Note di Matematica
Note Mat. **32** (2012) n. 1, 159–173.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v32n1p159

Onset of convection in porous layers salted from above and below

Salvatore Rionero
University of Naples Federico II,
Department of Mathematics and Applications 'R. Caccioppoli',
Complesso Universitario Monte S. Angelo, Via Cinzia, 80126 Naples, Italy.
rionero@unina.it

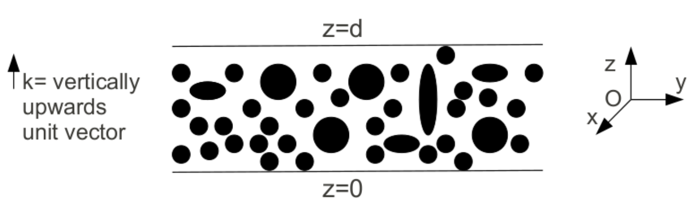
Abstract. For a porous layer heated from below and salted from above and below, the non existence of subcritical instabilities and conditions of global stability - for special values of the Prandtl numbers - are found.

Keywords: Multi-component fluid mixtures, Porous media, Convection, Stability.

1 Introduction and aims

A porous medium is schematized via a body (generally rigid and called *skeleton*) having interconnected pores everywhere. Generally, the fluid occupying the pores is a mixture since are dissolved in chemical species ("salts") and the layer is embedded in a temperature field.

The behaviour of convective-diffusive fluid mixtures in porous layer presents a picture of behaviours increasing with the number of components. Although the



subject of double-diffusive convection is still a very active research area [1]-[22], the same subject with more than two components, although more difficult – in the past as nowadays has also attracted the attention of many authors [23]-[30].

The present paper is concerned with an horizontal layer heated from below and salted by two salts, either from below ("salt 1") or from above ("salt 2").

Denoting by

$$\begin{cases} T = \text{temperature field}, \quad C_i = \text{concentration field of the "salt } i\text{"}, (i = 1, 2) \\ \\ \mathbf{v} = \text{ seepage velocity}, \quad p = \text{ pressure field}, \end{cases} \tag{1}$$

in the case of the boundary conditions

$$\begin{cases} T(0) = T_1, \quad T(d) = T_2, \quad T_1 - T_2 > 0, \\ \\ C_i(0) = C_{il}, \quad C_i(d) = C_{iu}, \quad (i = 1, 2), \\ \\ \mathbf{v} \cdot \mathbf{k} = 0, \quad \text{at } z = 0, \ z = d, \end{cases} \tag{2}$$

it follows that the perturbations $(\mathbf{u}, \Phi_1, \Phi_2, \Pi)$ to the conduction solution {cfr. [29]}, with

$$\begin{cases} \mathbf{u} = \text{ perturbation to } \mathbf{v}, \\ \Phi_i = \text{ perturbation to } C_i, \\ \Pi = \text{ perturbation to } p, \end{cases} \tag{3}$$

are governed by

$$\begin{cases} \nabla \Pi = -\mathbf{u} + (R\theta - R_1 \Phi_1 - R_2 \Phi_2)\mathbf{k}, \\ \\ \nabla \cdot \mathbf{u} = 0, \\ \\ \theta_t + \mathbf{u} \cdot \nabla \theta = R\omega + \Delta\theta, \\ \\ P_1(\Phi_{1t} + \mathbf{u} \cdot \nabla \Phi_1) = R_1 \omega + \Delta \Phi_1, \\ \\ P_2(\Phi_{2t} + \mathbf{u} \cdot \nabla \Phi_2) = -R_2 \omega + \Delta \Phi_2, \end{cases} \tag{4}$$

$$\omega = \theta = \Phi_1 = \Phi_2 = 0, \qquad \text{on } z = 0, z = 1, \tag{5}$$

with

$$\begin{cases} \omega = \mathbf{u} \cdot \mathbf{k}, \\ R = \text{Rayleigh thermal number}, \\ R_i = \text{Rayleigh concentration number of "salt } i\text{"}. \end{cases} \tag{6}$$

Assuming - as it is normally done - that

i) $\mathbf{u} = (u, v, \omega), \theta, \Phi_1, \Phi_2$ are periodic in the x and y directions respectively of periods $2\pi/a_x$, $2\pi/a_y$;

ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ is the periodicity cell;

iii) $\mathbf{u}, \theta, \Phi_1, \Phi_2$ belong to $W^{2,2}(\Omega)$ and are such that all their first derivatives and second spatial derivatives can be expanded in Fourier series uniformly convergent in Ω,

our aim - according to the results obtained in {[29]-[30],[33] } - is to show that when the layer is salted from below by "salt 1" and from above by "salt 2", then

1) do not exist subcritical instabilities;

2) exist physically relevant values of the "salts" Prandtl numbers such that the triply diffusive-convection can be reduced rigorously to the double diffusive-convection and the global stability condition is given by $R^2 < R_C^2$ with

$$\begin{cases} R_c^2 = \min\left[R_1^2 - \dfrac{R_2^2}{P_2} + 4\pi^2\left(1 + \dfrac{1}{P_2}\right), R_1^2 - R_2^2 + 4\pi^2\right], \\ \text{when } P_1 = 1 \end{cases} \tag{7}$$

$$\begin{cases} R_c^2 = \min\left[\dfrac{R_1^2}{P_1} - R_2^2 + 4\pi^2\left(1 + \dfrac{1}{P_1}\right), R_1^2 - R_2^2 + 4\pi^2\right], \\ \text{when } P_2 = 1, \end{cases} \tag{8}$$

$$\begin{cases} R_c^2 = \min\left[\dfrac{1}{P}(R_1^2 - R_2^2) + 4\pi^2\left(1 + \dfrac{1}{P}\right), R_1^2 - R_2^2 + 4\pi^2\right], \\ \text{when } P_1 = P_2 = P. \end{cases} \tag{9}$$

Section 2 is devoted to the boundary value problem of the problem at stake. The (Routh Hurwitz) conditions of linear stability are found in the subsequent Section while Sections 4-5 are devoted to the non existence of subcritical instabilities. Finally, (7)-(9) are shown in Section 6.

2 The boundary value problem at stake

We recall here a basic theorem - which proof is given either in [29] or [31] - concerned with the main boundary value problem $(4)_1$-$(4)_2$-(5).

Theorem 1. *Let* $(\mathbf{u}, \theta, \Phi_1, \Phi_2)$ *be solution of the b.v.p.*

$$\begin{cases} \nabla \Pi = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{k}, \\ \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{10}$$

$$\omega = \theta = \Phi_1 = \Phi_2 = 0, \qquad z = 0, 1, \tag{11}$$

then:

i) $(\mathbf{u}, \theta, \Phi_1, \Phi_2)$ *is solution of the b.v.p.*

$$\begin{cases} \Delta\omega = \Delta_1(R\theta - R_1\Phi_1 - R_2\Phi_2), \quad \textit{in } \Omega, \\ \mathbf{u} = \theta = \Phi_1 = \Phi_2 = 0, \qquad \textit{on } z = 0, 1, \\ \Delta_1 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}. \end{cases} \tag{12}$$

ii) a complete orthogonal system of solutions of (10) is given by

$$\begin{cases} \tilde{\omega}_n = \eta_n(R\tilde{\theta}_n - R_1\tilde{\Phi}_{1n} - R_2\tilde{\Phi}_{2n}), \\ \mathbf{u}_n = \dfrac{1}{a^2}\left(\dfrac{\partial^2\tilde{\omega}_n}{\partial x\partial z}\mathbf{i} + \dfrac{\partial^2\tilde{\omega}_n}{\partial y\partial z}\mathbf{j}\right) + \tilde{\omega}_n\mathbf{k}, \end{cases} \tag{13}$$

with

$$\begin{cases} a^2 = a_x^2 + a_y^2,\ \xi_n = a^2 + n^2\pi^2,\ \eta_n = \dfrac{a^2}{\xi_n}, \\ \omega = \sum_1^\infty \tilde{\omega}_n = \sum_1^\infty \omega_n(x, y, t)\sin(n\pi z), \\ \mathbf{u} = \sum_1^\infty \mathbf{u}_n = \sum_1^\infty (\tilde{u}_n\mathbf{i} + \tilde{v}_n\mathbf{j} + \tilde{\omega}_n\mathbf{k}), \\ \theta = \sum_1^\infty \tilde{\theta}_n = \sum_1^\infty \theta_n(x, y, t)\sin(n\pi z), \\ \Phi_i = \sum_1^\infty \tilde{\Phi}_{in} = \sum_1^\infty \Phi_{in}(x, y, t)\sin(n\pi z). \end{cases} \tag{14}$$

Remark 1. By virtue of theorem 1 the independent scalar unknown fields $(u, v, \omega, \theta, \Phi_1, \Phi_2)$ are reduced to (θ, Φ_1, Φ_2).

3 Linear instability

Linearizing (4) and taking into account that by virtue of theorem 1, it turns out that

$$\Delta\varphi = -\sum_1^\infty \xi_n\varphi_n, \qquad \varphi \in (\theta, \Phi_1, \Phi_2), \tag{15}$$

one obtains

$$\begin{cases} \theta_t = \sum_{n=1}^{\infty}(a_{1n}\theta_n + a_{2n}\Phi_{1n} + a_{3n}\Phi_{2n}), \\ \Phi_{1t} = \sum_{n=1}^{\infty}(b_{1n}\theta_n + b_{2n}\Phi_{1n} + b_{3n}\Phi_{2n}), \\ \Phi_{2t} = \sum_{n=1}^{\infty}(c_{1n}\theta_n + c_{2n}\Phi_{1n} + c_{3n}\Phi_{2n}), \end{cases} \tag{16}$$

with

$$\begin{cases} a_{1n} = R^2\eta_n - \xi_n, & a_{2n} = -RR_1\eta_n, & a_{3n} = -RR_2\eta_n, \\ b_{1n} = \dfrac{RR_1}{P_1}\eta_n, & b_{2n} = -\dfrac{(R_1^2\eta_n + \xi_n)}{P_1}, & b_{3n} = -\dfrac{R_1R_2}{P_1}\eta_n, \\ c_{1n} = -\dfrac{RR_2}{P_2}\eta_n, & c_{2n} = \dfrac{R_1R_2}{P_2}\eta_n, & c_{3n} = \dfrac{R_2^2\eta_n - \xi_n}{P_2}. \end{cases} \tag{17}$$

Setting

$$\begin{cases} \theta_n = \xi_{0n}(t)F(x,y)\sin(n\pi z), \\ \Phi_{in} = \xi_{in}(t)F(x,y)\sin(n\pi z), \end{cases} \tag{18}$$

by virtue of

$$\int_0^1 \sin(n\pi z)\sin(m\pi z)d\,z = \begin{cases} 0, & n \neq m \\ \dfrac{1}{2}, & n = m \end{cases} \tag{19}$$

(16) implies

$$\begin{cases} \dfrac{d\xi_{0n}}{dt} = a_{1n}\xi_{0n} + a_{2n}\xi_{1n} + a_{3n}\xi_{2n}, \\ \dfrac{d\xi_{1n}}{dt} = b_{1n}\xi_{0n} + b_{2n}\xi_{1n} + b_{3n}\xi_{2n}, \\ \dfrac{d\xi_{2n}}{dt} = c_{1n}\xi_{0n} + c_{2n}\xi_{1n} + c_{3n}\xi_{2n}. \end{cases} \tag{20}$$

Setting

$$\mathcal{L}_n = \begin{pmatrix} a_{1n} & a_{2n} & a_{3n} \\ b_{1n} & b_{2n} & b_{3n} \\ c_{1n} & c_{2n} & c_{3n} \end{pmatrix}$$

it easily follows that the characteristic equation of the $\mathcal{L}_n$ eigenvalues λ_n is

$$\lambda_n^3 - \mathtt{I}_{1n}\lambda_n^2 + \mathtt{I}_{2n}\lambda_n - \mathtt{I}_{3n} = 0, \tag{21}$$

with $\mathtt{I}_{1n}, \mathtt{I}_{2n}, \mathtt{I}_{3n}$ given by

$$\begin{cases} \mathtt{I}_{1n} = a_{1n} + b_{2n} + c_{3n} = \lambda_{1n} + \lambda_{2n} + \lambda_{3n}, \\[2ex] \mathtt{I}_{2n} = \begin{vmatrix} a_{1n} & a_{2n} \\ b_{1n} & b_{2n} \end{vmatrix} + \begin{vmatrix} a_{1n} & a_{3n} \\ c_{1n} & c_{3n} \end{vmatrix} + \begin{vmatrix} b_{2n} & b_{3n} \\ c_{2n} & c_{3n} \end{vmatrix}, \\[2ex] \mathtt{I}_{3n} = \begin{vmatrix} a_{1n} & a_{2n} & a_{3n} \\ b_{1n} & b_{2n} & b_{3n} \\ c_{1n} & c_{2n} & c_{3n} \end{vmatrix}. \end{cases} \tag{22}$$

By virtue of the Routh-Hurwitz stability-conditions [31], the following theorem holds.

Theorem 2. *The conduction solution is linearly stable if and only if,* $\forall n \in \mathbb{N}$*, the inequalities*

$$I_{1n} < 0,\ I_{3n} < 0,\ I_{1n} I_{2n} - I_{3n} < 0, \tag{23}$$

hold.

Since (23) requires $\mathtt{I}_{2n} > 0$, in view of

$$\begin{cases} \mathtt{I}_{1n} = \left[R^2 - \dfrac{R_1^2}{P_1} + \dfrac{R_2^2}{P_2} - \left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2} \right) \dfrac{\xi_n}{\eta_n} \right] \eta_n, \\[2ex] \mathtt{I}_{2n} = \left[\left(\dfrac{1}{P_1} + \dfrac{1}{P_2} + \dfrac{1}{P_1 P_2} \right) \dfrac{\xi_n}{\eta_n} + \dfrac{1}{P_1} \left(1 + \dfrac{1}{P_2} \right) R_1^2 - \dfrac{1}{P_2} \left(1 + \dfrac{1}{P_1} \right) R_2^2 + \right. \\[2ex] \qquad \left. - \left(\dfrac{1}{P_1} + \dfrac{1}{P_2} \right) R^2 \right] \xi_n \eta_n, \\[2ex] \mathtt{I}_{3n} = \dfrac{1}{P_1 P_2} \left(R^2 - R_1^2 + R_2^2 - \dfrac{\xi_n}{\eta_n} \right) \eta_n \xi_n^2, \\[2ex] \inf\limits_{(a^2,n) \in \mathbb{R}^+ \times \mathbb{N}} \dfrac{\xi_n}{\eta_n} = 4\pi^2, \end{cases} \tag{24}$$

it follows that

Theorem 3. *The conduction solution is linearly stable only if*

$$R_{C_i}^2 > 0,\ (i = 1, 2, 3), \quad R^2 < R_C^2, \tag{25}$$

with

$$\begin{cases} R^2_{C_1} = \dfrac{R_1^2}{P_1} - \dfrac{R_2^2}{P_2} + 4\pi^2\left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2}\right), \\ \\ R^2_{C_2} = \dfrac{1}{P_1+P_2}\left[(1+P_2)R_1^2 - (1+P_1)R_2^2 + 4\pi^2(1+P_1+P_2)\right], \\ \\ R^2_{C_3} = R_1^2 - R_2^2 + 4\pi^2,\ R^2_C = \min(R^2_{C_1}, R^2_{C_2}, R^2_{C_3}). \end{cases} \tag{26}$$

4 Preliminaries to nonexistence of subcritical instabilities

Lemma 1. *Let λ_1 be a real eigenvalue of the matrix*

$$L = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \qquad \begin{pmatrix} \alpha_{ij} = const \in \mathbb{R}) \\ i,j = 1,2,3 \end{pmatrix} \tag{27}$$

and let $\tilde{\mathbf{U}} = (1, \tilde{U}_2, \tilde{U}_3)$ be an associated eigenvector. Then the transformation

$$\mathbf{X} = L_1 \mathbf{Z}, \tag{28}$$

with

$$\mathbf{X} = (X_1, X_2, X_3)^T,\ \mathbf{Z} = (Z_1, Z_2, Z_3)^T,\ L_1 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{U}_2 & 1 & 0 \\ \tilde{U}_3 & 0 & 1 \end{pmatrix} \tag{29}$$

reduces the ternary system

$$\frac{d\mathbf{X}}{dt} = L\mathbf{X} + \mathbf{F}, \tag{30}$$

with $(\mathbf{F})_{\mathbf{X}=0} = 0$, to

$$\frac{d\mathbf{Z}}{dt} = \tilde{L}\mathbf{Z} + \tilde{\mathbf{F}}, \tag{31}$$

$$\tilde{L} = \begin{pmatrix} \lambda_1 & \tilde{\alpha}_{12} & \tilde{\alpha}_{13} \\ 0 & \tilde{\alpha}_{22} & \tilde{\alpha}_{23} \\ 0 & \tilde{\alpha}_{32} & \tilde{\alpha}_{33} \end{pmatrix}, \tilde{\mathbf{F}} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3)^T. \tag{32}$$

$$\begin{cases} Z_1 = X_1, Z_2 = X_2 - \tilde{U}_2 X_1, Z_3 = X_3 - \tilde{U}_3 X_1, \\ \tilde{F}_1 = F, \tilde{F}_2 = F_2 - \tilde{U}_2 F_1, \tilde{F}_3 = F_3 - \tilde{U}_3 F_1 \end{cases} \tag{33}$$

$$\begin{cases} \tilde{\alpha}_{12} = \alpha_{12},\ \tilde{\alpha}_{13} = \alpha_{13}, \\ \tilde{\alpha}_{22} = \alpha_{22} - \tilde{U}_2\alpha_{12},\ \tilde{\alpha}_{23} = \alpha_{23} - \tilde{U}_2\alpha_{13}, \\ \tilde{\alpha}_{32} = \alpha_{32} - \tilde{U}_3\alpha_{12},\ \tilde{\alpha}_{33} = \alpha_{33} - \tilde{U}_3\alpha_{13} \end{cases} \tag{34}$$

$$\tilde{\alpha}_{22} + \tilde{\alpha}_{33} = \lambda_2 + \lambda_3,\ \tilde{\alpha}_{22}\tilde{\alpha}_{33} - \tilde{\alpha}_{23}\tilde{\alpha}_{32} = \lambda_2\lambda_3, \tag{35}$$

λ_2, λ_3 *being other eigenvalues of* L.

Proof. The proof, based on [32, pp.194-197], can be found in [30], [33].

Lemma 2. *Let the eigenvalues* λ_i, $(i = 1, 2, 3)$ *of*

$$L = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{36}$$

have negative real part and let $\lambda_1 < 0$ *and*

$$\Psi = (\tilde{A}_1 Z_2 - \tilde{A}_3 Z_3)\tilde{F}_2 + (\tilde{A}_2 Z_3 - \tilde{A}_3 Z_2)\tilde{F}_3 + Z_1\tilde{F}_1 = 0, \tag{37}$$

with

$$\begin{cases} \tilde{A}_1 = \lambda_2\lambda_3 + \tilde{\alpha}_{32}^2 + \tilde{\alpha}_{33}^2, \quad \tilde{A}_2 = \lambda_2\lambda_3 + \tilde{\alpha}_{22}^2 + \tilde{\alpha}_{23}^2, \\ \\ \tilde{A}_3 = \tilde{\alpha}_{22}\tilde{\alpha}_{32} + \tilde{\alpha}_{23}\tilde{\alpha}_{33}. \end{cases} \tag{38}$$

Then the function

$$\tilde{W} = \frac{1}{2}\left[Z_1^2 + \lambda_2\lambda_3(Z_2^2 + Z_3^2) + (\tilde{\alpha}_{22}Z_3 - \tilde{\alpha}_{32}Z_2)^2 + (\tilde{\alpha}_{23}Z_3 - \tilde{\alpha}_{33}Z_2)^2\right], \tag{39}$$

has - along

$$\frac{d\mathbf{Z}}{dt} = \tilde{L}Z + \tilde{\mathbf{F}}, \tag{40}$$

the temporal derivative given by

$$\frac{d\tilde{W}}{dt} = \frac{1}{2}\left[\lambda_1 Z_1^2 + (\lambda_2 + \lambda_3)\lambda_2\lambda_3(Z_2^2 + Z_3^2)\right] < 0. \tag{41}$$

The null solution of (40) and hence of (30) is globally stable and subcritical instabilities do not exist.

Proof. A detailed proof can be found in [30], [33].

Remark 2. The eigenvalues of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

have negative real part if and only if

$$a_{11} < 0, \ I = a_{22} + a_{33} < 0, \ A = a_{22}a_{33} - a_{32}a_{23}.$$

(cfr. Remark 2.5 of [30]).

5 Nonexistence of subcritical instabilities and global stability

Following [33], we set

$$S_m^{(\theta)} = \sum_{n=1}^{m} \theta_n, \ S_m^{(\Phi_i)} = \sum_{n=1}^{m} \Phi_{in}, \ \mathbf{U}_m = \sum_{n=1}^{m} \mathbf{u}_n, \tag{42}$$

$$\theta = \lim_{m\to\infty} S_m^{(\theta)}, \ \Phi_i = \lim_{m\to\infty} S_m^{(\Phi_i)}, \ (i = 1, 2). \tag{43}$$

The nonexistence of subcritical instabilities and the global stability is guaranteed by showing that the asymptotic stability of the null solution of

$$\begin{cases} \dfrac{d}{dt} S_m^{(\theta)} = \displaystyle\sum_{n=1}^{m} (a_{1n}\theta_n + a_{2n}\Phi_{1n} + a_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla S_m^{(\theta)}, \\[2ex] \dfrac{d}{dt} S_m^{(\Phi_1)} = \displaystyle\sum_{n=1}^{m} (b_{1n}\theta_n + b_{2n}\Phi_{1n} + b_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla S_m^{(\Phi_1)}, \\[2ex] \dfrac{d}{dt} S_m^{(\Phi_2)} = \displaystyle\sum_{n=1}^{m} (c_{1n}\theta_n + c_{2n}\Phi_{1n} + c_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla S_m^{(\Phi_2)}, \end{cases} \tag{44}$$

$$\theta_n = \Phi_{1n} = \Phi_{2n} = 0, \forall n \in \{1, ..., m\} \text{ for } z = 0, 1. \tag{45}$$

$$\theta_n(0) = \theta_n^{(0)}, \ \Phi_{in}(0) = \Phi_{in}^{(0)}, \ \ i = 1, 2, n \in \{1, ..., m\}. \tag{46}$$

$\forall m \in \mathbb{N}$, is guaranteed by (23). On the other hand, introducing the evolution system governing the n-th component of the of the perturbation (θ, Φ_1, Φ_2)

$$\begin{cases} \dfrac{\partial}{\partial t}\theta_n = a_{1n}\theta_n + a_{2n}\Phi_{1n} + a_{3n}\Phi_{2n} - \mathbf{U}_m \cdot \nabla\theta_n, \\[2ex] \dfrac{\partial}{\partial t}\Phi_{1n} = b_{1n}\theta_n + b_{2n}\Phi_{1n} + b_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla\Phi_{1n}, \\[2ex] \dfrac{\partial}{\partial t}\Phi_{2n} = c_{1n}\theta_n + c_{2n}\Phi_{1n} + c_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla\Phi_{2n}, \end{cases} \tag{47}$$

$$\begin{cases} \mathbf{U}_m = \sum_{n=1}^{m} \mathbf{u}_n, \mathbf{u}_n = \frac{1}{a^2}\left(\frac{\partial^2 \omega_n}{\partial x \partial z}\mathbf{i} + \frac{\partial^2 \omega_n}{\partial y \partial z}\mathbf{j} + \omega_n \mathbf{k}\right), \\ \omega_n = \tilde{\omega}_n(x,y,t)\sin(n\pi z), \theta_n = \tilde{\theta}_n(x,y,t)\sin(n\pi z), \\ \Phi_{in} = \tilde{\Phi}_{in}(x,y,t)\sin(n\pi z), i = 1,2, n \in \{1,...,m\} \end{cases} \tag{48}$$

(44) are immediately obtained by adding with respect to n from $n = 1$ to $n = m$ each equations of (47). Therefore the nonexistence of subcritical instabilities and the global stability is guaranteed by showing that (23) imply the asymptotic stability of the null solution of (45)-(48) $\forall m \in \mathbb{N}$.

Theorem 4. *Let (23) hold. Then, for any $m \in \mathbb{N}$, the zero solution of (45)-(48) is asymptotically stable for any initial data*

Proof. Denoting by λ_{ni}, $(i = 1, 2, 3)$, the roots of (21), (23) guarantee that λ_{ni} have negative real part and at least one - say λ_{n1} - be a negative real number. Then denoting by $\tilde{\mathbf{U}}_n = (1, \tilde{U}_{n2}, \tilde{U}_{n3})$ an eigenvector associated to λ_{n1}, Lemma 1 can be applied by setting

$$\begin{cases} Z_{n1} = \theta_n, Z_{n2} = \Phi_{1n} - \tilde{U}_{n2}\theta_n, Z_{n3} = \Phi_{2n} - \tilde{U}_{n3}\theta_n, \\ F_1 = \mathbf{U}_m \cdot \nabla\theta_n,\ F_2 = \mathbf{U}_m \cdot \nabla\Phi_{1n}, F_3 = \mathbf{U}_m \cdot \nabla\Phi_{2n}, \\ \tilde{F}_1 = F_1, \tilde{F}_2 = F_2 - \tilde{U}_{n2}F_1, \tilde{F}_3 = F_3 - \tilde{U}_{n3}F_1 \end{cases} \tag{49}$$

Let

$$\tilde{V}_n = \int_\Omega \tilde{W}_n d\Omega, \tag{50}$$

with $\tilde{W}_n$ given by (39) with λ_{ni} at the place of λ_i. Then - instead of (41) - one has to show that

$$\tilde{\Psi} =< \tilde{A}_{1n}Z_{n2} - \tilde{A}_{3n}Z_{n3}, \tilde{F}_2 > + < \tilde{A}_{2n}Z_{n3} - \tilde{A}_{3n}Z_{n2}, \tilde{F}_3 > + < Z_{n1}, \tilde{F}_1 >= 0, \tag{51}$$

with $\tilde{A}_{1n}, \tilde{A}_{2n}, \tilde{A}_{3n}$ constants.
In view of (49) it follows that Z_{ni}, $(i = 1, 2, 3))$, is of the kind

$$Z_{ni} = \tilde{Z}_{ni}(x,y,t)\sin(n\pi z), \tag{52}$$

$$\mathbf{U}_m \cdot \nabla\theta_n = \sum_{p=1}^{m}\left[\frac{p\pi}{a^2}\left(\frac{\partial\tilde{\omega}_p}{\partial x}\frac{\partial\tilde{\theta}_n}{\partial x} + \frac{\partial\tilde{\omega}_p}{\partial y}\frac{\partial\tilde{\theta}_p}{\partial x}\right)\cos(p\pi z)\sin(n\pi z) + \right.$$
$$\left. n\pi\tilde{\omega}_p\tilde{\theta}_p\sin(p\pi z)\cos(n\pi z)\right] \tag{53}$$

and hence

$$< Z_m, \tilde{F}_1 >=< \frac{\tilde{Z}_{n1}\sin(n\pi z)}{U^*_{n1}}, \mathbf{U}_m \cdot \nabla\theta_n > . \tag{54}$$

Since it easily turns out that

$$\int_0^1 \sin(q\pi z)\cos(p\pi z)\sin(n\pi z)dz = 0, \text{for } p+q \neq n. \tag{55}$$

But one easily verifies that all the other scalar products appearing in $\tilde{\Psi}$ are linear combinations of terms of kind (54) (with Φ_{in} at the place of θ_n) and hence by virtue of (55) it turn out that $\tilde{\Psi} = 0$.

6 Proof of (7)

In the case $P_1 = 1$, (4) reduces to

$$\begin{cases} \nabla p = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{k}, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u}\cdot\nabla\theta, \\ \Phi_{1t} = R_1\omega + \Delta\Phi_1 - \mathbf{u}\cdot\Phi_1, \\ \Phi_{2t} = -\dfrac{R_2}{P_2}\omega + \dfrac{1}{P_2}\Delta\Phi_1 - \dfrac{\mathbf{u}}{P_2}\cdot\Phi_2. \end{cases} \tag{56}$$

Setting

$$\varphi = R_1\theta - R\Phi_1 = 0 \Leftrightarrow \Phi_1 = \frac{1}{R}(R_1\theta - \varphi), \tag{57}$$

it follows that

$$\nabla p = -\mathbf{u} + \left(\frac{R^2 - R_1^2}{R}\theta + \frac{R_1}{R}\varphi - R_2\Phi_2\right)\mathbf{k}, \tag{58}$$

$$\begin{cases} \varphi_t = \Delta\varphi - \mathbf{u}\cdot\nabla\varphi, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u}\cdot\nabla\theta, \\ \Phi_{2t} = -\dfrac{R_2}{P_2}\omega + \dfrac{1}{P_2}\Delta\Phi_2 - \dfrac{\mathbf{u}\cdot\nabla\Phi_2}{P_2}. \end{cases} \tag{59}$$

In view of theorem 1, one obtains

$$\tilde{\omega}_n = \eta_n\left(\frac{R^2 - R_1^2}{R}\tilde{\theta}_n + \frac{R_1}{R}\tilde{\varphi}_n - R_2\tilde{\Phi}_{2n}\right), \tag{60}$$

and hence

$$\begin{cases} \varphi_t = \displaystyle\sum_1^\infty (\bar\alpha_{1n}\varphi_n + \bar\alpha_{2n}\theta_n + \bar\alpha_{3n}\Phi_{2n}) - \mathbf{u}\cdot\nabla\varphi, \\ \theta_t = \displaystyle\sum_1^\infty \left(\bar\beta_{1n}\varphi_n + \bar\beta_{2n}\theta_n + \bar\beta_{3n}\Phi_{2n}\right) - \mathbf{u}\cdot\nabla\theta, \\ \Phi_{2t} = \displaystyle\sum_1^\infty (\bar\gamma_{1n}\varphi_n + \bar\gamma_{2n}\theta_n + \bar\gamma_{3n}\Phi_{2n}) - \mathbf{u}\cdot\nabla\Phi_2, \end{cases} \tag{61}$$

with

$$\begin{cases} \bar\alpha_{1n} = -\xi_n, \qquad \bar\alpha_{2n} = \bar\alpha_{3n} = 0, \\ \bar\beta_{1n} = R_1\eta_n, \bar\beta_{2n} = (R^2 - R_1^2)\eta_n - \xi_n, \bar\beta_{3n} = -RR_2\eta_n, \\ \bar\gamma_{1n} = -\dfrac{R_1R_2}{RP_2}\eta_n, \bar\gamma_{2n} = -\dfrac{R_2(R^2-R_1^2)}{RP_2}\eta_n, \bar\gamma_{3n} = \dfrac{R_2^2\eta_n - \xi_n}{P_2}. \end{cases} \tag{62}$$

The auxiliary system, governing the $n-$th component of the fields $(\varphi, \theta, \Phi_{2n})$ analogous to (47), can easily found to be

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_n = -\xi_n\varphi_n + \ 0 \quad + \quad 0 \ - \mathbf{U}_m\cdot\nabla\varphi_n, \\ \dfrac{\partial}{\partial t}\theta_n = \bar\beta_{1n}\varphi_n + \bar\beta_{2n}\theta_n + \bar\beta_{3n}\Phi_{2n} - \mathbf{U}_m\cdot\nabla\theta_n, \\ \dfrac{\partial}{\partial t}\Phi_{2n} = \bar\gamma_{1n}\varphi_n + \bar\gamma_{2n}\theta_n + \bar\gamma_{3n}\Phi_{2n} - \mathbf{U}_m\cdot\nabla\Phi_{2n}. \end{cases} \tag{63}$$

Setting

$$\begin{cases} \mathcal{I}_n = \bar\beta_{2n} + \bar\gamma_{3n} = \eta_n\left[R^2 - R_1^2 + \dfrac{R_2^2}{P_2} - \left(1 + \dfrac{1}{P_2}\right)\dfrac{\xi_n^2}{a^2}\right], \\ \mathcal{A}_n = \bar\beta_{2n}\bar\gamma_{3n} + \bar\beta_{3n}\bar\gamma_{2n} = \eta_n\xi_n\left[-(R^2 - R_1^2) - R_2^2 - \dfrac{\xi_n^2}{a^2}\right], \end{cases} \tag{64}$$

$$\tilde V_n = \frac{1}{2}\int_\Omega \left\{\varphi_n^2 + \mathcal{A}_n(\theta_n^2 + \Phi_{2n}^2) + (\bar\beta_{2n}\Phi_{2n} - \bar\gamma_{2n}\theta_n)^2 + (\bar\beta_{3n}\Phi_{2n} - \bar\gamma_{2n}\theta_n)^2\right\} ds \tag{65}$$

it follows that

$$\frac{d\tilde{V}_n}{dt} \leq \frac{1}{2}\int_\Omega \left[-\xi_n\varphi_n^2 + \mathcal{I}_n\mathcal{A}_n(\theta_n^2 + \Phi_{2n}^2)\right] d\Omega \tag{66}$$

with

$$\xi_n = a^2 + n^2\pi^2 \geq \pi^2. \tag{67}$$

Therefore

$$\begin{cases} \mathcal{I}_n < 0 \Leftrightarrow R^2 < R_1^2 - \dfrac{R_2^2}{P_2} + 4\pi^2\left(1 + \dfrac{1}{P_2}\right) \\ \\ \mathcal{A}_n > 0 \Leftrightarrow R^2 < R_1^2 - R_2^2 + 4\pi^2 \end{cases} \tag{68}$$

i.e. (7) guarantee the global stability.

Remark 3. We remark that:

1) The same procedure can be applied for obtaining (8)-(9) [29];

2) in the cases $P_2 \geq 1$, $P_1 \leq 1$, (7)-(8) reduce to

$$R_C^2 = R_1^2 - R_2^2 + 4\pi^2. \tag{69}$$

Acknowledgements. This paper has been performed under the auspices of G.N.F.M. of I.N.D.A.M. and was supported in part from the Leverhulm Trust, “Tipping points: mathematics metaphors and meanings”.

References

[1] B. Straughan: *Stability and wave motion in porous media*. Appl. Math. Sci., **165**, Springer (2008).

[2] D.A. Nield, A. Bejan: Convection in Porous Media. Springer-Verlag, Berlin Heidelberg New York, (1999).

[3] J. N. Flavin, S. Rionero: Qualitative estimates for partial differential equations: an introduction. Boca Raton (FL) : CRC Press, (1996);

[4] S. Lombardo, G. Mulone, S. Rionero: *Global Stability of the Bénard problem for a mixture with superimposed plane parallel shear flows*. Math. Methods Appl. Sci. **23** (2000) 1447.

[5] S. Lombardo, G. Mulone, S. Rionero: *Global nonlinear exponential stability of the conduction-diffusion solution for Schmidt numbers greater than Prandtl numbers*. J. Math. Anal. Appl. **262** (2001) 1229.

[6] G. Mulone: *On the nonlinear stability of a fluid layer of a mixture heated and salted from below*. Contin. Mech. Thermodyn. **6** (1994) 161.

[7] S. Lombardo, G. Mulone, B. Straughan: *Nonlinear stability in the Bénard problem for a double-diffusive mixture in a porous medium*. Math. Methods Appl. Sci. **24** (2001) 1229.

[8] G. Mulone, S. Rionero: *Unconditional nonlinear exponential stability in the Bénard problem for a mixture: Necessary and sufficient conditions*. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei **9** (1998) 221.

[9] F. Capone, S. Rionero: *Nonlinear stability of a convective motion in a porous layer driven by horizontally periodic temperature gradient*. Contin. Mech. Thermodyn. **15** (2003) 529.

[10] S. Lombardo, G. Mulone: *Nonlinear stability convection for laminar flows in a porous medium with Brinkman law*. Math. Methods Appl. Sci. **26** (2003) 453.

[11] F. Capone, S. Rionero: *On the instability of double diffusive convection in porous media under boundary data periodic in space*, in: S. Rionero, G. Romano (Eds.), Trends and Applications of Mathematics to Mechanics, STAMM, 2002, Springer, 2004, pp. 1-8.

[12] F. Capone, M. Gentile, S. Rionero: *Influence of linear concentration heat source and parabolic density on penetrative convection onset*, in: R. Monaco, G. Mulone, S. Rionero, T. Ruggeri (Eds.), Proceedings "Wascom 2005" 13th Conference on Waves and Stability in Continuum Media, World Scientific, 2006, pp. 77-82.

[13] F. Capone, M. Gentile, S. Rionero: *On penetrative convection in porous media driven by quadratic sources*, in: R. Monaco, G. Mulone, S. Rionero, T. Ruggeri (Eds.), Proceedings "Wascom 2005" 13th Conference on Waves and Stability in Continuum Media, World Scientific, 2006, pp. 83-88.

[14] S. Lombardo, G. Mulone: *Necessary and sufficient conditions of global nonlinear sta- bility for rotating double-diffusive convection in a porous medium*. Continuum Mech. Thermodyn. **14**: 527-540, 2002.

[15] S. Rionero: *A new approach to nonlinear L^2 -stability of double diffusive convection in porous media: Necessary and sufficient conditions for global stability via a linearization principle*. J. Math. Anal. Appl. **333**, 1036-1057 (2007).

[16] S. Rionero, L. Vergori: *Long time behaviour of fluid motions in porous media in the presence of Brinkman law*. Acta Mechanica, Springer Wien, **210**, no. 2-3, pp.221-240 (2009).

[17] A. A. Hill, S. Rionero, B. Straughan: *Global stability for penetrative convection with throughflow in a porous material*. IMA J. Appl. Math. **72** (2007), no. 5, 635–643.

[18] F. Capone, M. Gentile, A. Hill: *Anisotropy and symmetry in porous media convection*. Acta Mechanica, Springer Wien, **208**,no. 3-4, pp. 205-214 (2009).

[19] B. Straughan: *Oscillatory convection and the Cattaneo law of heat conduction*. Ricerche mat. **58**, pp.157-162 (2009);

[20] K. R. Rajagopal, G. Saccomandi, L. Vergori: *Stability analysis of the Rayleigh-Bénard convection for a fluid with temperature and pressure dependent viscosity*. Z. Angew. Math. Phys. **60**, no. 4, pp. 739–755 (2009).

[21] S. Rionero: *A new approach to nonlinear L^2-stability of double diffusive convection in porous media: necessary and sufficient conditions for global stability via a linearization principle*. J. Math. Anal. Appl. **333**, no. 2, pp. 1036-1057, (2007).

[22] S. Rionero: *Global nonlinear stability in double diffusive convection via hidden symmetries*. Int. J. Nonlinear Mech. **47**, 61-66, (2012).

[23] A.J. Pearlstein, R.M: Harris, G. Terrones: *The onset of convective instability in a triply diffusive fluid layer*. J.Fluid Mech. **202**, pp.443-465 (1989).

[24] R.A. Noutly, D.G. Leaist: *Quaternary diffusion in acqueous $KCl - KH_2PO_4 - H_3PO_4$ mixtures*. J. Phys. Chem. **91**, pp. 1655-1658 (1987).

[25] J. Tracey, *Multi-component convection-diffusion in a porous medium.* Continuum Mech. Thermodyn. vol.**8**, pp. 361-381 (1996);

[26] B. Straughan, DW Walker, *Multi-component convection-diffusion and penetrative convection.* Fluid Dyn. Res. vol.**19**, pp. 77-89 (1997);

[27] B. Straughan, J. Tracey, *Multi-component convection-diffusion with internal heating or cooling.* Acta Mechanica vol.**133**,pp. 219-239 (1999);

[28] S. Rionero: *Long time behaviour of multicomponent fluid mixture in porous media*, Journal of Engineering Science, **48**, (2010), 1519-1533.

[29] S. Rionero: *Global nonlinear stability for a triply diffusive-convection in a porous layer.* In press on Continuum Mechanics and Thermodynamics DOI: 10.1007/s00161-011-0219-4O .

[30] S. Rionero: *A peculiar Liapunov functional for ternary reaction-diffusion dynamical systems* Boll. U.M.I (9) IV (2011), 393-407.

[31] D.R. Merkin: Introduction to the theory of stability. Text in Applied Mathematic v. **24**, Springer (1997);

[32] F. R. Gantmacher: Lezioni di Meccanica Analitica. Editori riuniti, pp. 194-194, (1980).

[33] S. Rionero: *Absence of subcritical instabilities in triply diffusive-convection in porous layers via a new strategy* (submitted).