

# The contact Whitney sphere

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**Abstract.** In this paper, we introduce the contact Whitney sphere as an imbedding of the  $n$ -dimensional unit sphere as an integral submanifold of the standard contact structure on  $\mathbb{R}^{2n+1}$ . We obtain a general inequality for integral submanifolds in  $\mathbb{R}^{2n+1}$ , involving both the scalar curvature and the mean curvature, and we use the equality case in order to characterize the contact Whitney sphere. We also study a similar problem for anti-invariant submanifolds of  $\mathbb{R}^{2n+1}$ , tangent to the structure vector field.

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## Introduction

One of the most interesting topics in contact geometry is the study of integral submanifolds, i.e., submanifolds immersed in a contact manifold, such that the contact form restricted to the submanifold vanishes. In particular, 1-dimensional integral submanifolds are called *Legendre curves*.

A well-known property of Legendre curves of the standard contact structure  $dz - ydx$  on  $\mathbb{R}^3$  is that the projection  $\bar{\gamma}$  of a closed Legendre curve  $\gamma$  in  $\mathbb{R}^3$  to the  $xy$ -plane must have self-intersections and algebraic (signed) area zero. On the other hand, we can think of the pair of  $\gamma$  and its projection  $\bar{\gamma}$  in the following terms. Suppose that  $\gamma$  itself does not have self-intersections and regard  $\bar{\gamma}$  as a Lagrangian submanifold in  $\mathbb{C} \cong \mathbb{R}^2$  with self-intersections; then think of going from  $\bar{\gamma}$  to  $\gamma$  as a way of removing the singularity but preserving the “Lagrangian-Legendre” property.

A generalization of this can be done with the Whitney spheres. The Whitney spheres are usually defined as a family of Lagrangian immersions of the unit

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sphere  $\mathbb{S}^n$ , centered at the origin of  $\mathbb{R}^{n+1}$ , into  $\mathbb{C}^n \cong \mathbb{R}^{2n}$  given by

$$(u_0, u_1, \dots, u_n) \rightarrow \frac{r}{1 + u_0^2}(u_1, \dots, u_n, u_0 u_1, \dots, u_0 u_n) + B, \quad (1)$$

where  $r$  is a positive number and  $B$  is a vector of  $\mathbb{C}^n$ . The number  $r$  and the vector  $B$  are called the radius and the center of the Whitney sphere, respectively. From a topological point of view, it is well-known that the sphere can not be imbedded in  $\mathbb{C}^n$  as a Lagrangian submanifold. The Whitney spheres have the best possible behaviour, because they have only one double point at the poles of  $\mathbb{S}^n$ .

In the contact manifold  $\mathbb{R}^{2n+1}$  with its usual contact metric structure, we have the following presentation of the Whitney spheres as a family of imbedded spheres and integral submanifolds of the contact structure

$$(u_0, u_1, \dots, u_n) \rightarrow \frac{r}{1 + u_0^2}(u_0 u_1, \dots, u_0 u_n, u_1, \dots, u_n, \frac{r u_0}{1 + u_0^2} + C(1 + u_0^2)) + B,$$

where  $r$  is a positive number,  $B$  is a vector of  $\mathbb{R}^{2n+1}$  and  $C$  is a real constant. We refer to these spheres as the *contact Whitney spheres*. Therefore, with this presentation we have removed the previous singularity at the poles of  $\mathbb{S}^n$ .

On the other hand, the Whitney spheres in  $\mathbb{C}^n$  have an interesting geometric property. It was proven by Borrelli, Chen and Morvan [3] and independently by Ros and Urbano [6] that if  $M^n$  is a Lagrangian submanifold of  $\mathbb{C}^n$ , with mean curvature vector  $H$  and scalar curvature  $\tau$ , then  $|H|^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau$ , with equality if and only if  $M$  is either totally geodesic or a (piece of a) Whitney sphere. Moreover, in [4], Castro constructs a one-parameter family of Lagrangian spheres including the Whitney sphere, such that they satisfy a geometric equality of type  $\tau = \mu|H|^2$ , with  $\mu > 0$ .

In this paper, we establish an analogue of the above result for integral submanifolds in  $\mathbb{R}^{2n+1}$  with its standard Sasakian structure, such that the equality case holds for the contact Whitney spheres. We also give a characterization by the second fundamental form, similar to that of Ros and Urbano in [6].

Finally, we study the same problem for anti-invariant submanifolds tangent to the structure vector field on  $\mathbb{R}^{2n+1}$  and obtain the corresponding results.

## 1 Preliminaries.

Let  $(\mathbb{R}^{2n+1}, \phi, \xi, \eta, g)$  denote the manifold  $\mathbb{R}^{2n+1}$  with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^n y_i dx_i), \quad \xi = 2\frac{\partial}{\partial z},$$

$$g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i),$$

$$\phi \left( \sum_{i=1}^n \left( X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i} \right) + Z \frac{\partial}{\partial z} \right) = \sum_{i=1}^n \left( Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i} \right) + \sum_{i=1}^n Y_i y_i \frac{\partial}{\partial z},$$

where  $(x_i, y_i, z)$ ,  $i = 1 \dots n$  are the cartesian coordinates.

It is well-known that  $(\mathbb{R}^{2n+1}, \phi, \xi, \eta, g)$  is a Sasakian-space-form, with constant  $\phi$ -sectional curvature equal to  $-3$ . Hence, its curvature tensor  $\tilde{R}$  is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= -\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X - g(X, Z)\eta(Y)\xi + \\ &+ g(Y, Z)\eta(X)\xi - g(Z, \phi Y)\phi X + g(Z, \phi X)\phi Y - 2g(X, \phi Y)\phi Z, \end{aligned} \quad (2)$$

for any vector fields  $X, Y, Z$ . For more details and background, we refer to [2].

Let  $M$  be an  $n$ -dimensional Riemannian manifold isometrically immersed in the Sasakian-space-form  $\mathbb{R}^{2n+1}$ . We also denote by  $g$  the metric on  $M$ .

Let  $\nabla$  (resp.  $\tilde{\nabla}$ ) be the Levi-Civita connection of  $M$  (resp.  $\mathbb{R}^{2n+1}$ ). Then, the Gauss-Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \tilde{\nabla}_X V = -A_V X + D_X V,$$

for any tangent vector fields  $X$  and  $Y$  and any normal vector field  $V$ , where  $D$  is the connection in the normal bundle,  $\sigma$  is the second fundamental form of  $M$  and  $A$  is the shape operator. The mean curvature vector  $H$  is defined by  $H = (1/n)$  trace  $\sigma$ .

A submanifold  $M$  is called an *integral* submanifold if  $\eta$  restricted to  $M$  vanishes. It is well-known that the contact subbundle  $\{\eta = 0\}$  admits integral submanifolds up to and including dimension  $n$ , but of no higher dimension, see e.g. [2]. A direct consequence of this definition is that  $\phi X$  is a normal vector field, for any tangent vector field  $X$ , i.e.,  $M$  is an *anti-invariant* submanifold of  $\mathbb{R}^{2n+1}$ . Hence, in a neighborhood of every point  $p \in M$ , we can consider a local orthonormal frame  $\{e_1, \dots, e_n, e_{1*}, \dots, e_{n*}, \xi\}$ , such that  $e_1, \dots, e_n$  are tangent to  $M$  and  $e_{i*} = \phi e_i$ , for any  $i = 1, \dots, n$ . Such a frame is called a *Legendre frame*. If we put  $\sigma_{ij}^k = g(\sigma(e_i, e_j), e_{k*})$ , it can be proved that

$$\sigma_{jk}^i = \sigma_{ji}^k = \sigma_{ik}^j, \quad (3)$$

for any  $i, j, k = 1, \dots, n$ .

On the other hand, we have [2, p. 128]

$$A_\xi = 0, \quad (4)$$

equivalently  $\sigma(X, Y)$  is perpendicular to  $\xi$ , for any tangent vector fields  $X$  and  $Y$ .

## 2 Characterizing the Contact Whitney Sphere

Let  $M$  be an  $n$ -dimensional submanifold of  $\mathbb{R}^{2n+1}$ . Given a tangent orthonormal frame  $\{e_1, \dots, e_n\}$ , the scalar curvature  $\tau$  of  $M$  is defined by

$$\tau = \sum_{i < j} K(e_i \wedge e_j),$$

where  $K(e_i \wedge e_j)$  is the sectional curvature of the plane section spanned by  $e_i$  and  $e_j$ .

By following the same steps as in the proof of Lemma 1 of [3], and by virtue of (2), (3) and (4), we obtain the following inequality for integral submanifolds, involving both curvatures  $|H|$  and  $\tau$ :

**Proposition 1.** *Let  $M^n$  be an integral submanifold of  $\mathbb{R}^{2n+1}$ . Then, the square of its mean curvature  $|H|^2$  and its scalar curvature  $\tau$  satisfy at each point the following inequality:*

$$|H|^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau. \quad (5)$$

The equality holds if and only if there exists a real function  $\lambda$  defined on  $M$ , such that the second fundamental form  $\sigma$  of  $M$  satisfies

$$\begin{aligned} \sigma(e_1, e_1) &= 3\lambda e_{1*}, & \sigma(e_2, e_2) &= \dots = \sigma(e_n, e_n) = \lambda e_{1*}, \\ \sigma(e_1, e_j) &= \lambda e_{j*}, & \sigma(e_j, e_k) &= 0, \quad 2 \leq j \neq k \leq n, \end{aligned} \quad (6)$$

where  $\{e_1, \dots, e_n, e_{1*}, \dots, e_{n*}, \xi\}$  is a Legendre frame such that  $e_{1*}$  is parallel to  $H$ .

On the other hand, we can also characterize the equality case of the above inequality by the behavior of the second fundamental form of the submanifold. We obtain a formula similar to that of [6]:

**Proposition 2.** *Let  $M^n$  be an integral submanifold of  $\mathbb{R}^{2n+1}$ . Then,  $M$  satisfies the equality case of (5) at every point, if and only if*

$$\sigma(X, Y) = \frac{n}{n+2} \{g(X, Y)H + g(\phi X, H)\phi Y + g(\phi Y, H)\phi X\}, \quad (7)$$

for any tangent vector fields  $X$  and  $Y$ .

**PROOF.** If the equality case of (5) holds for every  $p \in M$ , then, in a neighborhood of every point, we can find a Legendre frame with  $e_{1*}$  parallel to  $H$  and such that it satisfies (6), with  $\lambda = \frac{n}{n+2}|H|$ . Hence, it follows that (7) holds for any tangent vector fields  $X$  and  $Y$ . The converse can be verified directly.  $\square$

We now proceed to show that the only non-trivial example of an integral submanifold satisfying the equality in (5), is the contact Whitney sphere. First, we state the following two lemmas. The first one can be easily proved by straightforward computation.

**Lemma 1.** *Let  $\pi : \mathbb{R}^{2n+1} \rightarrow \mathbb{C}^n$  be the differential map given by:*

$$\pi(x_1, \dots, x_n, y_1, \dots, y_n, z) = \frac{1}{2}(y_1, \dots, y_n, x_1, \dots, x_n). \quad (8)$$

*Then,  $\pi : (\mathbb{R}^{2n+1}, \phi, \xi, \eta, g) \rightarrow (\mathbb{C}^n, J, G)$  is a Riemannian submersion, where we denote by  $(\mathbb{C}^n, J, G)$  the usual Kaehlerian structure on  $\mathbb{C}^n$ . Moreover, it satisfies the following conditions:*

- i) The vertical subspace  $\mathcal{V}_p$  of the submersion at  $p \in \mathbb{R}^{2n+1}$  is equal to the span of  $\xi_p$ ;*
- ii)  $g = \pi^*G + \eta \otimes \eta$ ;*
- iii)  $\phi X = (J\pi_*X)^*$ , for any vector field  $X$  on  $\mathbb{R}^{2n+1}$ , where  $*$  denotes the horizontal lift with respect to  $\eta$ .*

**Lemma 2 (Uniqueness of the lift of the Whitney sphere).** *Let  $\pi : \mathbb{R}^{2n+1} \rightarrow \mathbb{C}^n$  be the Riemannian submersion given by (8), and  $w : \mathbb{S}^n \rightarrow \mathbb{C}^n$  a Whitney immersion given by (1). If  $\psi : \mathbb{S}^n \rightarrow \mathbb{R}^{2n+1}$  is an integral immersion, such that  $\pi \circ \psi = w$ , then  $\psi$  is the contact Whitney immersion given by*

$$\psi(u_0, u_1, \dots, u_n) = \frac{2r}{1+u_0^2}(u_0u_1, \dots, u_0u_n, u_1, \dots, u_n, \frac{2ru_0}{1+u_0^2} + C(1+u_0^2)) + \tilde{B},$$

*where  $C$  is a constant and  $\tilde{B}$  is a vector of  $\mathbb{R}^{2n+1}$  such that  $\pi(\tilde{B}) = B$ .*

PROOF. Since  $\pi \circ \psi = w$  and  $\sum_{i=0}^n u_i^2 = 1$ , we can write  $\psi$  as

$$\psi(u_0, u_1, \dots, u_n) = \frac{2r}{1+u_0^2}(u_0u_1, \dots, u_0u_n, u_1, \dots, u_n, f(u_1, \dots, u_n)) + \tilde{B}, \quad (9)$$

for any constant vector  $\tilde{B}$  such that  $\pi(\tilde{B}) = B$ .

Then, differentiating (9), we obtain, for any  $i = 1, \dots, n$ :

$$\begin{aligned} \frac{\partial}{\partial u_i} = & 2r \left[ \frac{u_0^2(1+u_0^2) - u_i^2(1-u_0^2)}{u_0(1+u_0^2)^2} \frac{\partial}{\partial x_i} - \sum_{j \neq i} \frac{u_i u_j (1-u_0^2)}{u_0(1+u_0^2)^2} \frac{\partial}{\partial x_j} + \right. \\ & \left. + \frac{1+u_0^2+2u_i^2}{(1+u_0^2)^2} \frac{\partial}{\partial y_i} + 2 \sum_{j \neq i} \frac{u_i u_j}{(1+u_0^2)^2} \frac{\partial}{\partial y_j} + \frac{2u_i f + (1+u_0^2) \frac{\partial f}{\partial u_i}}{(1+u_0^2)^2} \frac{\partial}{\partial z} \right]. \quad (10) \end{aligned}$$

Since  $\psi$  is an integral immersion in  $\mathbb{R}^{2n+1}$ , we have  $\eta(\frac{\partial}{\partial u_i}) = 0$ , for any  $i = 1, \dots, n$ , where  $\eta$  is the contact form on  $\mathbb{R}^{2n+1}$ . Hence, it follows from (10) that the function  $f$  must satisfy the system of partial differential equations

$$\frac{\partial}{\partial u_i} \left( \frac{f}{1+u_0^2} \right) = \frac{2ru_i(3u_0^2-1)}{u_0(1+u_0^2)^3}, \quad i = 1, \dots, n,$$

which implies that

$$f = \frac{2ru_0}{1+u_0^2} + C(1+u_0^2),$$

where  $C$  is a real constant.  $\square$

We can now state the main theorem:

**Theorem 1.** *Let  $M^n$  be an integral submanifold of the standard contact structure on  $\mathbb{R}^{2n+1}$ . Then, the equality case of (5) holds at every point  $p \in M$ , if and only if either  $M$  is a totally geodesic submanifold or it is a portion of a contact Whitney sphere.*

PROOF. Let  $\psi : M \rightarrow \mathbb{R}^{2n+1}$  be an integral immersion and put  $\hat{\psi} = \pi \circ \psi$ . Then, it follows from Lemma 1 that  $\hat{\psi} : M \rightarrow \mathbb{C}^n$  is a Lagrangian immersion, and that the metrics induced on  $M$  by both  $\psi$  and  $\hat{\psi}$  agree.

Denote by  $\hat{\sigma}$  and  $\bar{\nabla}$  the second fundamental form of  $\hat{\psi}$  and the Levi-Civita connection of  $\mathbb{C}^n$ , respectively. Then, it follows from the well-known O'Neill equations [5], that

$$\tilde{\nabla}_{X^*} Y^* = (\bar{\nabla}_X Y)^* + \frac{1}{2} \eta([X^*, Y^*]) \xi$$

for any vector fields  $X, Y$  on  $\mathbb{C}^n$  tangent to  $M$ . Since  $M$  is normal to  $\xi$ , we have  $\eta([X^*, Y^*]) = 0$ , and therefore

$$\sigma(X, Y) = (\hat{\sigma}(X, Y))^* \text{ and } \hat{H}^* = H, \quad (11)$$

where  $\hat{H}$  is the mean curvature vector of  $\hat{\psi}$ .

Suppose that the equality case of (5) holds for any  $p \in M$ . Then, Proposition 2 implies that  $\sigma$  satisfies (7). Hence, by virtue of (11), we have that

$$\hat{\sigma}(X, Y) = \frac{n}{n+2} \left\{ G(X, Y) \hat{H} + G(JX, \hat{H}) JY + G(JY, \hat{H}) JX \right\}$$

for any vector fields  $X, Y$  on  $\mathbb{C}^n$  tangent to  $M$ . Therefore, it follows from [6, Theorem 2], that either  $\hat{\psi}$  is totally geodesic or  $\hat{\psi}(M)$  is an open portion of a Whitney sphere.

If the first case holds, (11) implies that  $\psi$  is also totally geodesic. On the other hand, if  $\hat{\psi}(M)$  is a portion of a Whitney sphere, Lemma 2 implies that  $\psi(M)$  must be a portion of a corresponding contact Whitney sphere.

The converse can be proved by straightforward computation.  $\square$

### 3 Anti-invariant Submanifolds.

We now consider an  $(n + 1)$ -dimensional anti-invariant submanifold  $M^{n+1}$  of  $\mathbb{R}^{2n+1}$ ; such a submanifold is tangent to the structure vector field  $\xi$ . In particular, from the Sasakian condition on  $\mathbb{R}^{2n+1}$ ,  $\tilde{\nabla}_X \xi = -\phi X$ , and therefore

$$\sigma(X, \xi) = -\phi X, \quad (12)$$

for any vector field  $X$  tangent to  $M$ .

In this case, it is possible to state an inequality similar to (5):

**Proposition 3.** *Let  $M^{n+1}$  be an  $(n + 1)$ -dimensional anti-invariant submanifold of  $\mathbb{R}^{2n+1}$ . Then, the square of its mean curvature  $|H|^2$  and its scalar curvature  $\tau$  satisfy at each point the following inequality:*

$$|H|^2 \geq \frac{2(n+2)}{(n+1)^2(n-1)}\tau. \quad (13)$$

The equality holds if and only if there exists a real function  $\lambda$  defined on  $M$ , such that the second fundamental form  $\sigma$  of  $M$  satisfies

$$\begin{aligned} \sigma(e_1, e_1) &= 3\lambda e_{1*}, & \sigma(e_2, e_2) &= \dots = \sigma(e_n, e_n) = \lambda e_{1*}, \\ \sigma(e_1, e_j) &= \lambda e_{j*}, & \sigma(e_j, e_k) &= 0, \quad 2 \leq j \neq k \leq n, \end{aligned}$$

where  $\{e_1, \dots, e_n, \xi, e_{1*}, \dots, e_{n*}\}$  is a local orthonormal frame such that  $e_1, \dots, e_n, \xi$  are tangent to  $M$ ,  $e_{i*} = \phi e_i$ , for any  $i = 1, \dots, n$ , and  $e_{1*}$  is parallel to  $H$ .

We can also describe the second fundamental form of the submanifolds satisfying the equality case of (13) at every point:

**Proposition 4.** *Let  $M^{n+1}$  be an  $(n + 1)$ -dimensional anti-invariant submanifold of  $\mathbb{R}^{2n+1}$ . Then,  $M$  satisfies the equality case of (13) at every point, if and only if*

$$\begin{aligned} \sigma(X, Y) &= \frac{n+1}{n+2} \{(g(X, Y) - \eta(X)\eta(Y))H + \\ &+ (g(\phi X, H) - \frac{n+2}{n+1}\eta(X))\phi Y + (g(\phi Y, H) - \frac{n+2}{n+1}\eta(Y))\phi X\}, \end{aligned} \quad (14)$$

for any tangent vector fields  $X$  and  $Y$ .

A submanifold  $M$  of  $\mathbb{R}^{2n+1}$ , tangent to  $\xi$ , is said to be *totally contact geodesic* [1, p.110] if

$$\sigma(X, Y) = \eta(X)\sigma(Y, \xi) + \eta(Y)\sigma(X, \xi), \quad (15)$$

for any tangent vector fields  $X$  and  $Y$ . It is clear that every totally contact geodesic submanifold is minimal. Then, it follows from (12), (14) and (15) that

every anti-invariant totally contact geodesic submanifold satisfies the equality case of (13). With an additional condition, we can characterize the non-trivial anti-invariant submanifolds satisfying that equality.

Let  $\pi : \mathbb{R}^{2n+1} \rightarrow \mathbb{C}^n$  be the Riemannian submersion given by (8), and suppose that there exists a Lagrangian submanifold  $N$  of  $\mathbb{C}^n$ , such that the following diagram commutes

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{R}^{2n+1} \\ \downarrow & & \downarrow \pi \\ N & \longrightarrow & \mathbb{C}^n, \end{array}$$

where  $M$  is the set of fibres over  $N$ . Now, we state the following theorem:

**Theorem 2.** *Under the above conditions, the equality case of (13) holds for any  $p \in M$ , if and only if either  $M$  is a totally contact geodesic submanifold or is locally isometric to the Riemannian product of a portion of a Whitney sphere and  $\mathbb{R}$ .*

PROOF. As in the proof of Theorem 1, it follows from the O'Neill equations of the Riemannian submersion  $\pi$  that

$$\sigma(X^*, Y^*) = (\hat{\sigma}(X, Y))^* \text{ and } \hat{H}^* = \frac{n+1}{n}H, \quad (16)$$

where  $\hat{\sigma}$  and  $\hat{H}$  are the second fundamental form and the mean curvature vector of  $N$ , respectively. On the other hand, we also know that

$$\eta(\nabla_{X^*} Y^*) = -g(\nabla_{X^*} \xi, Y^*) = g(\phi X^*, Y^*) = 0,$$

since  $M$  is anti-invariant. This implies that  $M$  is locally isometric to the Riemannian product of  $N$  and  $\mathbb{R}$ .

Suppose now that the equality case of (13) holds for any  $p \in M$ . Then, Proposition 4 implies that  $\sigma$  satisfies (14). Hence, it follows from (11) that  $\hat{\sigma}$  satisfies the equation of [6, Theorem 2], and so, either  $N$  is totally geodesic or is a portion of a Whitney sphere.

Finally, it only remains to remark that, if  $N$  is totally geodesic, then, we obtain from (16) that  $\sigma(X, Y) = 0$ , for any tangent vector fields  $X$  and  $Y$ , perpendicular to  $\xi$ , and therefore,  $M$  is totally contact geodesic.

The converse can be proved by straightforward computation.  $\square$

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