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Separated and almost periodic solutions for impulsive differential equations

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Abstract. The present paper investigates the existence of almost periodic solutions of systems of impulsive differential equations with variable impulsive perturbations. The investigations are carried out by means of the notion for separated solutions.

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1 Preliminary notes and definitions

It is characteristic for the evolution of many real processes that at certain moments they change their state by jumps. For instance, if the population of a given species is regulated by some impulsive factors at certain moments, then we have no reasons to expect that the process will be simulated by regular control. On the contrary, the solutions must have jumps at these moments and the jumps are given beforehand. During the last few years impulsive systems of differential equations have been an object of numerous investigations [1-7,9] related to the applications of these system to physics, biology , chemistry, control theory, etc..

In the present paper by means of the concept of separated solutions, some sufficient conditions for the existence of almost periodic solution of systems of impulsive differential equations with variable impulsive perturbations are found.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $|.|$ and scalar product $\langle .,. \rangle$, $\mathbb{R} = \{-\infty, \infty\}$, $\mathbb{Z}$ be set of integers.

By B, $B = \{\{t_i\}_{i=-\infty}^{\infty} : \ t_i \in \mathbb{R}, \ t_i < t_{i+1}, \ i \in \mathbb{Z}\}$ we denote the family of countable sets of all sequences unbounded and strictly increasing with distance $\rho(\{t_i\}^{(1)}, \{t_i\}^{(2)}) = \inf_{\varepsilon > 0}\{|t_i^{(1)} - t_i^{(2)}| < \varepsilon, \ i \in \mathbb{Z}\}$.

We denote by $PC^\kappa = PC^\kappa(J, \mathbb{R}^n)$, where $J \subset \mathbb{R}$, $\kappa = 1, 2, \ldots$ the space of all piecewise continuous functions $x : J \to \mathbb{R}^n$ such that:

(1) The set $T \subset B$, $T = \{t_i \in J, \ i \in \mathbb{Z}\}$ is from all points of discontinuity of x.

(2) For any $t_i \in T$, $x(t_i - 0) = x(t_i)$, and the limit $\lim\limits_{t \to t_i+0} x(t) = x(t_i + 0)$ is finite.

(3) x is a C^κ continuous function in $J \setminus T$.

Let Ω be a domain in $\mathbb{R}^n$, $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $I_i : \Omega \to \mathbb{R}^n$, $i \in \mathbb{Z}$, $\tau_i : \Omega \to \mathbb{R}$.

Introduce the following assumptions:

H1. $f \in C^1(\mathbb{R} \times \Omega, \mathbb{R}^n)$.

H2. $I_i \in C^1(\Omega, \mathbb{R}^n)$, $i \in \mathbb{Z}$.

H3. If $x \in \Omega$, then $x + I_i(x) \in \Omega$, $L_i(x) = x + I_i(x)$ are invertible in Ω, and $(L_i(x))^{-1} \in \Omega$ for $i \in \mathbb{Z}$.

H4. $\tau_i(x) \in C^1(\Omega, \mathbb{R})$ and $\lim\limits_{i \to \infty} \tau_i(x) = \infty$ uniformly on $x \in \Omega$.

H5. The following inequalities hold:

$$\begin{aligned}
&\sup\{\|f(t,x)\| \,:\, (t,x) \in \mathbb{R} \times \Omega\} \le A < \infty,\\
&\sup\Big\{\|\tfrac{\partial \tau_i(x)}{\partial x}\| \,:\, x \in \Omega,\ i = \pm 1, \pm 2, \dots\Big\} \le B < \infty,\ AB < 1,\\
&\sup\Big\{\langle \tfrac{\partial \tau_i(x + sI_i(x))}{\partial x}, I_i(x)\rangle \,:\, s \in [0,1], x \in \Omega,\ i \in \mathbb{Z}\Big\} \le 0.
\end{aligned}$$

Let conditions H1-H5 are satisfied. Consider the system of impulsive differential equations

$$\dot{x} = f(t,x),\ t \ne \tau_i(x),\ i \in \mathbb{Z}, \tag{1}$$

$$\Delta x = I_i(x),\ t = \tau_i(x),\ i \in \mathbb{Z}. \tag{2}$$

We recall [7] that for any point $(t_0, x_0) \in \mathbb{R} \times \Omega$ which is not in hypersurface $\xi_i = \{(t,x) \in \mathbb{R} \times \Omega \,:\, t = \tau_i(x)\}$, the function $x(t) = x(t; t_0, x_0)$ is called a solution of the system (1), (2) with initial condition $x(t_0 + 0) = x_0$, if:

1. $x(t; t_0, x_0) \in PC^1(J, \mathbb{R}^n)$ and for any $t_i \in J$,

$$x(t_i + 0; t_0, x_0) = x(t_i; t_0, x_0) + I_i(x(t_i; t_0, x_0)).$$

2. $x(t)$ satisfies the equation (1) in $J \setminus T$.

We also note too that from H5 it follows that the phenomenon "beating" is absent for the system (1), (2) i.e. the integral orbit of any solution of the system (1), (2) meets each hypersurface ξ_i, $i \in \mathbb{Z}$ at most once [8].

Definition 1 (10). The function $f(t,x)$, $f \in C(\mathbb{R} \times \Omega, \mathbb{R}^n)$ is said to be *almost periodic in t* uniformly with respect to $x \in \Omega$ if and only if for every sequence of real numbers $\{\alpha'_m\}$ the sequence $f(t + \alpha'_m, x)$ has a subsequence $f(t + \alpha_n, x)$, $\alpha_n = \alpha'_{m_n}$ which converges uniformly with respect to $t \in \mathbb{R}$, $x \in \Omega$.

Definition 2. The sequences $\{I_i(x)\},\ I_i \in C(\Omega, \mathbb{R}^n),\ i \in \mathbb{Z}$ is said to be *almost periodic* uniformly with respect to $x \in \Omega$ if and only if for every sequence of integer numbers s'_m there exists subsequence $s_k,\ s_k = s'_{m_k}$ such that the sequence $I_{i+s_k}(x)$ converges uniformly for $k \to \infty$ with respect to $x \in \Omega$.

Let $D \subset \mathbb{R}$. Introduce the sets $\theta_\varepsilon = \{t + \varepsilon,\ t \in D,\ \varepsilon \in \mathbb{R}\}$, $F_\varepsilon(D) = \cap\{\theta_\varepsilon(D),\ \varepsilon > 0\}$ and let for $T,\ P \in B$, $s(T \cup P) : B \to B$ be a map such that the set $s(T \cup P)$ form strictly increasing sequence. By $\phi = (\varphi(t), T)$ we denote the element from the space $PC \times B$, and for every sequence of real number $\{\alpha_n\},\ n = 1, 2, \ldots$ with $\theta_{\alpha_n}\phi$ denote the sets $\{\varphi(t + \alpha_n), T + \alpha_n\} \subset PC \times B$, where $T + \alpha_n = \{t_i + \alpha_n,\ i \in \mathbb{Z},\ n = 1, 2, \ldots\}$.

Definition 3. The sets of sequences $\{t_i^j\},\ t_i^j = t_{i+j} - t_i,\ i \in \mathbb{Z},\ j \in \mathbb{Z}$ is said to be *uniformly almost periodic* if for any $\varepsilon > 0$ there exists a relatively dance set in R of ε-almost periods common for all the sequence $\{t_i^j\}$.

Lemma 1 (8). *The set of sequences $\{t_i^j\}$ is uniformly almost periodic, if and only if from each infinite sequences of shifts $\{t_i + h_n\},\ i \in \mathbb{Z},\ n = 1, 2, \ldots$ we can choose a subsequence, converges in B.*

Definition 4. The sequence $\{\phi_n\}$, $\phi_n = (\varphi_n(t), T_n) \in PC \times B$ converges to ϕ, $\phi = (\varphi(t), T),\ (\varphi(t), T) \in PC \times B$ if and only if for any $\varepsilon > 0$ there exists $n_0 > 0$ such that for $n \geq n_0$,

$$\rho(T, T_n) < \varepsilon, |\ \varphi_n(t) - \varphi(t)| < \varepsilon$$

holds uniformly on $t \in \mathbb{R} \setminus F_\varepsilon(s(T_n \cup T))$.

Definition 5. The function $\varphi \in PC(\mathbb{R}, \Omega)$ is said to be an *almost periodic piecewisecontinuousfunction* with points of discontinuity of the first kind from the set T if for every sequence of real numbers $\{\alpha'_m\}$ it follows that there exists a subsequence $\{\alpha_n\}$, $\alpha_n = \alpha'_{m_n}$ such that $\theta_{\alpha_n}(\phi)$ is compact in $PC \times B$.

Introduce the following further assumptions:
H6. The function $f(t, x)$ is almost periodic in t uniformly with respect to $x \in \Omega$, Ω is compact subset of $\mathbb{R}^n$.
H7. The sequences $\{I_i(\Omega)\}$ and $\{\tau_i(x)\}$ are almost periodic uniformly with respect to $x \in \Omega$, Ω is subset of $\mathbb{R}^n$.
H8. The set of sequences $\left\{\tau_i^j\right\}$, $\tau_i^j = \tau_{i+j} - \tau_i$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}$ is uniformly almost periodic.

Let the conditions H1-H8 hold and let $\{\alpha_m'\}$ be an arbitrary sequence of real numbers. Then there exist a subsequence $\{\alpha_n\}$, $\alpha_n = \alpha_{m_n}'$ such that the limit $\lim_{n\to\infty}\{f(t + \alpha_n, x)\} = f^\alpha(t, x)$ exists uniformly on $(t, x) \in \mathbb{R} \times \Omega$, and the limit $\lim_{n\to\infty}\{\tau_i(x) - \alpha_n\} = \tau_i^\alpha(x)$ exists uniformly on $i \in \mathbb{Z}$, $x \in \Omega$.

By $\{i_{n_k}\}$ we denote the sequence of integer numbers such that the subsequence $\{\tau_{i_{n_k}}(x)\}$ converges to $\tau_i^\alpha(x)$ uniformly as $k \to \infty$ with respect to i.

From H7 it follows that there exists a subsequence of the sequence $\{i_{n_k}\}$ such that the sequence $\{I_{i_{n_k}}(x)\}$ convergent uniformly to the limit denoted by $I_i^{\alpha}(x)$.

Then for every sequence $\{\alpha_m{}'\}$, the system (1), (2) is moving to the system

$$\dot{x} = f^{\alpha}(t,x),\ t \neq \tau_i^{\alpha}(x),\ i \in \mathbb{Z}, \tag{3}$$

$$\Delta x = I_i^{\alpha}(x),\ t = \tau_i^{\alpha}(x),\ i \in \mathbb{Z}. \tag{4}$$

Definition 6. The set of all systems in the form (3), (4) is said to be *module* of the system (1), (2).

Further we denote this module by $\mathrm{mod}(f, I_i, \tau_i)$.

We introduce the following operator notation. If $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is a subsequence of the sequence $\alpha' = \{\alpha_n\}_{n=0}^{\infty}$ we write $\alpha \subset \alpha'$. Also $\alpha + \beta$ will denote $\{\alpha_n + \beta_n\}_{n=0}^{\infty}$. By $\alpha > 0$ we mean $\alpha_n > 0$ for each n. If $\alpha \subset \alpha'$ and $\beta \subset \beta'$, then α and β are said to have matching subscripts if $\alpha = \{\alpha_{n_k}{}'\}$ and $\beta = \left\{\beta_{n_k}{}'\right\}$.

We denote by $S_{\alpha+\beta}\phi$ and $S_\alpha S_\beta \phi$ the limits

$$\lim_{n\to\infty} \theta_{\alpha_n+\beta_n}(\phi), \qquad \lim_{n\to\infty} \theta_{\alpha_n}(\lim_{m\to\infty} \theta_{\beta_m}\phi).$$

Lemma 2. *The function $\varphi(t)$ is almost periodic if and only if from every pair of sequences α', β' one can extract some common subsequences $\alpha \subset \alpha'$, $\beta \subset \beta'$ such that we have*

$$S_{\alpha+\beta}\phi = S_\alpha S_\beta \phi, \tag{5}$$

pointwise.

Proof. Let (5) exist pointwise and let γ' be a sequence such that $\gamma \subset \gamma'$ and $S_\gamma \phi$ exist. If $S_\gamma \phi$ is uniform, we have complete the proof. We can find $\varepsilon > 0$ and sequences $\beta \subset \gamma$, $\beta' \subset \gamma$ such that

$$\rho(T_n^{\beta}, T_n^{\beta'}) < \varepsilon,$$

but

$$\sup_{t \in \mathbb{R} \setminus F_\varepsilon(s(T_n^{\beta} \cup T_n^{\beta'}))} |\varphi(t+\beta_n) - \varphi(t+\beta'{}_n)| \geq \varepsilon > 0,$$

where T_n^{β}, $T_n^{\beta'}$ we denote the points of discontinuity of the functions $\varphi(t+\beta_n)$, $\varphi(t+\beta'_n)$ $n = 1, 2, \ldots$.

Using the intermediate value theorem for the common intervals of continuity of the function $\varphi(t+\beta_n)$, and $\varphi(t+\beta'_n)$ and fact that

$$\lim_{n\to\infty} |\varphi(\beta_n) - \varphi(\beta'_n)| = 0,$$

we may assume that there is a sequence α such that

$$\sup_{t\in\mathbb{R}\setminus F_\varepsilon(s(T_n^\beta\cup T_n^{\beta'}))}|\varphi(\alpha_n+\beta_n)-\varphi(\alpha_n+\beta'_n)|\geq\varepsilon>0. \tag{6}$$

Let for α there exist a common subsequences $\alpha_1\subset\alpha$, $\beta_1\subset\beta$, $\beta_2\subset\beta$ such that

$$S_{\alpha_1+\beta_1}\phi=R_1,\ S_{\alpha_1+\beta_2}\phi=R_2,$$

where $R_j=(r_j(t),P_j),\ r_j(t)\in PC,\ P_j\in B,\ j=1,2,$ exist pointwise.

Then from (5) we get

$$\begin{aligned}R_1=S_{\alpha_1+\beta_1}\phi=S_{\alpha_1}S_{\beta_1}\phi=S_{\alpha_1}S_\gamma\phi=\\ =S_{\alpha_1}S_{\beta_2}\phi=S_{\alpha_1+\beta_2}\phi=R_2,\end{aligned} \tag{7}$$

for $t\in\mathbb{R}\setminus F_\varepsilon(s(P_1\cup P_2))$.

On the other hand from (6) it follows that

$$|r_1(0)-r_2(0)|>0,$$

which is contradiction of (7).

Let $\varphi(t)$ be almost periodic. If α' and β' are given, we take some subsequences $\alpha\subset\alpha'$, $\beta\subset\beta'$ successively such that they are common subsequences and such that $S_\alpha\phi=\phi_1$, $S_\beta\phi_1=\phi_2$ and $S_{\alpha+\beta}\phi=\phi_3$, where $\phi_j=(\varphi_j,T_j)\in PC\times B$, $j=1,2,3$ exist uniformly for $t\in\mathbb{R}\setminus F_\varepsilon(s(T\cup T_1\cup T_2\cup T_3))$.

If $\varepsilon>0$ is given, then

$$|\varphi(t+\alpha_n+\beta_n)-\varphi_3(t)|<\frac{\varepsilon}{3}$$

for n large and for all $t\in\mathbb{R}\setminus F_\varepsilon(s(T_{n,n}\cup T_3))$, where $T_{n,n}$ is the set of points of discontinuity of the functions $\varphi(t+\alpha_n+\beta_n)$.

Also

$$|\varphi(t+\alpha_n+\beta_m)-\varphi_1(t+\beta_n)|<\frac{\varepsilon}{3}$$

for n, m large and for all $t\in\mathbb{R}\setminus F_\varepsilon(s(T_{n,m}\cup T_{1,n}))$, where $T_{n,m}$ is the set of points of discontinuity of the functions $\varphi(t+\alpha_n+\beta_m)$ and the set $T_{1,n}$ is formed by the points of discontinuity of the functions $\varphi_1(t+\beta_n)$.

Finally

$$|\varphi_1(t+\beta_m)-\varphi_2(t)|<\frac{\varepsilon}{3},$$

for m large and all $t\in\mathbb{R}\setminus F_\varepsilon(s(T_{1,m}\cup T_2))$, where $T_{1,m}$ is the set of points of discontinuity of the functions $\varphi_1(t+\beta_m)$.

By the triangle inequality for $n = m$ and large we have $|\varphi_2(t) - \varphi_3(t)| < \varepsilon$ for all $t \in \mathbb{R} \setminus F_\varepsilon(s(T_2 \cup T_3))$. Since ε is arbitrary we get $\varphi_2(t) = \varphi_3(t)$ for all $t \in \mathbb{R} \setminus F_\varepsilon(s(T_{1,m} \cup T_2))$, i.e. (5) holds.

The proof of Lemma 2 is complete. $\boxed{QED}$

Definition 7. The function $\varphi(t)$ is said to *satisfy the Condition G* if, given for any sequence γ' with $\lim\limits_{n\to\infty} \gamma'_n = \infty$, there exists $\gamma \subset \gamma'$ and a number $d(\gamma) > 0$ such that $S_\gamma\phi,\ \phi = (\varphi, T)$ exists pointwise and if α is a sequence with $\alpha > 0$, $\beta' \subset \gamma$ and $\beta'' \subset \gamma$ are such that $S_{\alpha+\beta'}\phi = (r_1(t), P_1),\ S_{\alpha+\beta''}\phi = (r_2(t), P_2)$, $(r_j, P_j) \in PC \times B,\ j = 1, 2$, then either $r_1(t) = r_2(t)$ or $|r_1(t) - r_2(t)| > d(\gamma)$, for $t \in \mathbb{R} \setminus F_\varepsilon(s(P_1 \cup P_2))$.

Definition 8. Let $K \subset \Omega$ is compact in $\mathbb{R}^n$. The bounded solution $x(t)$ of the system (1), (2) with point of discontinuity in the set T is said to be *separated in* K if for any other solution $x_1(t)$ in K of the same system with points of discontinuity in the set P,there exists a number $d(x_1(t)) > 0$ such that $|x(t) - x_1(t)| > d(x_1(t))$ for $t \in \mathbb{R} \setminus F_\varepsilon(s(T))$.

2 Main results

Theorem 1. *The function $\varphi(t)$ is almost periodic if and only if $\varphi(t)$ satisfies the Condition G.*

Proof. Let φ satisfy *Condition G* and let γ' be a sequence such that $\lim\limits_{n\to\infty} \gamma'_n = \infty$. There exists, by *Condition G*, $\gamma \subset \gamma'$ such that $S_\gamma\phi$ exist pointwise. If the convergence is not uniform in $\mathbb{R}$, then there exist sequences $\delta' > 0$, $\alpha' \subset \gamma$, $\beta' \subset \gamma$ and $\varepsilon > 0$ such that $|\varphi(\alpha'_n + \delta'_n) - \varphi(\beta'_n + \delta'_n)| \geq \varepsilon$, where we may pick $\varepsilon < d(\gamma)$. Since $S_\gamma(\varphi(0), T)$ exists, we have

$$|\varphi(\alpha'_n) - \varphi(\beta'_n)| < d(\gamma) \tag{8}$$

for large n.

Consequently, $k(t) = \varphi(t+\alpha'_n) - \varphi(t+\beta'_n)$ satisfies $|k(0)| < d(\gamma)$ and $|k(\delta'_n)| \geq \varepsilon$ for large n. Thus there exists δ''_n such that $\delta''_n \subset \delta'_n$ and $\varepsilon \leq |k(\delta''_n)| < d(\gamma)$.

Consider the sequences $\alpha' + \delta''$ and $\beta' + \delta''$. By *Gondition G* there exist sequences $\alpha + \delta \subset \alpha' + \delta''$ and $\beta + \delta \subset \beta' + \delta''$ with matching subscripts such that $S_{\alpha+\delta}\phi = \phi_1$, $S_{\alpha+\delta}\phi = \phi_2$, $\phi_j = (\varphi_j, T_j)$ exist pointwise, and $\varphi_1(t) = \varphi_2(t)$ or $|\varphi_1(t) - \varphi_2(t)| > 2d(\gamma)$, for $t \in \mathbb{R} \setminus F_\varepsilon(s(T_1 \cup T_2))$.

On the other hand

$$|\varphi_1(0) - \varphi_2(0)| = \lim_{n\to\infty} |\varphi(\alpha_n + \delta_n) - \varphi(\beta_n + \delta_n)|$$

and from (8) it follows that $|\varphi_1(0) - \varphi_2(0)| \leq d(\gamma)$. This contradiction shows that $S_\gamma\phi$ exists uniformly on $t \in \mathbb{R} \setminus F_\varepsilon(s(T))$.

Conversely if $\varphi(t)$ is almost periodic, let γ' be given with $\lim\limits_{n\to\infty}\gamma'_n = \infty$. There exists $\gamma \subset \gamma'$ such that $S_\gamma\phi$ exists uniformly on $t \in \mathbb{R} \setminus F_\varepsilon(s(T))$ and $S_\gamma\phi = (k(t), Q),\ (k(t), Q) \in PC \times B$. Let $\beta' \subset \gamma$, $\beta'' \subset \gamma$ and $\alpha > 0$ such that $S_{\alpha+\beta'}\phi = (r_1(t), P_1)$, $S_{\alpha+\beta''}\phi = (r_2(t), P_2)$.

From Lemma 2 it follows that there exist $\alpha' \subset \alpha$, $\overline{\beta'} \subset \beta'$, $\overline{\beta''} \subset \beta''$ such that

$$(r_1(t), P_1) = S_{\alpha'+\overline{\beta'}}(\varphi(t), T) = S_{\alpha'}S_{\overline{\beta'}}(\varphi(t), T) = S_{\alpha'}S_\gamma(\varphi(t), T) =$$

$$= S_{\alpha'}(k(t), Q) = S_\alpha(k(t), Q). \tag{9}$$

$$(r_2(t), P_2) = S_{\alpha'+\overline{\beta''}}(\varphi(t), T) = S_{\alpha'}S_{\overline{\beta''}}(\varphi(t), T) = S_{\alpha'}S_\gamma(p(t), T) =$$

$$= S_{\alpha'}(k(t), Q) = S_\alpha(k(t), Q). \tag{10}$$

From (9) and (10) we get $r_1(t) = r_2(t)$ for $t \in \mathbb{R} \setminus F_\epsilon(s(P_1 \cup P_2))$. Then $\varphi(t)$ satisfy *Condition G*.

The proof of Theorem 1 is complete. $\boxed{QED}$

Consider the system

$$\dot{x} = g(t, x),\ t \neq \sigma_i(x),\ i \in \mathbb{Z}, \tag{11}$$

$$\Delta x = G_i(x),\ t = \sigma_i(x),\ i \in \mathbb{Z}, \tag{12}$$

where $(g, G_i, \sigma_i) \in \mathrm{mod}(f, I_i, \tau_i)$.

Theorem 2. *Assume that:*

1. Conditions H1-H8 be fulfilled.

2. Every solution of the system (11), (12) in K is separated.

Then every equation in $\mathrm{mod}(f, I_i, \tau_i)$ *has only a finite number of solutions and the separated constant d may be picked to be independent of solutions.*

Proof. The fact that each equation has only a finite number solutions in K is a consequence of compactness of K and the resulting compactness of the solutions in K. But no solution can be the limit of others by the separated condition. Consequently, the number of solutions of any equations of $\mathrm{mod}(f, I_i, \tau_i)$ is finite and d may be picked as a function of equations in $\mathrm{mod}(f, I_i, \tau_i)$.

Let $(h, L_i, l_i) \in \mathrm{mod}(f, I_i, \tau_i)$ and $S_{\alpha'}(g, G_i, \sigma_i) = (h, L_i, l_i)$, with $\lim\limits_{n\to\infty}{\alpha_n}' = \infty$. If $(\varphi(t), T)$, $(\varphi_0(t), T_0)$ are two solutions in K, let $\alpha \subset \alpha'$ such that $S_\alpha(\varphi(t), T)$ and $S_\alpha(\varphi_0(t), T_0)$ exist uniformly on K and are solutions of the system

$$\dot{x} = h(t, x),\ t \neq l_i(x),\ i \in \mathbb{Z} \tag{13}$$

$$\Delta x = L_i(x),\ t = l_i(x),\ i \in \mathbb{Z}. \tag{14}$$

Then $|S_\alpha(\varphi(t), T) - S_\alpha(\varphi_0(t), T_0)| \geq d(g, G_i, \sigma_i)$.

So, if $\varphi_1, \dots, \varphi_n$ are the solutions of (11), (12) in K, then $S_\alpha(\varphi_j, T_j)$, $j = 1, 2, \dots, n$ are distinct solutions of (13), (14) in K such that

$$|S_\alpha(\varphi_j, T_j) - S_\alpha(\varphi_i, T_i)| \geq d(g, G_i, \sigma_i),\ i \neq j.$$

Hence the number of solutions of (13), (14) in K is greater than or equal to n. By a "symmetry" argument the reverse is true, hence each equation has the same number.

On the other hand $S_\alpha(\varphi_i, T_i)$ exhaust the solutions of (13), (14) in K so that $d(g, G_i, \sigma_i) \leq d(h, L_i, l_i)$. Again by simmetry, $d(h, L_i, l_i) \leq d(g, G_i, \sigma_i)$.

The proof of Theorem 2 is complete. $\boxed{QED}$

Theorem 3. *Assume that:*

1. Conditions H1-H8 be fulfilled.

2. Each equation in $\mathrm{mod}(f, I_i, \tau_i)$ *has only separated solution on* K*.*

Then:

1. If some equation in $\mathrm{mod}(f, I_i, \tau_i)$ *has a solution in* K*, then every equation in* $\mathrm{mod}(f, I_i, \tau_i)$ *has a solution in* K*.*

2. Every equation in $\mathrm{mod}(f, I_i, \tau_i)$ *has an almost periodic solution in* K*.*

Proof. The first statement has been noted above in Theorem 2. Let $\varphi(t)$ be a solution in K of (11), (12) and δ be the separation constant. Let γ' be any sequence with $\lim\limits_{n\to\infty} \gamma' = \infty$ and $\gamma \subset \gamma'$ such that $S_\gamma(g, G_i, \sigma_i) = (h, L_i, l_i)$ and $S_\gamma(\varphi(t), T)$ exist. Let $\beta' \subset \gamma$, $\beta'' \subset \gamma$ and $\alpha > 0$ be such that

$$S_{\alpha+\beta'}(\varphi(t), T) = (\varphi_1(t), T_1),$$

$$S_{\alpha+\beta''}(\varphi(t), T) = (\varphi_2(t), T_2).$$

Again take further subsequences with matching subscripts so that (without changing notation)

$$S_{\alpha+\beta'}(g, G_i, \sigma_i) = S_\alpha S_{\beta'}(g, G_i, \sigma_i) =$$
$$= S_\alpha S_\gamma(g, G_i, \sigma_i) = S_\alpha(h, L_i, l_i),$$

and similarly

$$S_{\alpha+\beta''}(g, G_i, \sigma_i) = S_\alpha(h, L_i, l_i).$$

Consequently $\varphi_1(t)$ and $\varphi_2(t)$ are solutions of the same equation and for each $\varepsilon > 0$, $\varphi_1 \equiv \varphi_2$, for $\mathbb{R} \setminus F_\varepsilon(s(T_1 \bigcup T_2))$ or $|\varphi_1(t) - \varphi_2(t)| \geq \delta = 2d$ on $\mathbb{R} \setminus F_\varepsilon(s(T_1 \bigcup T_2))$.

Consequently $\varphi(t)$ satisfies *Condition G* and from Theorem 1 it follows that $\varphi(t)$ is an almost periodic function.

Let $\varphi(t)$ be a solution of (11), (12) in K which by the above is an almost periodic function. Set $\alpha'_n = n$. Then there exists $\alpha \subset \alpha'$ such that the limits $S_\alpha(g, G_i, \sigma_i) = (h, L_i, l_i)$, $S_{-\alpha}(h, L_i, l_i) = (g, G_i, \sigma_i)$ exist uniformly and $S_\alpha(\varphi(t), T) = (r(t), P)$, $S_{-\alpha}(r(t), P)$ exist uniformly on K, where $S_{-\alpha}(r(t), P)$ is solution of (11), (12). From condition 2 of the theorem it is easy to see that $(r(t), P) = S_\alpha(\varphi(t), T)$ and thus $S_{-\alpha}(r(t), P)$ exists uniformly, and so is almost periodic.

The proof of Theorem 3 is complete. $\boxed{QED}$

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