Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **32** (2012) n. 1, 227–238. doi:10.1285/i15900932v32n1p227

Fluctuating hydrodynamics based on extended thermodynamics

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Abstract. We summarize the recent theory of fluctuating hydrodynamics based on extended thermodynamics, which is developed through the study of the 13-variable theory for a monatomic rarefied gas as a representative case. The relationship between the present theory and the Landau-Lifshitz theory is shown, and the hierarchy structure of the hydrodynamic fluctuations is discussed. The Landau-Lifshitz theory is included in the present theory as a limiting case.

Keywords: Fluctuating hydrodynamics , Extended thermodynamics , Landau-Lifshitz theory , Hierarchy structure

MSC 2000 classification: primary 82B35, secondary 76P05, 35L40, 82C31

1 Introduction

Landau and Lifshitz developed the theory of fluctuating hydrodynamics for viscous, heat-conducting fluids with constitutive equations of Navier-Stokes and Fourier type [1-3] basing on thermodynamics of irreversible processes (TIP) [4, 5]. They introduced additional stochastic flux terms (generalized random forces) into the constitutive equations of the viscous stress and the heat flux by applying the fluctuation-dissipation theorem [6-8]. See also review articles on fluctuating hydrodynamics [9-11].

In recent years, the Landau-Lifshitz (LL) theory has been applied to, in particular, nano-technology [12, 13] and molecular biology [14, 15]. Numerical analyses of the fluctuations by using the theory have been made extensively [16-22]. The fluctuating-hydrodynamic approach can also contribute to the study of fluctuations in nonequilibrium states [11, 23, 24].

[i]This work is partially supported by Japan Society of Promotion of Science (JSPS) No. 08J08281 (S. Taniguchi) and No. 20560054 (M. Sugiyama).
[ii]This work is partially supported by National Natural Science Foundation of China (NSFC) No. 20973119.

However, as TIP rests essentially on the local equilibrium assumption [4, 5] that is valid for nonequilibrium phenomena near equilibrium, it is highly probable that TIP may no longer be valid for highly nonequilibrium cases such as the cases where nanoflows are involved, or the cases where rarefied gases play a role. As for the discussion on the validity criterion of the assumption, see, for example, Ref. [25].

Extended thermodynamics (ET) [26] is a generalized theory being applicable to such cases where physical quantities undergo evident changes in a small spatio-temporal scale. ET for rarefied monatomic gases has a counterpart in the kinetic theory of gases. For example, ET of 13 variables (ET-13) coincides with the moment theory of the Boltzmann equation within the Grad's 13-moment approximation [27]. In this respect, we may say that the effect of the higher moments neglected in the theory manifests itself in the form of the fluctuations.

The purpose of the present paper is to summarize the newly developed theory of fluctuating hydrodynamics based on ET through the study of the 13-variable theory as a representative case [28]. Some new results that could not be included in the paper [28] are also shown and discussed.

2 Theory of fluctuating hydrodynamics based on ET

The basic equations in the present study are the linearized equations of ET-13 for a monatomic rarefied gas [26] around an equilibrium state. The independent variables are the mass density ρ, velocity v_i, temperature T, shear stress $t_{\langle ij\rangle}$ (angular brackets stand for the symmetric traceless part with respect to the suffixes inside), and heat flux q_i, where suffixes $i, j = 1, 2, 3$. Note that the dynamic pressure vanishes identically in this case.

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial v_i}{\partial x_i} = 0,\\
&\frac{\partial v_i}{\partial t} + \frac{a T_0}{\rho_0}\frac{\partial \rho}{\partial x_i} + a\frac{\partial T}{\partial x_i} - \frac{1}{\rho_0}\frac{\partial t_{\langle ij\rangle}}{\partial x_j} = 0,\\
&a\frac{\partial T}{\partial t} + \frac{2}{3} a T_0 \frac{\partial v_k}{\partial x_k} + \frac{2}{3\rho_0}\frac{\partial q_k}{\partial x_k} = 0,\\
&\frac{\partial t_{\langle ij\rangle}}{\partial t} - \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} - 2a\rho_0 T_0 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -s_{\langle ij\rangle},\\
&\frac{\partial q_i}{\partial t} - a T_0 \frac{\partial t_{\langle ij\rangle}}{\partial x_j} + \frac{5}{2} a^2 \rho_0 T_0 \frac{\partial T}{\partial x_i} = \frac{s_{ppi}}{2},
\end{aligned}
\tag{1}
$$

where $a \equiv k_B/m$ with k_B being the Boltzmann constant and m the mass of a molecule, and $s_{\langle ij\rangle}$ and s_{ppi} are the source terms. The quantities with and without the suffix 0 are, respectively, the quantities at the equilibrium state and the deviations from the equilibrium state. The first three equations represent, respectively, the mass, momentum and energy conservation laws, and the last two are the equations of balance type for the irreversible fluxes $t_{\langle ij\rangle}$ and q_i. Owing to the presence of the second part that have been neglected in the traditional hydrodynamic analysis, the rapidly changing (deterministic) modes can be taken into account. We may call these modes fast modes. The specific entropy production Σ is obtained as follows:

$$\Sigma = \lambda_{\langle ij\rangle} s_{\langle ij\rangle} + \lambda_{ppi} s_{qqi} \geq 0, \tag{2}$$

where $\lambda_{\langle ij\rangle}$ and λ_{ppi} are so-called Lagrange multipliers.

Within the linear constitutive equations, we have

$$s_{\langle ij\rangle} = b\lambda_{\langle ij\rangle}, \qquad s_{ppi} = c\lambda_{ppi}, \tag{3}$$

where b and c are positive phenomenological coefficients. Furthermore we can prove the following relations [26]:

$$\lambda_{\langle ij\rangle} = \frac{1}{2a\rho_0 T_0^2} t_{\langle ij\rangle}, \qquad \lambda_{ppi} = -\frac{1}{5a^2\rho_0 T_0^3} q_i. \tag{4}$$

Let us now try to introduce the random forces into ET. Following the general theory [29] and introducing the new phenomenological coefficients (relaxation times) α and β instead of the coefficients b and c above, we can arrive at the final expressions for $s_{\langle ij\rangle}$ and s_{ppi} in terms of $t_{\langle ij\rangle}$, q_i and the Gaussian white random forces $\mathfrak{r}_{\langle ij\rangle}$, $\mathfrak{s}_i$ [28]:

$$\begin{aligned} s_{\langle ij\rangle} &= \frac{1}{\alpha} t_{\langle ij\rangle} + \mathfrak{r}_{\langle ij\rangle}, \\ s_{ppi} &= -\frac{2}{\beta} q_i + \mathfrak{s}_i. \end{aligned} \tag{5}$$

The means of the random forces $\mathfrak{r}_{\langle ij\rangle}$ and $\mathfrak{s}_i$ vanish. And their correlations are given by

$$\begin{aligned} \langle \mathfrak{r}_{\langle ij\rangle}(\boldsymbol{x},t)\mathfrak{r}_{\langle mn\rangle}(\boldsymbol{x}',t')\rangle &= k_B \frac{2a\rho_0 T_0^2}{\alpha} \\ &\quad \times(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} - \frac{2}{3}\delta_{ij}\delta_{mn})\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \\ \langle \mathfrak{s}_i(\boldsymbol{x},t)\mathfrak{s}_j(\boldsymbol{x}',t')\rangle &= k_B \frac{20a^2\rho_0 T_0^3}{\beta}\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \\ \langle \mathfrak{r}_{\langle ij\rangle}(\boldsymbol{x},t)\mathfrak{s}_m(\boldsymbol{x}',t')\rangle &= 0, \end{aligned} \tag{6}$$

where the brackets $\langle \ \rangle$ in the left-hand side stand for the statistical average at the reference equilibrium state.

Equations (1) with (5) and (6) constitute ***the basic system of equations for fluctuating hydrodynamics based on ET (ET-13)***.

The relaxation times α and β can be evaluated by experiments or kinetic-theoretical analyses. For gases with Maxwellian interatomic potential, in particular, we have the relation $3\alpha = 2\beta$ [26]. Other realistic monatomic gases satisfy this relation approximately.

3 Two subsystems of the stochastic field equations

The basic system of equations obtained above may be decomposed into two uncoupled subsystems, that is, the subsystem composed of longitudinal modes (System-L) and the subsystem of transverse modes (System-T).

System-L

The relevant quantities of the system are given by

$$\begin{gathered}\rho, \quad T, \quad \psi\left(\equiv \frac{\partial v_i}{\partial x_i}\right), \quad \tau\left(\equiv \frac{\partial^2 t_{\langle ij\rangle}}{\partial x_i \partial x_j}\right), \quad \varphi\left(\equiv \frac{\partial q_i}{\partial x_i}\right), \\ \mathfrak{v}\left(\equiv -\frac{\partial^2 \mathfrak{r}_{\langle ij\rangle}}{\partial x_i \partial x_j}\right), \quad \text{and} \quad \mathfrak{w}\left(\equiv \frac{1}{2}\frac{\partial \mathfrak{s}_i}{\partial x_i}\right).\end{gathered} \tag{7}$$

The spatial Fourier transform of the system is the system of the rate-type differential equations in the space of the wave number $\boldsymbol{k}$ and time t ($\boldsymbol{k}t$-representation) as follows:

$$\begin{aligned}&\frac{\partial \rho(\boldsymbol{k},t)}{\partial t} + \rho_0 \psi(\boldsymbol{k},t) = 0, \\ &\frac{\partial \psi(\boldsymbol{k},t)}{\partial t} - \frac{aT_0k^2}{\rho_0}\rho(\boldsymbol{k},t) - ak^2T(\boldsymbol{k},t) - \frac{1}{\rho_0}\tau(\boldsymbol{k},t) = 0, \\ &a\frac{\partial T(\boldsymbol{k},t)}{\partial t} + \frac{2}{3}aT_0\psi(\boldsymbol{k},t) + \frac{2}{3\rho_0}\varphi(\boldsymbol{k},t) = 0, \\ &\frac{\partial \tau(\boldsymbol{k},t)}{\partial t} + \frac{8}{15}k^2\varphi(\boldsymbol{k},t) + \frac{4}{3}a\rho_0T_0k^2\psi(\boldsymbol{k},t) = -\frac{1}{\alpha}\tau(\boldsymbol{k},t) + \mathfrak{v}(\boldsymbol{k},t), \\ &\frac{\partial \varphi(\boldsymbol{k},t)}{\partial t} - aT_0\tau(\boldsymbol{k},t) - \frac{5}{2}a^2\rho_0T_0k^2T(\boldsymbol{k},t) = -\frac{1}{\beta}\varphi(\boldsymbol{k},t) + \mathfrak{w}(\boldsymbol{k},t),\end{aligned} \tag{8}$$

where $\rho(\boldsymbol{k},t)$ is the spatial Fourier transform of $\rho(\boldsymbol{x},t)$ defined as

$$\rho(\boldsymbol{k},t) \equiv \frac{1}{(2\pi)^3}\int \rho(\boldsymbol{x},t)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{9}$$

and the other quantities are similarly defined.

From Eq. (6), the quantities $\mathfrak{v}(\boldsymbol{k},t)$ and $\mathfrak{w}(\boldsymbol{k},t)$ are the Gaussian white random forces with null means and correlations:

$$\begin{aligned}
&\langle \mathfrak{v}(\boldsymbol{k},t)\mathfrak{v}(\boldsymbol{k}',t')\rangle = k_B \frac{a\rho_0 T_0^2}{3\pi^3\alpha} k^4 \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
&\langle \mathfrak{w}(\boldsymbol{k},t)\mathfrak{w}(\boldsymbol{k}',t')\rangle = k_B \frac{5a^2\rho_0 T_0^3}{8\pi^3\beta} k^2 \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
&\langle \mathfrak{v}(\boldsymbol{k},t)\mathfrak{w}(\boldsymbol{k}',t')\rangle = 0.
\end{aligned} \tag{10}$$

System-T

The relevant quantities of the system are given by

$$\begin{aligned}
&\omega_i \left(\equiv (\mathrm{curl}\boldsymbol{v})_i\right), \quad \sigma_i \left(\equiv \epsilon_{ijk}\frac{\partial^2 t_{\langle kn\rangle}}{\partial x_j \partial x_n}\right), \quad \pi_i(\equiv (\mathrm{curl}\boldsymbol{q})_i),\\
&\mathfrak{x}_i \left(\equiv -\epsilon_{ijk}\frac{\partial^2 \mathfrak{r}_{\langle kn\rangle}}{\partial x_j \partial x_n}\right), \quad \text{and} \quad \mathfrak{y}_i \left(\equiv \frac{1}{2}(\mathrm{curl}\mathfrak{s})_i\right).
\end{aligned} \tag{11}$$

The field equations in the $\boldsymbol{k}t$-representation are as follows:

$$\begin{aligned}
&\frac{\partial \omega_i(\boldsymbol{k},t)}{\partial t} - \frac{1}{\rho_0}\sigma_i(\boldsymbol{k},t) = 0,\\
&\frac{\partial \sigma_i(\boldsymbol{k},t)}{\partial t} + \frac{2}{5}k^2\pi_i(\boldsymbol{k},t) + a\rho_0 T_0 k^2 \omega_i(\boldsymbol{k},t) = -\frac{1}{\alpha}\sigma_i(\boldsymbol{k},t) + \mathfrak{x}_i(\boldsymbol{k},t),\\
&\frac{\partial \pi_i(\boldsymbol{k},t)}{\partial t} - aT_0\sigma_i(\boldsymbol{k},t) = -\frac{1}{\beta}\pi_i(\boldsymbol{k},t) + \mathfrak{y}_i(\boldsymbol{k},t).
\end{aligned} \tag{12}$$

Note that, for given $\mathfrak{x}_i$ and $\mathfrak{y}_i$, the equations for the set of variables $(\omega_i, \sigma_i, \pi_i)$ with the same suffix i can be solved separately from those with the different suffix $j(\neq i)$. In view of Eq. (6), $\mathfrak{x}_i$ and $\mathfrak{y}_i$ are the Gaussian white random forces with null means and correlations:

$$\begin{aligned}
\langle \mathfrak{x}_i(\boldsymbol{k},t)\mathfrak{x}_m(\boldsymbol{k}',t')\rangle &= k_B \frac{a\rho_0 T_0^2}{4\pi^3\alpha} k^4 \left(\delta_{im} - \frac{k_i k_m}{k^2}\right)\\
&\qquad \times \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\langle \mathfrak{y}_i(\boldsymbol{k},t)\mathfrak{y}_m(\boldsymbol{k}',t')\rangle &= k_B \frac{5a^2\rho_0 T_0^3}{8\pi^3\beta} k^2 \left(\delta_{im} - \frac{k_i k_m}{k^2}\right)\\
&\qquad \times \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\langle \mathfrak{x}_i(\boldsymbol{k},t)\mathfrak{y}_m(\boldsymbol{k}',t')\rangle &= 0.
\end{aligned} \tag{13}$$

4 Elimination of the fast modes

Let us express the shear stress and the heat flux in terms of the other quantities so as to eliminate the fast modes in the basic system of equations.

We solve the last two equations of (8) and (12) with respect to (τ, φ) and (σ_i, π_i), respectively, assuming, for the moment, that the other 3 variables in the case of Eq. (8) (or 1 variable in the case of Eq. (12)) are some given functions of $\boldsymbol{k}$ and t.

The solutions can be expressed in a generic way because the last two equations of both the systems can be written in the following matrix form:

$$\frac{d\boldsymbol{y}(\boldsymbol{k},t)}{dt} + \boldsymbol{M}(\boldsymbol{k}) \cdot \boldsymbol{y}(\boldsymbol{k},t) = \boldsymbol{d}(\boldsymbol{k},t) + \mathfrak{a}(\boldsymbol{k},t), \tag{14}$$

where $\boldsymbol{y}(\boldsymbol{k},t)$, $\boldsymbol{M}(\boldsymbol{k})$, $\boldsymbol{d}(\boldsymbol{k},t)$ and $\mathfrak{a}(\boldsymbol{k},t)$ are given explicitly in Eq. (25) or (28) below. The quantity $\mathfrak{a}(\boldsymbol{k},t)$ is a Gaussian white random force vector with two components. It is important to notice that the matrix $\boldsymbol{M}(\boldsymbol{k})$ satisfies the relation: $\boldsymbol{M}(-\boldsymbol{k}) = \boldsymbol{M}(\boldsymbol{k})$.

The solution of (14) is given by

$$\begin{aligned}\boldsymbol{y}(\boldsymbol{k},t) = &\exp[-(t-t_0)\boldsymbol{M}(\boldsymbol{k})] \cdot \boldsymbol{y}(\boldsymbol{k},t_0)\\ &+ \int_{t_0}^{t} \mathrm{d}\theta \exp[-(t-\theta)\boldsymbol{M}(\boldsymbol{k})] \cdot \boldsymbol{d}(\boldsymbol{k},\theta)\\ &+ \int_{t_0}^{t} \mathrm{d}\theta \exp[-(t-\theta)\boldsymbol{M}(\boldsymbol{k})] \cdot \mathfrak{a}(\boldsymbol{k},\theta),\end{aligned} \tag{15}$$

where t_0 is an initial time. The third term on the right-hand side may be regarded as a Gaussian random force, which is denoted by $\mathfrak{b}$, i.e.,

$$\mathfrak{b}(\boldsymbol{k},t) \equiv \int_{t_0}^{t} \mathrm{d}\theta \exp[-(t-\theta)\boldsymbol{M}(\boldsymbol{k})] \cdot \mathfrak{a}(\boldsymbol{k},\theta). \tag{16}$$

The random force $\mathfrak{b}$ is, in general, not white.

The correlation matrix $\hat{\boldsymbol{\chi}}(\boldsymbol{k}_2,\boldsymbol{k}_1,t_2,t_1)$ ($t_2 > t_1$) of the random force $\mathfrak{b}$ is given by

$$\begin{aligned}&\hat{\chi}_{ij}(\boldsymbol{k}_2,\boldsymbol{k}_1,t_2,t_1)\\ &\equiv \langle \mathfrak{b}_i(\boldsymbol{k}_2,t_2)\mathfrak{b}_j(\boldsymbol{k}_1,t_1)\rangle\\ &= \int_{t_0}^{t_2} \mathrm{d}\theta_2 \int_{t_0}^{t_1} \mathrm{d}\theta_1 \left(\exp[-(t_2-\theta_2)\boldsymbol{M}(\boldsymbol{k}_2)]\right)_{il} \langle \mathfrak{a}_l(\boldsymbol{k}_2,\theta_2)\mathfrak{a}_k(\boldsymbol{k}_1,\theta_1)\rangle\\ &\qquad\qquad \times \left(\exp[-(t_1-\theta_1)\boldsymbol{M}(\boldsymbol{k}_1)]\right)_{jk},\end{aligned} \tag{17}$$

where the correlation matrix of the random force $\mathfrak{a}$ is given in the following form:

$$\langle \mathfrak{a}_l(\boldsymbol{k}_2,\theta_2)\mathfrak{a}_k(\boldsymbol{k}_1,\theta_1)\rangle = C_{lk}(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2+\boldsymbol{k}_1)\delta(\theta_2-\theta_1) \tag{18}$$

with a matrix $\boldsymbol{C}$. See the relations (10) and (13).

After some calculations, we may summarize the solution compactly as follows [30]: The solution $\boldsymbol{y}$ is expressed by

$$\boldsymbol{y}(\boldsymbol{k},t) = \int_{-\infty}^{t} \mathrm{d}\theta \boldsymbol{\Phi}(\boldsymbol{k},t-\theta) \cdot \boldsymbol{d}(\boldsymbol{k},\theta) + \mathfrak{b}(\boldsymbol{k},t) \tag{19}$$

with the memory function given by

$$\boldsymbol{\Phi}(\boldsymbol{k},t) = \exp[-t\boldsymbol{M}(\boldsymbol{k})]. \tag{20}$$

Here we have neglected the transient effect that depends on an initial condition by taking the limit: $t_0 \to -\infty$. The mean of the random force $\mathfrak{b}$ vanishes. And its correlation matrix is expressed by

$$\boldsymbol{\chi}(\boldsymbol{k}_1, t_2 - t_1) = \boldsymbol{\Phi}(\boldsymbol{k}_1, t_2 - t_1)\boldsymbol{\chi}^0(\boldsymbol{k}_1), \tag{21}$$

where the quantity $\boldsymbol{\chi}$ is introduced by the relation:

$$\hat{\boldsymbol{\chi}}(\boldsymbol{k}_2, \boldsymbol{k}_1, t_2, t_1) = \boldsymbol{\chi}(\boldsymbol{k}_1, t_2 - t_1)\delta(\boldsymbol{k}_2 + \boldsymbol{k}_1), \tag{22}$$

and

$$\boldsymbol{\chi}^0(\boldsymbol{k}_1) = \boldsymbol{\chi}(\boldsymbol{k}_1, 0). \tag{23}$$

Finally the relation between $\boldsymbol{\chi}^0$ and $\boldsymbol{C}$ is given by

$$\boldsymbol{M}(\boldsymbol{k})\boldsymbol{\chi}^0(\boldsymbol{k}) + \boldsymbol{\chi}^0(\boldsymbol{k})\boldsymbol{M}(\boldsymbol{k})^T = \boldsymbol{C}(\boldsymbol{k}). \tag{24}$$

We summarize some noticeable points:

(i) We may regard the relation (19) as a constitutive relation between $\boldsymbol{y}$ and $\boldsymbol{d}$ with the random force $\mathfrak{b}$. The quantity $\boldsymbol{y}$ is the functional of the history of the quantity $\boldsymbol{d}$. It should be emphasized that the constitutive relation has been obtained from the dynamic equations of the fast modes by neglecting the dependence on the initial condition. Such constitutive relations have been proposed in the theory of so-called *generalized hydrodynamics* where the transport coefficients, in general, take into account the effect of non-locality in space and time.

(ii) As we pointed out above, we notice clearly, from Eq. (21), that the random force $\mathfrak{b}$ is the Gaussian non-white (or colored) random force. And, in accordance with the general considerations [8, 34] on such a case, the memory function $\boldsymbol{\Phi}$ defined in Eq. (20) plays an essential role in both Eq. (19) for the constitutive equation and Eq. (21) for the correlation of the random force. In other words, if we adopt some approximation in the form of the memory function, both the constitutive equation and the correlation of the random force are affected by the approximation simultaneously in order to keep the consistency in the theory. See also the remark (iv) below.

(iii) The relation (24) is the one that connects the random force $\mathfrak{b}$ and the random force $\mathfrak{a}$. This is the key relation in the following analysis of the hierarchy structure of the random forces in different levels of description in fluctuating hydrodynamics.

(iv) In comparison with the characteristic relaxation times for the conserved quantities such as mass, momentum and energy, the fast modes have much smaller relaxation times, and decay quickly. If we describe hydrodynamic phenomena in such a way that the relaxation times α and β for the fast modes are sufficiently small, the memory function can be well approximated by the Dirac's delta function with a suitable proportional constant. See Refs. [28, 31, 32, 33] for such a coarse graining approximation. At the same time, the random force $\mathfrak{b}$ in this approximation becomes to be a white random force. This case will be studied in the next section.

5 Relationship to the Landau-Lifshitz theory

In what follows, we adopt the coarse graining approximation explained above and show explicitly the coarse-grained solutions for the System-L and System-T [28]. We will see that these solutions are just the ones in the LL theory.

System-L

The quantities in Eq. (14) are given by

$$\boldsymbol{y}(\boldsymbol{k},t) = \begin{bmatrix} \tau(\boldsymbol{k},t) \\ \varphi(\boldsymbol{k},t) \end{bmatrix}, \quad \boldsymbol{M}(\boldsymbol{k}) = \begin{bmatrix} \dfrac{1}{\alpha} & \dfrac{8}{15}k^2 \\ -aT_0 & \dfrac{1}{\beta} \end{bmatrix},$$
$$\boldsymbol{d}(\boldsymbol{k},t) = \begin{bmatrix} -\dfrac{4}{3}a\rho_0 T_0 k^2 \psi(\boldsymbol{k},t) \\ \dfrac{5}{2}a^2\rho_0 T_0 k^2 T(\boldsymbol{k},t) \end{bmatrix}, \quad \mathfrak{a}(\boldsymbol{k},t) = \begin{bmatrix} \mathfrak{v}(\boldsymbol{k},t) \\ \mathfrak{w}(\boldsymbol{k},t) \end{bmatrix}. \tag{25}$$

Denoting $\mathfrak{b} = [\mathfrak{g},\ \mathfrak{h}]^T$, we have the following relation up to the leading term with respect to α and β [28]:

$$\begin{bmatrix} \tau(\boldsymbol{k},t) \\ \varphi(\boldsymbol{k},t) \end{bmatrix} = \begin{bmatrix} -\dfrac{4}{3}a\rho_0 T_0 k^2 \alpha\psi(\boldsymbol{k},t) + \mathfrak{g}(\boldsymbol{k},t) \\ \dfrac{5}{2}a^2\rho_0 T_0 k^2 \beta T(\boldsymbol{k},t) + \mathfrak{h}(\boldsymbol{k},t) \end{bmatrix}. \tag{26}$$

The Gaussian white random forces $\mathfrak{g}$ and $\mathfrak{h}$ have null means and correlations:

$$\begin{aligned}
\left\langle \mathfrak{g}(\boldsymbol{k},t)\mathfrak{g}(\boldsymbol{k}',t')\right\rangle &= \frac{1}{3\pi^3}k_B a\rho_0 T_0^2 k^4 \alpha\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\left\langle \mathfrak{h}(\boldsymbol{k},t)\mathfrak{h}(\boldsymbol{k}',t')\right\rangle &= \frac{5}{8\pi^3}k_B a^2\rho_0 T_0^3 k^2 \beta\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\left\langle \mathfrak{g}(\boldsymbol{k},t)\mathfrak{h}(\boldsymbol{k}',t')\right\rangle &= 0.
\end{aligned} \tag{27}$$

System-T

The quantities in Eq. (14) are given by

$$\begin{aligned}
\boldsymbol{y}(\boldsymbol{k},t) &= \begin{bmatrix}\sigma_i(\boldsymbol{k},t)\\ \pi_i(\boldsymbol{k},t)\end{bmatrix}, \quad \boldsymbol{M}(\boldsymbol{k}) = \begin{bmatrix}\dfrac{1}{\alpha} & \dfrac{2}{5}k^2\\ -aT_0 & \dfrac{1}{\beta}\end{bmatrix},\\
\boldsymbol{d}(\boldsymbol{k},t) &= \begin{bmatrix}-a\rho_0 T_0 k^2\omega_i(\boldsymbol{k},t)\\ 0\end{bmatrix}, \quad \mathfrak{a}(\boldsymbol{k},t) = \begin{bmatrix}\mathfrak{x}_i(\boldsymbol{k},t)\\ \mathfrak{y}_i(\boldsymbol{k},t)\end{bmatrix}.
\end{aligned} \tag{28}$$

Denoting $\mathfrak{b} = [\mathfrak{k}_i, \mathfrak{l}_i]^T$, we obtain the following relations in a similar way as above [28]:

$$\begin{bmatrix}\sigma_i(\boldsymbol{k},t)\\ \pi_i(\boldsymbol{k},t)\end{bmatrix} = \begin{bmatrix}-a\rho_0 T_0 k^2\alpha\omega_i(\boldsymbol{k},t)+\mathfrak{k}_i(\boldsymbol{k},t)\\ \mathfrak{l}_i(\boldsymbol{k},t)\end{bmatrix}. \tag{29}$$

Note that there is no deterministic part in $\pi_i(\boldsymbol{k},t)$, therefore, only the random force plays a role. The correlations between the zero-mean Gaussian white random forces are given by

$$\begin{aligned}
\left\langle \mathfrak{k}_i(\boldsymbol{k},t)\mathfrak{k}_m(\boldsymbol{k}',t')\right\rangle &= \frac{1}{4\pi^3}k_B a\rho_0 T_0^2 k^4\alpha\left(\delta_{im}-\frac{k_i k_m}{k^2}\right)\\
&\qquad\times\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\left\langle \mathfrak{l}_i(\boldsymbol{k},t)\mathfrak{l}_m(\boldsymbol{k}',t')\right\rangle &= \frac{5}{8\pi^3}k_B a^2\rho_0 T_0^3 k^2\beta\left(\delta_{im}-\frac{k_i k_m}{k^2}\right)\\
&\qquad\times\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(t-t'),\\
\left\langle \mathfrak{k}_i(\boldsymbol{k},t)\mathfrak{l}_m(\boldsymbol{k}',t')\right\rangle &= 0.
\end{aligned} \tag{30}$$

The relationship between the present theory and the LL theory: We can now confirm that the expressions in Eqs. (26), (27), (29) and (30) are exactly the same as those derived from the LL theory where the shear viscosity μ and the heat conductivity κ are identified by the relations [26]:

$$\mu = a\rho_0 T_0\alpha, \qquad \kappa = \frac{5}{2}a^2\rho_0 T_0\beta. \tag{31}$$

Thus we have proved that the LL theory can be derived from the present theory by using the coarse graining approximation, and that the LL theory is included in the present theory as a limiting case.

The present theory and the LL theory belong to the two different levels of description of fluctuating hydrodynamics. As we analyzed above, the rapidly changing deterministic modes (fast modes) in ET have been consistently renormalized into the random forces in the LL theory. Therefore, from a physical point of view, the delta functions appeared in the correlations have their own validity range depending on the spatio-temporal resolution of their description levels.

6 Discussion and concluding remarks

In the present paper, we have summarized the recent theory of fluctuating hydrodynamics based on ET. And we have made clear the link between the two levels of description of fluctuating hydrodynamics, that is, the present theory based on ET-13 and the LL theory. This link has been established through introducing another intermediate level of description, which is characterized by the conservation equations with the memory-type constitutive equation and the colored Gaussian random force. In this way, we notice that there are at least three levels of description of fluctuating hydrodynamics in the present work. And we notice, in particular, the explicit hierarchy structure of the random forces.

Generally speaking, there are many such levels. Boillat and Ruggeri [26, 35] found the hierarchy structure of ET and the important concept called the "main subsystem" of field equations. Each main subsystem gives us one level of description with different resolution from each other. And, in a similar way as above, we can develop the corresponding fluctuating hydrodynamics basing on a given main subsystem. Detailed study of the hierarchy structure in the hydrodynamic fluctuations based on ET will be presented in the next paper.

Finally we summarize the concluding remarks:

(i) In ET, Navier-Stokes and Fourier constitutive equations are obtained as its limit case by using an iterative scheme called the Maxwellian iteration [26]. If we apply this scheme formally to the present basic system with random forces, we can also obtain the results of the LL theory.

(ii) In the present paper, we have studied a monatomic rarefied gas only. Fluctuating hydrodynamics can also be established in a similar way by using recently developed ET for a polyatomic rarefied gas and for a real gas [36] where the dynamic pressure exists. We hope that we will soon show their details.

(iii) As the basic system of equations in ET is of hyperbolic type, the propagation speed of information is finite. In this respect, ET is in sharp contrast to the traditional theory of Navier-Stokes and Fourier type that predicts infinite speeds for the propagation of heat and shear stress. It is, therefore, quite reasonable to adopt ET in order to develop, in particular, the relativistic fluctuating hydrodynamics. See the pioneering work by Calzetta [37].

(iv) Numerical analyses based on the present theory in various situations are highly expected. We can expect qualitatively different effects predicted by the present theory from those by the LL theory, especially when we study the fluctuations in a small spatio-temporal scale.

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