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A Non-Isothermal Spinning Magneto-gasdynamic Cloud System A Hamiltonian-Ermakov Integrable Reduction

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Abstract. A 2+1-dimensional magneto-gasdynamic version of a gas cloud system originating in work of Ovsiannikov and Dyson is shown, when adiabatic index $\gamma = 2$, to admit an integrable reduction to a subsystem with underlying Hamiltonian-Ermakov structure. A class of exact solutions of the original nonlinear magneto-gasdynamic system is thereby constructed.

1 Introduction

In a series of papers on 2+1-dimensional magnetogasdynamics, Neukirch *et al* [1, 2, 3] introduced a novel procedure wherein the nonlinear acceleration terms in the Lundquist momentum equation either vanish or are conservative. Here, by contrast, an elliptic vortex ansatz approach is adopted based on that originally introduced in [4] in the context of elliptic warm core eddy theory. In recent work, this procedure has been applied in [5] to a rotating shallow water system with underlying circular paraboloidal bottom topography as well as in [6] to a non-isothermal gasdynamic system with origin in work of Ovsiannikov [7] and Dyson [8] on spinning gas clouds. The procedure of [4] has also been applied to certain magneto-gasdynamic systems in [9] and [10]. In the latter work, a magneto-gasdynamic analogue of the pulsrodon of f-plane shallow water theory [4] was isolated. This corresponds to a pulsating, rotating elliptical plasma cylinder bounded by a vacuum state. It is noted that the pulsrodon of [4] and associated solutions were shown to be orbitally Lyaponov stable to perturbations within the class of elliptical vortex solutions by Holm [11]. The temporal evolution of the pulsrodon in the context of the nonlinear reduced gravity shallow water system of [4] is consistent with experimental work of Rubino and Brandt [12].

The present work concerns a non-isothermal magneto-gasdynamic version of the gas cloud system of Ovsiannikov and Dyson in [7, 8]. A nonlinear dynamical subsystem is derived which is shown to be integrable provided the adiabatic index $\gamma = 2$. The subsystem is shown to have underlying Hamiltonian Ermakov-Ray-Reid structure. A Lax pair for the system is derived in the manner described for non-conducting gas clouds in [6].

2 The magnetogasdynamic system

Here, we consider the anisentropic magnetogasdynamic system

$$\operatorname{div}\mathbf{q} = -\frac{1}{\gamma-1}\frac{d}{dt}\ \ln\ T\ , \quad \gamma \neq 1\ , \tag{1}$$

$$\frac{\partial\,\mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\,\mathbf{q} - (\,\mu/\rho\,)\ \text{curl}\ \mathbf{H}\times\mathbf{H} + f\,(\mathbf{k}\times\mathbf{q}) = T\,\nabla\,S - \nabla\left(\frac{\gamma\,T}{\gamma-1}\right)\ , \tag{2}$$

$$\operatorname{div}\mathbf{H} = 0\ , \tag{3}$$

$$\frac{\partial\mathbf{H}}{\partial t} = \text{curl}\ (\mathbf{q}\times\mathbf{H})\ , \tag{4}$$

$$\frac{\partial S}{\partial t} + \mathbf{q}.\nabla S = 0 \tag{5}$$

with polytropic gas law

$$S = -\ln\ \rho + \frac{1}{\gamma-1}\ \ln\ T\ , \quad \gamma \neq 1\ . \tag{6}$$

The above represents a rotating magneto-gasdynamic version of a non-conducting gas cloud system originally investigated by Ovsiannikov [7], Dyson [8] and subsequently by Gaffet (see [13] and works cited therein). It was observed therein that the gasdynamic system is compatible with an ansatz in which the entropy S is quadratic and the velocity linear is spatial variables (time-modulated) and the temperature T is dependent on time alone. In that case, the 3+1-dimensional gasdynamic system reduces to an eighteen-dimensional dynamical subsystem.

On elimination of the temperature T in (1) via the gas law (6), and use of the convective entropy condition (5) the continuity equation

$$\frac{\partial\rho}{\partial t} + \text{div}\ (\rho\,\mathbf{q}) = 0 \tag{7}$$

is retrieved while the momentum equation (2) may, in turn, be re-written as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\,\mathbf{q} - (\,\mu/\rho\,)\ \text{curl}\ \mathbf{H}\times\mathbf{H} + f\,(\mathbf{k}\times\mathbf{q}) + T\,\nabla\ \ln\,\rho + \nabla T = \mathbf{0}\ . \quad (8)$$

Here, attention is restricted to 2+1-dimensional motions in which all the magneto-gas variables are dependent only on x, y and t. In particular, the magnetic induction equation (3) then implies that $\mathbf{H}$ admits the representation

$$\mathbf{H} = \nabla A \times \mathbf{k} + h\,\mathbf{k} \quad (9)$$

where $A(\mathbf{x}, t)$ is the magnetic flux. Insertion of (9) into Faraday's law (4) produces the convective constraint

$$\frac{\partial A}{\partial t} + \mathbf{q}.\nabla A = 0 \quad (10)$$

together with

$$\frac{\partial h}{\partial t} + \text{div}(h\,\mathbf{q}) = 0 \quad (11)$$

which holds identically if we set

$$h = \lambda\rho\ , \quad \lambda \in \mathbb{R}\ . \quad (12)$$

Substitution of the representation (9) into the momentum equation (8) now yields

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\,\mathbf{q} + (\,\mu/\rho\,)\ (\nabla^2 A)\nabla A + f\,(\mathbf{k}\times\mathbf{q}) + T\nabla\ \ln\,\rho + \mu\lambda^2\nabla\rho + \nabla T = \mathbf{0} \quad (13)$$

together with

$$A_y\rho_x - A_x\rho_y = 0\ , \quad (14)$$

if $\lambda \neq 0$, whence

$$A = A(\rho, t)\ . \quad (15)$$

In the sequel, attention is restricted to the separable case

$$A = \Phi(\rho)\Psi(t) \quad (16)$$

whence, on substitution into (10) and use of the continuity equation it is seen that

$$\dot{\Psi} = \rho\,\frac{\Phi'}{\Phi}\Psi\,\text{div}\,\mathbf{q} \quad (17)$$

If as in the non-conducting studies of [7], [8] and [13], the temperature T is assumed to depend on temperature alone then (1) together with (17) show that

$$\frac{\dot{\Psi}}{\Psi} = -\frac{1}{\gamma - 1}\left(\frac{\rho\,\Phi'}{\Phi}\right)\frac{\dot{T}}{T}\ , \quad \gamma \neq 1 \quad (18)$$

whence, we set $\Phi = \rho$ and

$$\Psi = \nu T^{\frac{1}{1-\gamma}} \ , \quad \nu \in \mathbb{R} \tag{19}$$

so that

$$A = \nu\rho \, T^{\frac{1}{1-\gamma}} = \nu e^{-S} \ . \tag{20}$$

The momentum equation (13) now reduces to

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\,\mathbf{q} + \left(\mu\,\nu^2 T^{\frac{2}{1-\gamma}} \nabla^2 \rho + T\right) \nabla \ln\ \rho + \mu\lambda^2 \nabla\rho + f(\mathbf{k} \times \mathbf{q}) = \mathbf{0} \tag{21}$$

to be solved in conjunction with the continuity equation (7) and the time evolution

$$\operatorname{div} \mathbf{q} = -\frac{1}{\gamma - 1}\,\frac{\dot{T}}{T} \tag{22}$$

It is this constrained nonlinear coupled system that is the object of the subsequent analysis.

3 A Nonlinear Dynamical Sub-system

In companion studies of isothermal magnetogasdynamics systems in [9, 10] an elliptic vortex ansatz was introduced under which the logarithmic term in the momentum equation is removed and reduction made to an analogous f-plane shallow water system. Therein, by contrast to the polytropic gas law (6), a parabolic constitutive relation

$$p = p_0 + \delta\rho + \epsilon\,\rho^2 \ , \quad \frac{\partial p}{\partial \rho} > 0$$

was adopted.

Here, an integrable nonlinear dynamical subsystem is sought via the elliptic vortex ansatz

$$\begin{aligned} \mathbf{q} &= \mathbf{L}(t)\mathbf{x} + \mathbf{M}(t) \ , \\ \rho &= \frac{\mathbf{x}^T \mathbf{E}(t)\mathbf{x} + h_0(t)}{\mu\lambda^2} \ , \end{aligned} \qquad \mathbf{x} = \begin{pmatrix} x - \bar{q}(t) \\ y - \bar{p}(t) \end{pmatrix} \tag{23}$$

where

$$\mathbf{L} = \begin{pmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{pmatrix} \ , \quad \mathbf{E} = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix} \ , \quad \mathbf{M} = \begin{pmatrix} \dot{\bar{q}}(t) \\ \dot{\bar{p}}(t) \end{pmatrix} \tag{24}$$

Insertion of the above ansatz into the continuity equation yields

$$\begin{pmatrix} \dot a \\ \dot b \\ \dot c \end{pmatrix} + \begin{pmatrix} 3u_1+v_2 & 2v_1 & 0 \\ u_2 & 2(u_1+v_2) & v_1 \\ 0 & 2u_2 & u_1+3v_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{0} \tag{25}$$

together with

$$\dot h_0 = -(u_1+v_2)\,h_0\ . \tag{26}$$

It is subsequently required that

$$2(a+c)\left(\frac{\nu}{\lambda}\right)^2 = -T^{\frac{\gamma+1}{\gamma-1}} \tag{27}$$

so that the term in $\nabla \ln\ \rho$ in the momentum equation is thereby removed and (21) reduces to

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\,\mathbf{q} + f(\mathbf{k}\times\mathbf{q}) + \mu\lambda^2\nabla\rho = 0 \tag{28}$$

Insertion of (23) into (28) now gives

$$\begin{pmatrix} \dot u_1 \\ \dot u_2 \\ \dot v_1 \\ \dot v_2 \end{pmatrix} + \begin{pmatrix} \mathbf{L}^T & -f\mathbf{I} \\ f\mathbf{I} & \mathbf{L}^T \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} + 2\begin{pmatrix} a \\ b \\ b \\ c \end{pmatrix} = \mathbf{0} \tag{29}$$

augmented by the linear auxiliary equations

$$\ddot p + f\dot q = 0\ , \quad \ddot q - f\dot p = 0 \tag{30}$$

Hence, this reduction of the magnetogasdynamic system is determined by the 7-dimensional dynamical sub-system given by (25) and (29). Once the solution of this system is known, the quantities h_0 and T are obtained via (26) and (1), that is

$$\dot T/T = (1-\gamma)(u_1+v_2) \tag{31}$$

However, the requirement (27) remains. The admissibility of this constraint on the dynamical system is examined in the sequel.

In what follows, it proves convenient to proceed in terms of new variables as previously employed in a shallow water hydrodynamics context in [4], namely

$$\begin{gathered} G = u_1+v_2\ , \quad G_R = \frac{1}{2}(v_1-u_2) \\ G_S = \frac{1}{2}(v_1+u_2)\ , \quad G_N = \frac{1}{2}(u_1-v_2) \\ B = a+c\ , \quad B_S = b\ , \quad B_N = \frac{1}{2}(a-c) \end{gathered} \tag{32}$$

Here, G and G_R represent, in turn, the divergence and spin of the velocity field, while G_S and G_N represent shear and normal deformation rates.

On use of the expressions (32), the system (25)−(26) together with (29) produce the dynamical system

$$\begin{aligned}
&\dot{h}_0 + h_0 G = 0\ ,\\
&\dot{B} + 2\,[\,BG + 2\,(\,B_N G_N + B_S G_S\,)\,] = 0\ ,\\
&\dot{B}_S + 2B_S G + G_S B - 2B_N G_R = 0\ ,\\
&\dot{B}_N + 2B_N G + G_N B + 2B_S G_R = 0\ ,\\
&\dot{G} + \frac{G^2}{2} + 2\,(\,G_N^2 + G_S^2 - G_R^2\,) - 2fG_R + 2B = 0\\
&\dot{G}_R + GG_R + \frac{f}{2}G = 0\\
&\dot{G}_N + GG_N - fG_S + 2B_N = 0\\
&\dot{G}_S + GG_S + fG_N + 2B_S = 0
\end{aligned} \tag{33}$$

If we now introduce Ω via

$$G = \frac{2\dot{\Omega}}{\Omega} \tag{34}$$

then $(33)_1$ and $(33)_6$ yield, in turn,

$$h_0 = c_{\mathrm{I}}\,\Omega^{-2} \tag{35}$$

and

$$G_R + \frac{f}{2} = c_0\,\Omega^{-2} \tag{36}$$

where $c_0,\ c_{\mathrm{I}}$ are arbitrary constants of integration. The relation (31) shows that the temperature T is given in terms of Ω by

$$T = T_0\,\Omega^{2(1-\gamma)} \tag{37}$$

where T_0 is a constant of integration.

New Ω-modulated variables are now introduced according to

$$\begin{gathered}
\bar{B} = \Omega^4 B\ ,\quad \bar{B}_S = \Omega^4 B_S\ ,\quad \bar{B}_N = \Omega^4 B_N\\
\bar{G}_S = \Omega^2 G_S\ ,\quad \bar{G}_N = \Omega^2 G_N
\end{gathered} \tag{38}$$

whence the system (33) reduces to

$$\begin{aligned}
&\dot{\bar{B}} + 4\,(\,\bar{B}_N\bar{G}_N + \bar{B}_S\bar{G}_S\,)/\Omega^2 = 0\ ,\\
&\dot{\bar{B}}_S + (\,\bar{B}\bar{G}_S - 2c_0\bar{B}_N\,)/\Omega^2 + f\bar{B}_N = 0\ ,\\
&\dot{\bar{B}}_N + (\,\bar{B}\bar{G}_N + 2c_0\bar{B}_S\,)/\Omega^2 - f\bar{B}_S = 0\ ,\\
&\dot{\bar{G}}_N - f\bar{G}_S + 2\bar{B}_N/\Omega^2 = 0\ ,\\
&\dot{\bar{G}}_S + f\bar{G}_N + 2\bar{B}_S/\Omega^2 = 0\ ,\\
&\Omega^3\ddot{\Omega} + \frac{f^2}{4}\Omega^4 - c_0^2 + \bar{G}_N^2 + \bar{G}_S^2 + \bar{B} = 0
\end{aligned} \tag{39}$$

augmented by the relations (35) and (36).

The constraint (27) may now be re-written as

$$2\left(\frac{\nu}{\lambda}\right)^2 \bar{B} = -T^{\frac{\gamma+1}{\gamma-1}}\,\Omega^4 = -T_0^{\frac{2}{\gamma-1}}\,T$$

so that

$$\bar{B} = \epsilon T\ , \quad \epsilon < 0 \tag{40}$$

that is,

$$\bar{B} = \delta\,\Omega^{2(1-\gamma)}\ , \quad \delta \neq 0 \tag{41}$$

where $\delta = \epsilon T_0$.

Combination of $(39)_2$ and $(39)_3$ with use of $(39)_1$ produces the integral of motion

$$\bar{B}_S^2 + \bar{B}_N^2 - \frac{\bar{B}^2}{4} = c_{\rm II} \tag{42}$$

while $(39)_4$ and $(39)_5$ together give a further integral of motion

$$\bar{G}_S^2 + \bar{G}_N^2 - \bar{B} = c_{\rm III} \tag{43}$$

where $c_{\rm II}$, $c_{\rm III}$ are constants of integration.

4 A Parametrisation

The integrals of motion (42) and (43) may be conveniently parametrised, in turn according to

$$\begin{aligned}
&\bar{B}_S = \pm\sqrt{c_{\rm II} + \frac{1}{4}\bar{B}^2}\ \cos\phi(t)\ , \quad \bar{B}_N = \pm\sqrt{c_{\rm II} + \frac{1}{4}\bar{B}^2}\ \sin\phi(t)\\
&\bar{G}_S = \pm\sqrt{c_{\rm III} + \bar{B}}\ \sin\theta(t)\ , \qquad \bar{G}_N = \pm\sqrt{c_{\rm III} + \bar{B}}\ \cos\theta(t)
\end{aligned} \tag{44}$$

Here, we set

$$\bar{B}_S = -\sqrt{c_{\rm II} + \frac{1}{4}\bar{B}^2}\ \cos\phi(t)\ , \quad \bar{B}_N = -\sqrt{c_{\rm II} + \frac{1}{4}\bar{B}^2}\ \sin\phi(t)\ ,$$
$$\bar{G}_S = -\sqrt{c_{\rm III} + \bar{B}}\ \sin\theta(t)\ , \qquad \bar{G}_N = +\sqrt{c_{\rm III} + \bar{B}}\ \cos\theta(t)\ . \tag{45}$$

Substitution of the parametrisation (45) into $(39)_1$ yields

$$\dot{\bar{B}} + \frac{4}{\Omega^2}\sqrt{(\,c_{\rm II} + \bar{B}^2/4\,)(\,c_{\rm III} + \bar{B}\,)}\ \sin(\theta - \phi) = 0 \tag{46}$$

while conditions $(39)_{2,3}$ reduces to the single requirement

$$\sqrt{c_{\rm II} + \bar{B}^2/4}\ \left[\dot{\phi} + \frac{2}{\Omega^2} - f\right] - \frac{\bar{B}}{\Omega^2}\ \sqrt{c_{\rm III} + \bar{B}}\ \cos(\theta - \phi) = 0 \tag{47}$$

and similarly, $(33)_{4,5}$ produce the single additional condition

$$\sqrt{c_{\rm III} + \bar{B}}\ \left[\,f - \dot{\theta}\,\right] - \frac{2}{\Omega^2}\ \sqrt{c_{\rm II} + \bar{B}^2/4}\ \cos(\theta - \phi) = 0\ . \tag{48}$$

Two conditions which are key to the subsequent development and which may be established by appeal to the system (33) are now recorded:

Theorem 1.

$$\dot{M} = -3GM\ , \tag{49}$$

$$\dot{Q} = -3GQ\ , \tag{50}$$

where

$$M = a(u_2 - \frac{f}{2}) + b(v_2 - u_1) - c(v_1 + \frac{f}{2})\ ,$$
$$Q = -a(u_2^2 + v_2^2) + 2b(u_1u_2 + v_1v_2) - c(u_1^2 + v_1^2) + 4\Delta \tag{51}$$

and

$$\triangle = ac - b^2 \tag{52}$$

Corollary 1. *On use of (32) and (34) it is seen that*

$$M = c_{\rm IV}\Omega^{-6}\ , \tag{53}$$

$$Q = c_{\rm V}\Omega^{-6}\ , \tag{54}$$

where

$$M = 2(B_NG_S - B_SG_N) - B\left(\frac{1}{2}\,f + G_R\right)\ , \tag{55}$$

$$Q = -B\left(G_S^2 + G_N^2 + G_R^2 + \frac{1}{4}G^2\right) + 4G_R(B_NG_S - B_SG_N) + 2G(B_SG_S + B_NG_N) + 4\triangle \tag{56}$$

and $c_{\rm IV}$, $c_{\rm V}$ *are arbitrary constants of integration.*

In particular, the relations (36), (53) and (55) show that

$$c_0 \bar{B} = -c_{\rm IV} + 2(\bar{B}_N \bar{G}_S - \bar{B}_S \bar{G}_N) \tag{57}$$

whence one obtains:

Corollary 2.

$$c_0 \bar{B} = -c_{\rm IV} + 2\sqrt{\left(c_{\rm II} + \frac{1}{4}\bar{B}^2\right)\left(c_{\rm III} + \bar{B}\right)}\ \cos(\theta - \phi) \tag{58}$$

Elimination of $\cos(\theta - \phi)$ between (58) and (47), (48) in turn, yields

$$\dot{\phi} = f + \frac{2}{\Omega^2}\left[-1 + \bar{B}\left(\frac{c_0 \bar{B} + c_{\rm IV}}{4\,c_{\rm II} + \bar{B}^2}\right)\right] \tag{59}$$

and

$$\dot{\theta} = f - \frac{1}{\Omega^2}\left(\frac{c_0 \bar{B} + c_{\rm IV}}{c_{\rm III} + \bar{B}}\right) . \tag{60}$$

It remains to consider the nonlinear equation $(39)_6$ for Ω, namely

$$\Omega^3\,\ddot{\Omega} + \frac{f^2}{4}\,\Omega^4 + c_{\rm III} + 2\bar{B} - c_0^2 = 0 \tag{61}$$

where $\bar{B}$ is given in terms of Ω by (41). On use of **Theorem 1** it is readily shown that

$$\ddot{(\Omega^2 \bar{B}\,)} + f^2\,\Omega^2 \bar{B} = -2(Q + fM)\,\Omega^6 = -2(c_V + f c_{\rm IV}) \tag{62}$$

whence

$$\Omega^2 \bar{B} = \begin{cases} c_{\rm VI} \cos ft + c_{\rm VII} \sin ft - 2(c_{\rm V} + f c_{\rm IV})/f^2 \ , & f \neq 0 \\ -c_{\rm V} t^2 + c_{\rm VI} t + c_{\rm VII} \ , & f = 0 \ . \end{cases} \tag{63}$$

On elimination of $\theta - \phi$ and Ω in (46) via the relations (58) and (63) it is seen that, if $\bar{B} \neq const$ then $\bar{B}$ obeys the elliptic integral relation

$$\int_{c_{\rm VIII}}^{\bar{B}} \frac{d\bar{B}^*}{\bar{B}^* \sqrt{(\bar{B}^{*2} + 4c_{\rm II})\,(\bar{B}^* + c_{\rm III}) - (c_0 \bar{B}^* + c_{\rm IV})^2}}$$

$$= \begin{cases} -2\displaystyle\int_0^t \frac{dt^*}{c_{\rm VI} \cos ft^* + c_{\rm VII} \sin ft^* - 2(c_{\rm V} + f c_{\rm IV})/f^2} \ , & \text{if } f \neq 0 \\ = -2\displaystyle\int_0^t \frac{dt^*}{-c_{\rm V}\, t^{*2} + c_{\rm VI}\, t^* + c_{\rm VII}} \ , & \text{if } f = 0 \end{cases} \tag{64}$$

where $\bar{B}|_{t=0} = c_{\mathrm{VIII}}$.

It remains to consider the compatibility of the constraint (34) and the nonlinear equation (61) with the above elliptic integral expression involving $\bar{B}$.

Substitution of (41) into (61) produces the nonlinear equation

$$\Omega^3\ddot{\Omega} + \frac{f^2}{4}\,\Omega^4 + 2\,\delta\,\Omega^{2-2\gamma} + c_{\mathrm{III}} - c_0^2 = 0\ , \tag{65}$$

while (62) yields

$$\delta\,(\ddot{\Omega^{4-2\gamma}}) + \delta\,f^2\,(\Omega^{4-2\gamma}) = -2(c_{\mathrm{V}} + f c_{\mathrm{IV}})\ . \tag{66}$$

These equations are required to be compatible and admit, in turn the integrals

$$\dot{\Omega}^2 + \frac{f^2}{4}\,\Omega^2 - 2\delta\gamma^{-1}\,\Omega^{-2\gamma} - (\,c_{\mathrm{III}} - c_0^2\,)\,\Omega^{-2} + k_1 = 0\ , \tag{67}$$

$$\delta(4-2\gamma)^2\,\dot{\Omega}^2\,\Omega^{6-4\gamma} + \delta\,f^2\,\Omega^{8-4\gamma} + 4(c_{\mathrm{V}} + f c_{\mathrm{IV}})\,\Omega^{4-2\gamma} + k_2 = 0 \tag{68}$$

where k_1, k_2 are arbitrary constants of integration. It is seen that, since it has been assumed that $\delta \neq 0$ and $\gamma \neq 1$, compatibility requires that the adiabatic index $\gamma = 2$ in which case (67), (68) reduce to

$$\dot{\Omega}^2 + \frac{f^2}{4}\,\Omega^2 - \delta\,\Omega^{-4} - (\,c_{\mathrm{III}} - c_0^2\,)\,\Omega^{-2} + k_1 = 0 \tag{69}$$

and the relation

$$\delta\,f^2 + 4(c_{\mathrm{V}} + f c_{\mathrm{IV}}) + k_2 = 0 \tag{70}$$

while, from (41) and (63),

$$\Omega^2\bar{B} = const = \delta = \begin{cases} -2(c_{\mathrm{V}} + f c_{\mathrm{IV}})/f^2\ , & f \neq 0 \\ c_{\mathrm{VII}}\ , & f = 0\ . \end{cases} \tag{71}$$

and

$$T = \frac{T_0}{\Omega^2}\ . \tag{72}$$

Elimination of Ω in (69) via the relation (71) yields

$$\dot{\bar{B}}^2 + f^2\bar{B}^2 - 4\,\delta^{-2}\bar{B}^5 - 4\,\delta^{-2}(\,c_{\mathrm{III}} - c_0^2\,)\,\bar{B}^4 + 4\,\delta^{-1}k_1\,\bar{B}^3 = 0\ , \tag{73}$$

while the elliptic integral relation (64) gives

$$\dot{\bar{B}}^2 + 4\,\delta^{-2}(\,c_{\mathrm{IV}}^2 - 4\,c_{\mathrm{II}}\,c_{\mathrm{III}}\,)\,\bar{B}^2 - 4\,\delta^{-2}\bar{B}^5 - 4\,\delta^{-2}(\,c_{\mathrm{III}} - c_0^2\,)\bar{B}^4 - 8\,\delta^{-2}(\,2c_{\mathrm{II}} - c_0 c_{\mathrm{IV}}\,)\bar{B}^3 = 0\ . \tag{74}$$

Thus, compatibility requires that

$$4\, c_{\rm II}\, c_{\rm III}\, f^2 + c_{\rm V}^2 + 2 f\, c_{\rm V}\, c_{\rm IV} = 0\ , \tag{75}$$

and

$$k_1 = -2\,\delta^{-1}(2\, c_{\rm II} - c_0\, c_{\rm IV})\ . \tag{76}$$

With $\bar{B} = \epsilon\, T(t)$, determined by the compatible elliptic integral representation (64) Ω is then given by the relation (71). The angles $\phi,\ \theta$ are obtained by integration, in turn, of (59) and (60). The velocity components u_1, u_2, v_1, v_2 and the quantities a, b, c are given, in turn, by

$$\begin{aligned} u_1 &= \frac{\dot{\Omega}}{\Omega} + \frac{1}{\Omega^2}\sqrt{c_{\rm III} + \bar{B}}\ \cos\theta(t), \\ v_1 &= -\frac{1}{\Omega^2}\sqrt{c_{\rm III} + \bar{B}}\ \sin\theta(t) + \frac{c_0}{\Omega^2} - \frac{f}{2}, \\ u_2 &= -\frac{1}{\Omega^2}\sqrt{c_{\rm III} + \bar{B}}\ \sin\theta(t) - \frac{c_0}{\Omega^2} + \frac{f}{2}, \\ v_2 &= \frac{\dot{\Omega}}{\Omega} - \frac{1}{\Omega^2}\sqrt{c_{\rm III} + \bar{B}}\ \cos\theta(t) \end{aligned} \tag{77}$$

together with

$$\begin{aligned} a &= \frac{1}{\Omega^4}\left[\ \frac{1}{2}\,\bar{B} - \sqrt{c_{\rm II} + \frac{1}{4}\,\bar{B}^2}\ \sin\phi(t)\ \right] \\ b &= -\frac{1}{\Omega^4}\sqrt{c_{\rm II} + \frac{1}{4}\,\bar{B}^2}\ \cos\phi(t), \\ c &= \frac{1}{\Omega^4}\left[\ \frac{1}{2}\,\bar{B} + \sqrt{c_{\rm II} + \frac{1}{4}\,\bar{B}^2}\ \sin\phi(t)\ \right], \\ h_0 &= \frac{c_{\rm I}}{\Omega^2}. \end{aligned} \tag{78}$$

The magnetic field is given by

$$\mathbf{H} = \nu\, T^{-1}\nabla\rho + \lambda\rho\,\mathbf{k} \tag{79}$$

and the magneto-gas density by

$$\rho = \frac{1}{\mu\lambda^2}\left[(x - \bar{q}(t),\ y - \bar{p}(t))\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x - \bar{q}(t) \\ y - \bar{p}(t) \end{pmatrix} + h_0(t)\right] \tag{80}$$

where $\bar{p}(t), \bar{q}(t)$ are determined by the auxiliary equations (30). The entropy distribution is determined by

$$S = \ln(T/\rho) \tag{81}$$

while the pressure p is obtained via the gas law

$$p = \rho\, T\ . \tag{82}$$

as in [8].

5 Hamiltonian-Ermakov Structure

It turns out that the nonlinear dynamical system (33) has remarkable underlying structure in that it may be reduced to consideration of a Ermakov-Ray-Reid system

$$\begin{aligned} \ddot{\alpha} + \omega^2(t)\alpha &= \frac{1}{\alpha^2\beta}\, F(\beta/\alpha) \,, \\ \ddot{\beta} + \omega^2(t)\beta &= \frac{1}{\alpha\beta^2}\, G(\alpha/\beta) \,. \end{aligned} \tag{83}$$

Such systems have their origin in the work of Ermakov [14] and were introduced by Ray and Reid in [15, 16]. Extension to 2+1-dimensions was presented in [17] and to multi-component systems in [18]. The main theoretical interest in the system (83) resides in its admittance of a distinctive integral of motion, namely, the Ray-Reid invariant

$$I = \frac{1}{2}(\alpha\dot{\beta} - \beta\dot{\alpha})^2 + \int^{\beta/\alpha} F(z)dz + \int^{\alpha/\beta} G(w)dw \,. \tag{84}$$

Applications of such Ermakov-Ray-Reid systems arise, most notably, in nonlinear optics (see [19] and literature cited therein).

In the sequel, it proves convenient to proceed with $\bar{p}(t) = \bar{q}(t) = 0$ in the ansatz (23). However, the terms are readily re-introduced by use of a Lie group invariance of the magneto-gasdynamic system.

The semi-axes of the time-modulated ellipse

$$\begin{gathered} a(t)x^2 + 2b(t)xy + cy^2 + h_0(t) = 0 \\ (ac - b^2 > 0) \end{gathered} \tag{85}$$

are given by

$$\Phi = \sqrt{\frac{2h_0}{[\sqrt{(a-c)^2 + 4b^2} - (a+c)\,]}} = \sqrt{\frac{h_0}{(B_N^2 + B_S^2)^{\frac{1}{2}} - \dfrac{B}{2}}} \tag{86}$$

and

$$\Psi = \sqrt{\frac{2h_0}{[-\sqrt{(a-c)^2 + 4b^2} - (a+c)\,]}} = \sqrt{\frac{h_0}{-(B_N^2 + B_S^2)^{\frac{1}{2}} - \dfrac{B}{2}}} \tag{87}$$

On use of the integral of motion (42) and the relation (40) it is seen that

$$\begin{aligned}\Phi &= \Omega\sqrt{c_{\mathrm{I}}}\;\Big/\sqrt{\left(c_{\mathrm{II}}+\frac{\bar{B}^2}{4}\right)^{1/2}-\frac{\bar{B}}{2}}\\ &= \Omega\sqrt{c_{\mathrm{I}}}\;\Big/\sqrt{\left(c_{\mathrm{II}}+\epsilon^2\frac{T^2}{4}\right)^{1/2}-\epsilon\frac{T}{2}}\end{aligned} \tag{88}$$

$$\begin{aligned}\Psi &= \Omega\sqrt{c_{\mathrm{I}}}\;\Big/\sqrt{-\left(c_{\mathrm{II}}+\frac{\bar{B}^2}{4}\right)^{1/2}-\frac{\bar{B}}{2}}\\ &= \Omega\sqrt{c_{\mathrm{I}}}\;\Big/\sqrt{-\left(c_{\mathrm{II}}+\epsilon^2\frac{T^2}{4}\right)^{1/2}-\epsilon\frac{T}{2}}\end{aligned} \tag{89}$$

whence, the ratio of the semi-axes is given by

$$\Phi/\Psi = \frac{\sqrt{-c_{\mathrm{II}}}}{\left(c_{\mathrm{II}}+\frac{\bar{B}^2}{4}\right)^{1/2}-\frac{\bar{B}}{2}} > 0\,, \tag{90}$$

where it is required that $0 < -c_{\mathrm{II}} < \frac{1}{4}\bar{B}^2$. Thus, $\bar{B} = \bar{B}(\Phi/\Psi)$ and the ratio of the semi-axes of the ellipse is constrained by the elliptic integral relation (64). By contrast, in the case of the pulsrodon elliptical plasma cylinder in the isothermic analysis of [10] the ratio of the semi-axes was shown to be constant.

It is readily established that the semi-axes Φ, Ψ of the ellipse (86) are governed by the Ermakov-Ray-Reid system

$$\begin{aligned}\ddot{\Phi}+\frac{1}{4}f^2\Phi &= \frac{1}{\Phi^2\Psi}\left[\frac{ZZ'}{1+(\Psi/\Phi)^2}-\left(\frac{\Psi}{\Phi}\right)\frac{(Z^2+\frac{k}{4})}{[1+(\Psi/\Phi)^2]^2}\right],\\ \ddot{\Psi}+\frac{1}{4}f^2\Psi &= \frac{1}{\Psi^2\Phi}\left[-\left(\frac{\Phi}{\Psi}\right)\frac{(Z^2+\frac{k}{4})}{[1+(\Phi/\Psi)^2]^2}-\frac{ZZ'}{1+(\Psi/\Phi)^2}\right]\end{aligned} \tag{91}$$

where

$$Z(\Phi/\Psi) = \Psi\dot{\Phi}-\dot{\Psi}\Phi = \frac{2c_{\mathrm{I}}}{\sqrt{-c_{\mathrm{II}}}}\sqrt{\frac{(\bar{B}^2+4c_{\mathrm{II}})(\bar{B}+c_{\mathrm{III}})-(\bar{B}+c_{\mathrm{IV}})^2}{\bar{B}^2+4c_{\mathrm{II}}}} \tag{92}$$

and

$$\bar{B} = -\sqrt{-c_{\rm II}}\left[\frac{\Psi}{\Phi} + \frac{\Phi}{\Psi}\right] \tag{93}$$

with the requirement that $0 < |\bar{B}| < c_{\rm III}$. In the above, the constant of integration k is given by

$$k = \left(\frac{c_{\rm I}}{c_{\rm II}}\right)^2 \left[f^2(\,c_{\rm IV}^2 + c_{\rm VII}^2\,) - \frac{4}{f^2}\,(\,c_{\rm V} + f c_{\rm IV}\,)^2 \right] . \qquad (f \neq 0) \tag{94}$$

In addition, the system (91), in addition to admitting a Ray-Reid integral of motion, is seen to be Hamiltonian with invariant

$$H = \frac{1}{2}(\dot{\Phi}^2 + \dot{\Psi}^2) - \frac{1}{2(\Phi^2+\Psi^2)}\left[Z^2 - \frac{1}{4}\,f^2(\Phi^2+\Psi^2)^2 + \frac{k}{4} \right] , \tag{95}$$

and is, accordingly, integrable.

6 A Lax Pair Formulation

Here, it is shown in the manner of [6] that the nonlinear dynamical system admits an associated Lax pair representation.

The eight-dimensional dynamical equations (25), (26) together with (29) arising from the elliptic vortex ansatz (23) and (24) may be reformulated as the nonlinear matrix system:

$$\begin{aligned} &\dot{\mathbf{E}} + \mathbf{E}\mathbf{L} + \mathbf{L}^T\,\mathbf{E} + \mathbf{E}\,tr\mathbf{L} = \mathbf{0}\ , \\ &\dot{\mathbf{L}} + \mathbf{L}^2 + f\,\mathbf{H}\mathbf{L} + 2\mathbf{E} = \mathbf{0} \end{aligned} \tag{96}$$

along with the linear system

$$\dot{h}_0 + h_0\,tr\mathbf{L} = 0\ , \quad \dot{\mathbf{M}} + f\,\mathbf{H}\mathbf{M} = \mathbf{0} \tag{97}$$

where $\mathbf{H}$ is given by

$$\mathbf{H} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{98}$$

Moreover, the anisentropic condition (1) may be re-written as

$$\dot{T} + (\gamma - 1)(tr\mathbf{L})\,T = 0\ . \tag{99}$$

It proves convenient to proceed with the gauge transformation (cf [6])

$$\tilde{\mathbf{L}} = \mathbf{D}\mathbf{L}\mathbf{D}^{-1} + \frac{1}{2}\,f\mathbf{H}\ , \quad \tilde{\mathbf{E}} = \mathbf{D}\mathbf{E}\mathbf{D}^{-1} \tag{100}$$

where

$$\mathbf{D} = \exp\left(\frac{1}{2}\,\mathbf{H} f t\right) \ . \tag{101}$$

whence (96) yields

$$\begin{aligned} &\dot{\tilde{\mathbf{E}}} + \tilde{\mathbf{E}}\tilde{\mathbf{L}} + \tilde{\mathbf{L}}^T\,\tilde{\mathbf{E}} + \tilde{\mathbf{E}}\,tr\tilde{\mathbf{L}} = \mathbf{0}\ , \\ &\dot{\tilde{\mathbf{L}}} + \tilde{\mathbf{L}}^2 + \frac{1}{4}\,f^2\,\mathbf{I} + 2\tilde{\mathbf{E}} = \mathbf{0}\ . \end{aligned} \tag{102}$$

On application of the Cayley-Hamilton identity

$$\tilde{\mathbf{L}}^2 - (tr\tilde{\mathbf{L}})\tilde{\mathbf{L}} + (\det\tilde{\mathbf{L}})\mathbf{I} = \mathbf{0} \tag{103}$$

the matrix equation $(102)_2$ becomes

$$\dot{\tilde{\mathbf{L}}} + (tr\tilde{\mathbf{L}})\tilde{\mathbf{L}} - (\det\tilde{\mathbf{L}})\mathbf{I} + \frac{1}{4}\,f^2\mathbf{I} + 2\tilde{\mathbf{E}} = \mathbf{0}\ . \tag{104}$$

Further, on introduction of a new trace-free matrix $\tilde{\mathbf{Q}}$ via

$$\tilde{\mathbf{Q}} = \mathbf{H}\tilde{\mathbf{E}} \tag{105}$$

and on use of the relation

$$\mathbf{H}\tilde{\mathbf{L}}\mathbf{H} = \tilde{\mathbf{L}}^T - (tr\tilde{\mathbf{L}})\,\mathbf{I} \tag{106}$$

the system $(102)_1$ results in

$$\dot{\tilde{\mathbf{Q}}} + 2(tr\tilde{\mathbf{L}})\tilde{\mathbf{Q}} + [\tilde{\mathbf{Q}},\ \tilde{\mathbf{L}}] = \mathbf{0}\ . \tag{107}$$

Since $tr\,\mathbf{L} = tr\,\tilde{\mathbf{L}} = 2\dot{\Omega}/\Omega$, it is natural to introduce the scaling

$$\bar{\mathbf{L}} = \tilde{\mathbf{L}}\Omega^2\ , \quad \bar{\mathbf{E}} = \tilde{\mathbf{E}}\Omega^4\ , \quad \bar{\mathbf{Q}} = \tilde{\mathbf{Q}}\Omega^4 \tag{108}$$

so that (104) and (107) adopt the form

$$\begin{aligned} &\dot{\bar{\mathbf{Q}}} + \Omega^{-2}\,[\bar{\mathbf{Q}},\bar{\mathbf{L}}] = \mathbf{0}\ , \\ &\dot{\bar{\mathbf{L}}} - \Omega^{-2}\,(\det\bar{\mathbf{L}})\mathbf{I} + \frac{f^2}{4}\,\Omega^2\mathbf{I} + 2\Omega^{-2}\bar{\mathbf{E}} = \mathbf{0}\ . \end{aligned} \tag{109}$$

At this stage, it is noticed that $(109)_1$ may be reformulated in terms of two trace-free matrixes $\bar{\mathbf{Q}}$ and $\bar{\mathbf{L}}^*$ as

$$\dot{\bar{\mathbf{Q}}} + \Omega^{-2}\,[\bar{\mathbf{Q}},\bar{\mathbf{L}}^*] = \mathbf{0}\ , \tag{110}$$

where $\bar{\mathbf{L}}^*$ denotes the trace-free part of $\bar{\mathbf{L}}$. Moreover, $(109)_2$ may be decomposed into the trace-free part

$$\dot{\bar{\mathbf{L}}}^* + \Omega^{-2}\,[\bar{\mathbf{Q}}, \mathbf{H}] = \mathbf{0}\ , \tag{111}$$

together with

$$(tr\dot{\bar{\mathbf{L}}}) - 2\Omega^{-2}\,(\det\bar{\mathbf{L}}^*) - \frac{1}{2}\,\Omega^{-2}(tr\bar{\mathbf{L}})^2 + \frac{1}{2}\,f^2\,\Omega^2 + 2\Omega^{-2}\,(tr\bar{\mathbf{E}}) = 0\ . \tag{112}$$

In general, the matrix system (110), (111) and the scalar equation (112) are coupled via the relation

$$\dot{h}_0 + h_0\,tr\tilde{\mathbf{L}} = 0\ . \tag{113}$$

A new time variable τ is now introduced via

$$\tau = \int \Omega^{-2} dt \tag{114}$$

whence the equations (110) and (111) reduce to (cf the non-conducting case of [6])

$$\bar{\mathbf{Q}}' + [\bar{\mathbf{Q}}, \bar{\mathbf{L}}^*] = \mathbf{0}\ , \quad \bar{\mathbf{L}}^{*'} + [\bar{\mathbf{Q}}, \bar{\mathbf{H}}] = \mathbf{0}\ . \tag{115}$$

It is now seen that the matrix system (115) constitutes the compatibility condition

$$\mathcal{M}'(\lambda) + [\mathcal{M}(\lambda)\ ,\ \mathcal{L}(\lambda)] = \mathbf{0} \tag{116}$$

associated with the linear pair

$$\Psi' = \mathcal{L}(\lambda)\Psi\ , \quad \mu\Psi = \mathcal{M}(\lambda)\Psi \tag{117}$$

where

$$\mathcal{L}(\lambda) = \bar{\mathbf{L}}^* + \lambda\mathbf{H}\ , \quad \mathcal{M}(\lambda) = \bar{\mathbf{Q}} + \lambda\bar{\mathbf{L}}^* + \lambda^2\mathbf{H} \tag{118}$$

and μ is an arbitrary parameter. Here, $\mathcal{L}$ and $\mathcal{M}$ represent Lax matrices for the nonlinear matrix system (115).

Finally, it is observed that, in terms of new time variable τ, (112) reduces to a classical Steen-Ermakov type equation [14, 20]

$$\Sigma'' + (\det\bar{\mathbf{L}}^* - tr\bar{\mathbf{E}})\Sigma = \frac{f^2}{4\Sigma^3}\ , \quad \Sigma = \Omega^{-1}\ . \tag{119}$$

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