30] M. Kobayashi: *On semi-invariant submanifolds of S-manifolds of constant f-sectional curvature*, Tensor **51** (1992), 41–47.

[31] M. Kobayashi, S. Tsuchiya: *Invariant submanifolds in an f-manifold with complemented frames*, Kodai Math. Sem. Rep. **24** (1972), 430–450.

[32] A. Lotta: *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie **39** (1996), 183–198.

[33] K. Matsumoto: *On contact CR-submanifolds of Sasakian manifolds*, Internat. J. Math. Math. Sci. **6**(1983), 313–326.

[34] K. Matsumoto, I. Mihai, R. Rosca *A certain locally conformal almost cosymplectic manifolds and its submanifolds*, Tensor **51** (1992), 91–102.

[35] K. Matsumoto, I. Mihai, M. H. Shahid: *Certain submanifolds of a Kenmotsu manifold*, Proc. Third Pacific Rim Geom. Conf., Seoul, (1996), Monograph Geom. Topology **25**, International Press, Cambridge, MA, (1998), 183–193.

[36] I. Mihai: *CR-submanifolds of a framed f-manifold*, Stud. Cerc. Mat. **35** (1983), 127–136.

[37] I. Mihai: *On generic submanifolds of a framed f-manifold*, Proc. Nat. Conf. Geom. Timişoara (1984), 153–158.

[38] I. Mihai: *Certain submanifolds of a Kaehler manifold*, Geometry and Topology of Submanifolds, Vol **VII**, World Scientific, 1995, 186-188.

[39] K. Nomizu: *Characteristic roots and vectors of a differentiable family of symmetric matrices*, Linear and Multilinear Algebra **1** (1973), 159–162.

[40] N. Papaghiuc: *Semi-slant submanifolds of a Kaehlerian manifold*, An. Ştiint. Univ. “Al. I. Cuza” Iaşi Secţ Ia Mat. **40** (1994) 55–61.

[41] G. S. Ronsse: *Generic and skew CR submanifolds of a Kaehler manifold*, Bull. Inst. Math. Acad. Sinica **18** (1990), 127–141.

[42] R. E. Stong: *The rank of an f-structure*, Kodai Math. Sem. Rep. **29** (1977), 207–209.

[43] M. H. Shahid, S. Ali *Generic submanifolds of a Sasakian manifold*, Math. Student **52** (1989), 205–210.

[44] M. M. Tripathi: *Semi-invariant submanifolds of an almost r-contact metric manifold*, Math. J. Toyama Univ. **14** (1991), 95–111.

[45] M. M. Tripathi: On almost semi-invariant submanifolds, Ph.D. Thesis, Lucknow University, 1996.

[46] M. M. Tripathi: *Almost semi-invariant submanifolds of trans-Sasakian manifolds*, J. Indian Math. Soc. **62** (1996), 220–245.

[47] M. M. Tripathi, K. D. Singh: *Almost semi-invariant submanifolds of an ϵ-framed metric manifold*, Demonstratio Math. **29** (1996), 413–426.

[48] M. M. Tripathi, K. D. Singh: *On submanifolds of S-manifolds*, Ganita **47** (1996), 51-=54

[49] J. Vanzura: *Almost r-contact structures*, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. **26** (1972), 97–115.

[50] K. Yano, S. Ishihara: *The f-structure induced on submanifolds of complex and almost complex spaces*, Kodai Math. Sem. Rep. **18** (1966), 271–292.

[51] K. Yano, S. Ishihara: Tangent and cotangent bundles, Lecture notes in pure and applied mathematics, n. **16**, Marcel Dekker Inc. NY, 1973.

[52] K. Yano, M. Kon: Anti-invariant submanifolds, Lecture notes in pure and applied mathematics, n. **21**, Marcel Dekker Inc. NY, 1976.

[53] K. Yano, M. Kon: CR submanifolds of Kaehlerian and Sasakian manifolds, Birkhäuser, Boston, 1983.

[54] K. Yano, M. Kon: Structures on manifolds, World Scientific, 1984.

[55] K. Yano, M. Kon: *On contact CR submanifolds*, J. Korean Math. Soc. **26** (1989), 231–262.