Onset of convection in porous layers salted from above and below

Salvatore Rionero
University of Naples Federico II,
Department of Mathematics and Applications ‘R. Caccioppoli’,
Complesso Universitario Monte S. Angelo, Via Cinzia, 80126 Naples, Italy.
rianero@unina.it

Abstract. For a porous layer heated from below and salted from above and below, the non existence of subcritical instabilities and conditions of global stability - for special values of the Prandtl numbers - are found.

Keywords: Multi-component fluid mixtures, Porous media, Convection, Stability.

1 Introduction and aims

A porous medium is schematized via a body (generally rigid and called skeleton) having interconnected pores everywhere. Generally, the fluid occupying the pores is a mixture since are dissolved in chemical species (“salts”) and the layer is embedded in a temperature field.

The behaviour of convective-diffusive fluid mixtures in porous layer presents a picture of behaviours increasing with the number of components. Although the subject of double-diffusive convection is still a very active research area [1]-[22], the same subject with more than two components, although more difficult – in the past as nowadays has also attracted the attention of many authors [23]-[30].

The present paper is concerned with an horizontal layer heated from below and salted by two salts, either from below (“salt 1”) or from above (“salt 2”).
Denoting by
\[
\begin{align*}
T &= \text{temperature field,} \quad C_i = \text{concentration field of the “salt} \ i', (i = 1, 2) \\
v &= \text{seepage velocity,} \quad p = \text{pressure field,}
\end{align*}
\]
in the case of the boundary conditions
\[
\begin{align*}
T(0) &= T_1, \quad T(d) = T_2, \quad T_1 - T_2 > 0, \\
C_i(0) &= C_{id}, \quad C_i(d) = C_{iu}, \quad (i = 1, 2), \\
v \cdot k &= 0, \quad \text{at} \ z = 0, \ z = d,
\end{align*}
\]
it follows that the perturbations \((u, \Phi_1, \Phi_2, \Pi)\) to the conduction solution \{cfr. \cite{29}\}, with
\[
\begin{align*}
\begin{align*}
u &= \text{perturbation to} \ v, \\
\Phi_i &= \text{perturbation to} \ C_i, \\
\Pi &= \text{perturbation to} \ p,
\end{align*}
\end{align*}
\]
are governed by
\[
\begin{align*}
\nabla \Pi &= -u + (R\theta - R_1 \Phi_1 - R_2 \Phi_2)k, \\
\nabla \cdot u &= 0, \\
\theta_t + u \cdot \nabla \theta &= R\omega + \Delta \theta, \\
P_1(\Phi_{1t} + u \cdot \nabla \Phi_1) &= R_1 \omega + \Delta \Phi_1, \\
P_2(\Phi_{2t} + u \cdot \nabla \Phi_2) &= -R_2 \omega + \Delta \Phi_2,
\end{align*}
\]
\[
\omega = \theta = \Phi_1 = \Phi_2 = 0, \quad \text{on} \ z = 0, \ z = 1,
\]
with
\[
\begin{align*}
\begin{align*}
\omega &= u \cdot k, \\
R &= \text{Rayleigh thermal number,} \\
R_i &= \text{Rayleigh concentration number of “salt} \ i'.
\end{align*}
\end{align*}
\]
Assuming - as it is normally done - that
i) \(u = (u, v, \omega), \theta, \Phi_1, \Phi_2\) are periodic in the \(x\) and \(y\) directions respectively of periods \(2\pi/a_x, 2\pi/a_y\);
ii) \(\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]\) is the periodicity cell;
iii) $u, \theta, \Phi_1, \Phi_2$ belong to $W^{2,2}(\Omega)$ and are such that all their first derivatives and second spatial derivatives can be expanded in Fourier series uniformly convergent in $\Omega$.

our aim - according to the results obtained in $[[29]-[30],[33]]$ - is to show that when the layer is salted from below by “salt 1” and from above by “salt 2”, then

1) do not exist subcritical instabilities;

2) exist physically relevant values of the “salts” Prandtl numbers such that the triply diffusive-convection can be reduced rigorously to the double diffusive-convection and the global stability condition is given by $R^2 < R_c^2$ with

$$R_c^2 = \min \left\{ \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + 4\pi^2 \left( 1 + \frac{1}{P_1} \right), R_1^2 - R_2^2 + 4\pi^2 \right\},$$  \hspace{1cm} (7)$$

$$R_c^2 = \min \left\{ \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + 4\pi^2 \left( 1 + \frac{1}{P_1} \right), R_1^2 - R_2^2 + 4\pi^2 \right\},$$  \hspace{1cm} (8)$$

$$R_c^2 = \min \left\{ \frac{1}{P_1^2} (R_1^2 - R_2^2) + 4\pi^2 \left( 1 + \frac{1}{P_1} \right), R_1^2 - R_2^2 + 4\pi^2 \right\},$$  \hspace{1cm} (9)$$

Section 2 is devoted to the boundary value problem of the problem at stake. The (Routh Hurwitz) conditions of linear stability are found in the subsequent Section while Sections 4-5 are devoted to the non existence of subcritical instabilities. Finally, (7)-(9) are shown in Section 6.

2 The boundary value problem at stake

We recall here a basic theorem - which proof is given either in [29] or [31] - concerned with the main boundary value problem $(4)_1-(4)_2-(5)$.

**Theorem 1.** Let $(u, \theta, \Phi_1, \Phi_2)$ be solution of the b.v.p.

$$\begin{align*}
\nabla \Pi &= -u + (R\theta - R_1\Phi_1 - R_2\Phi_2)k, \\
\nabla \cdot u &= 0,
\end{align*}$$  \hspace{1cm} (10)$$
\[ \omega = \theta = \Phi_1 = \Phi_2 = 0, \quad z = 0, 1, \quad (11) \]

then:

i) \((u, \theta, \Phi_1, \Phi_2)\) is solution of the b.v.p.

\[
\begin{align*}
\Delta \omega &= \Delta_1 (R\theta - R_1 \Phi_1 - R_2 \Phi_2), \quad \text{in } \Omega, \\
u &= \theta = \Phi_1 = \Phi_2 = 0, \quad \text{on } z = 0, 1, \\
\Delta_1 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. 
\end{align*}
\quad (12)
\]

ii) a complete orthogonal system of solutions of (10) is given by

\[
\begin{align*}
\tilde{\omega}_n &= \eta_n (R\tilde{\theta}_n - R_1 \tilde{\Phi}_1n - R_2 \tilde{\Phi}_2n), \\
u_n &= \frac{1}{a^2} \left( \frac{\partial^2\tilde{\omega}_n}{\partial x \partial z} + \frac{\partial^2\tilde{\omega}_n}{\partial y \partial z} \right) + \tilde{\omega}_nk, 
\end{align*}
\quad (13)
\]

with

\[
\begin{align*}
a^2 &= a_x^2 + a_y^2, \quad \xi_n = a^2 + n^2\pi^2, \quad \eta_n = \frac{a^2}{\xi_n}, \\
\omega &= \sum_{1}^{\infty} \hat{\omega}_n = \sum_{1}^{\infty} \omega_n(x, y, t) \sin(n\pi z), \\
u &= \sum_{1}^{\infty} u_n = \sum_{1}^{\infty} (\hat{u}_nk + \hat{v}_nj + \hat{\omega}_nk), \\
\theta &= \sum_{1}^{\infty} \hat{\theta}_n = \sum_{1}^{\infty} \theta_n(x, y, t) \sin(n\pi z), \\
\Phi_i &= \sum_{1}^{\infty} \tilde{\Phi}_in = \sum_{1}^{\infty} \Phi_{in}(x, y, t) \sin(n\pi z).
\end{align*}
\quad (14)
\]

Remark 1. By virtue of theorem 1 the independent scalar unknown fields \((u, v, \omega, \theta, \Phi_1, \Phi_2)\) are reduced to \((\theta, \Phi_1, \Phi_2)\).

3 Linear instability

Linearizing (4) and taking into account that by virtue of theorem 1, it turns out that

\[ \Delta \varphi = -\sum_{1}^{\infty} \xi_n \varphi_n, \quad \varphi \in (\theta, \Phi_1, \Phi_2), \quad (15) \]
one obtains

\[ \begin{align*}
\theta_t &= \sum_{n=1}^{\infty} (a_1 n \theta_n + a_2 n \Phi_1 n + a_3 n \Phi_2 n), \\
\Phi_{1t} &= \sum_{n=1}^{\infty} (b_1 n \theta_n + b_2 n \Phi_1 n + b_3 n \Phi_2 n), \\
\Phi_{2t} &= \sum_{n=1}^{\infty} (c_1 n \theta_n + c_2 n \Phi_1 n + c_3 n \Phi_2 n),
\end{align*} \tag{16} \]

with

\[ \begin{align*}
a_1 n &= R^2 \eta_n - \xi_n, & a_2 n &= -RR_1 \eta_n, & a_3 n &= -RR_2 \eta_n, \\
b_1 n &= RR_1 P \eta_n, & b_2 n &= -\left(\frac{R_1^2 \eta_n + \xi_n}{P_1}\right), & b_3 n &= -\frac{R_1 R_2}{P_1} \eta_n, \\
c_1 n &= \frac{R_2}{P_2} \eta_n, & c_2 n &= \frac{R_1 R_2}{P_2} \eta_n, & c_3 n &= \frac{R_2^2 \eta_n - \xi_n}{P_2}.
\end{align*} \tag{17} \]

Setting

\[ \begin{align*}
\theta_n &= \xi_0 n(t) F(x,y) \sin(n \pi z), \\
\Phi_{in} &= \xi_{in} n(t) F(x,y) \sin(n \pi z),
\end{align*} \tag{18} \]

by virtue of

\[ \int_0^1 \sin(n \pi z) \sin(m \pi z) \, dz = \begin{cases} 0, & n \neq m \\ \frac{1}{2}, & n = m \end{cases} \tag{19} \]

(16) implies

\[ \begin{align*}
\frac{d \xi_{0n}}{dt} &= a_1 n \xi_0 n + a_2 n \xi_1 n + a_3 n \xi_2 n, \\
\frac{d \xi_{1n}}{dt} &= b_1 n \xi_0 n + b_2 n \xi_1 n + b_3 n \xi_2 n, \\
\frac{d \xi_{2n}}{dt} &= c_1 n \xi_0 n + c_2 n \xi_1 n + c_3 n \xi_2 n.
\end{align*} \tag{20} \]

Setting

\[ \mathcal{L}_n = \begin{pmatrix} a_1 n & a_2 n & a_3 n \\ b_1 n & b_2 n & b_3 n \\ c_1 n & c_2 n & c_3 n \end{pmatrix} \]

it easily follows that the characteristic equation of the \( \mathcal{L}_n \) eigenvalues \( \lambda_n \) is

\[ \lambda_n^3 - I_{1n} \lambda_n^2 + I_{2n} \lambda_n - I_{3n} = 0, \tag{21} \]
with $I_{1n}, I_{2n}, I_{3n}$ given by

$$
\begin{align*}
I_{1n} &= a_{1n} + b_{2n} + c_{3n} = \lambda_{1n} + \lambda_{2n} + \lambda_{3n}, \\
I_{2n} &= \begin{vmatrix}
    a_{1n} & a_{2n} \\
    b_{1n} & b_{2n}
\end{vmatrix} + \begin{vmatrix}
    a_{1n} & a_{3n} \\
    c_{1n} & c_{3n}
\end{vmatrix} + \begin{vmatrix}
    b_{2n} & b_{3n} \\
    c_{2n} & c_{3n}
\end{vmatrix}, \\
I_{3n} &= \begin{vmatrix}
    a_{1n} & a_{2n} & a_{3n} \\
    b_{1n} & b_{2n} & b_{3n} \\
    c_{1n} & c_{2n} & c_{3n}
\end{vmatrix}.
\end{align*}
$$

(22)

By virtue of the Routh-Hurwitz stability-conditions [31], the following theorem holds.

**Theorem 2.** The conduction solution is linearly stable if and only if, $\forall n \in \mathbb{N}$, the inequalities

$$I_{1n} < 0, I_{3n} < 0, I_{1n}I_{2n} - I_{3n} < 0,$$

(23)

hold.

Since (23) requires $I_{2n} > 0$, in view of

$$
\begin{align*}
I_{1n} &= R^2 - \frac{R_2^2}{P_1} + \frac{R_1^2}{P_2} - \left(1 + \frac{1}{P_1} + \frac{1}{P_2}\right) R^2 R_1 - \frac{1}{P_2} R_2^2 + \\
I_{2n} &= \frac{1}{P_1} \left(1 + \frac{1}{P_2}\right) R^2 \xi_n \eta_n, \\
I_{3n} &= \frac{1}{P_1 P_2} \left(R^2 - \frac{R_2^2}{P_1} + \frac{R_1^2}{P_2} - \frac{\xi_n}{\eta_n}\right) \eta_n \xi_n, \\
\inf_{(a^2,n) \in \mathbb{R}^+ \times \mathbb{N}} \frac{\xi_n}{\eta_n} &= 4\pi^2,
\end{align*}
$$

(24)

it follows that

**Theorem 3.** The conduction solution is linearly stable only if

$$R_{C_i}^2 > 0, (i = 1, 2, 3), \quad R^2 < R_{C_i}^2,$$

(25)
with

\[
\begin{align*}
R_{C1}^2 &= R_1^2 - \frac{R_2^2}{P_1} + 4\pi^2 \left(1 + \frac{1}{P_1} + \frac{1}{P_2}\right), \\
R_{C2}^2 &= \frac{1}{P_1 + P_2} \left[(1 + P_2)R_1^2 - (1 + P_1)R_2^2 + 4\pi^2(1 + P_1 + P_2)\right], \\
R_{C3}^2 &= R_1^2 - R_2^2 + 4\pi^2, \\
R_C^2 &= \min(R_{C1}^2, R_{C2}^2, R_{C3}^2).
\end{align*}
\] (26)

4 Preliminaries to nonexistence of subcritical instabilities

Lemma 1. Let $\lambda_1$ be a real eigenvalue of the matrix

\[
L = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}, \quad (\alpha_{ij} = \text{const} \in \mathbb{R}) \quad i, j = 1, 2, 3
\] (27)

and let $\tilde{U} = (1, \tilde{U}_2, \tilde{U}_3)$ be an associated eigenvector. Then the transformation

\[
X = L_1 Z,
\] (28)

with

\[
X = (X_1, X_2, X_3)^T, \quad Z = (Z_1, Z_2, Z_3)^T, \quad L_1 = \begin{pmatrix} 1 & 0 & 0 \\
\tilde{U}_2 & 1 & 0 \\
\tilde{U}_3 & 0 & 1 \end{pmatrix}
\] (29)

reduces the ternary system

\[
\frac{dX}{dt} = LX + F,
\] (30)

with $(F)_{X=0} = 0$, to

\[
\frac{dZ}{dt} = \tilde{L} Z + \tilde{F},
\] (31)

\[
\tilde{L} = \begin{pmatrix}
\lambda_1 & \tilde{\alpha}_{12} & \tilde{\alpha}_{13} \\
0 & \tilde{\alpha}_{22} & \tilde{\alpha}_{23} \\
0 & \tilde{\alpha}_{32} & \tilde{\alpha}_{33}
\end{pmatrix}, \quad \tilde{F} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3)^T.
\] (32)

\[
\begin{align*}
\begin{cases}
Z_1 &= X_1, Z_2 = X_2 - \tilde{U}_2 X_1, Z_3 = X_3 - \tilde{U}_3 X_1, \\
\tilde{F}_1 &= F_1, \tilde{F}_2 &= F_2 - \tilde{U}_2 F_1, \tilde{F}_3 &= F_3 - \tilde{U}_3 F_1
\end{cases}
\end{align*}
\] (33)
\[
\begin{align*}
\dot{\alpha}_{12} &= \alpha_{12}, \quad \dot{\alpha}_{13} = \alpha_{13}, \\
\dot{\alpha}_{22} &= \alpha_{22} - \bar{U}_2\alpha_{12}, \quad \dot{\alpha}_{23} = \alpha_{23} - \bar{U}_2\alpha_{13}, \\
\dot{\alpha}_{32} &= \alpha_{32} - \bar{U}_3\alpha_{12}, \quad \dot{\alpha}_{33} = \alpha_{33} - \bar{U}_3\alpha_{13}
\end{align*}
\] (34)

\[
\dot{\alpha}_{22} + \dot{\alpha}_{33} = \lambda_2 + \lambda_3, \quad \dot{\alpha}_{22}\dot{\alpha}_{33} - \dot{\alpha}_{23}\dot{\alpha}_{32} = \lambda_2\lambda_3,
\] (35)

\(\lambda_2, \lambda_3\) being other eigenvalues of \(L\).

**Proof.** The proof, based on [32, pp.194-197], can be found in [30], [33].

**Lemma 2.** Let the eigenvalues \(\lambda_i, (i = 1, 2, 3)\) of

\[
L = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}
\] (36)

have negative real part and let \(\lambda_1 < 0\) and

\[
\Psi = (\hat{A}_1Z_2 - \hat{A}_3Z_3)\hat{F}_2 + (\hat{A}_2Z_3 - \hat{A}_3Z_2)\hat{F}_3 + Z_1\hat{F}_1 = 0,
\] (37)

with

\[
\begin{align*}
\hat{A}_1 &= \lambda_2\lambda_3 + \tilde{\alpha}_{32}^2 + \tilde{\alpha}_{33}^2, \\
\hat{A}_2 &= \lambda_2\lambda_3 + \tilde{\alpha}_{22}^2 + \tilde{\alpha}_{23}^2, \\
\hat{A}_3 &= \tilde{\alpha}_{22}\tilde{\alpha}_{32} + \tilde{\alpha}_{23}\tilde{\alpha}_{33}.
\end{align*}
\] (38)

Then the function

\[
\tilde{W} = \frac{1}{2} [Z_1^2 + \lambda_2\lambda_3(Z_2^2 + Z_3^2) + (\tilde{\alpha}_{22}Z_3 - \tilde{\alpha}_{32}Z_2)^2 + (\tilde{\alpha}_{23}Z_3 - \tilde{\alpha}_{33}Z_2)^2],
\] (39)

has - along

\[
\frac{dZ}{dt} = \tilde{L}Z + \tilde{F},
\] (40)

the temporal derivative given by

\[
\frac{d\tilde{W}}{dt} = \frac{1}{2} [\lambda_1Z_1^2 + (\lambda_2 + \lambda_3)\lambda_2\lambda_3(Z_2^2 + Z_3^2)] < 0.
\] (41)

The null solution of (40) and hence of (30) is globally stable and subcritical instabilities do not exist.

**Proof.** A detailed proof can be found in [30], [33].

**Remark 2.** The eigenvalues of the matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix}
\]
Onset of convection in porous layers salted from above and below

have negative real part if and only if

\[ a_{11} < 0, \quad I = a_{22} + a_{33} < 0, \quad A = a_{22}a_{33} - a_{32}a_{23}. \]

(cfr. Remark 2.5 of [30]).

5 Nonexistence of subcritical instabilities and global stability

Following [33], we set

\[
S_m^\theta = \sum_{n=1}^{m} \theta_n, \quad S_m^{\Phi_i} = \sum_{n=1}^{m} \Phi_{i,n}, \quad U_m = \sum_{n=1}^{m} u_n, \tag{42}
\]

\[
\theta = \lim_{m \to \infty} S_m^\theta, \quad \Phi_i = \lim_{m \to \infty} S_m^{\Phi_i}, \quad (i = 1, 2). \tag{43}
\]

The nonexistence of subcritical instabilities and the global stability is guaranteed by showing that the asymptotic stability of the null solution of

\[
\begin{cases}
\frac{d}{dt} S_m^\theta = \sum_{n=1}^{m} (a_{1n}\theta_n + a_{2n}\Phi_{1,n} + a_{3n}\Phi_{2,n}) - U_m \cdot \nabla S_m^\theta, \\
\frac{d}{dt} S_m^{\Phi_1} = \sum_{n=1}^{m} (b_{1n}\theta_n + b_{2n}\Phi_{1,n} + b_{3n}\Phi_{2,n}) - U_m \cdot \nabla S_m^{\Phi_1}, \\
\frac{d}{dt} S_m^{\Phi_2} = \sum_{n=1}^{m} (c_{1n}\theta_n + c_{2n}\Phi_{1,n} + c_{3n}\Phi_{2,n}) - U_m \cdot \nabla S_m^{\Phi_2},
\end{cases} \tag{44}
\]

\[
\begin{align*}
\theta_n &= \Phi_{1,n} = \Phi_{2,n} = 0, \quad \forall n \in \{1, \ldots, m\} \text{ for } z = 0, 1. \tag{45} \\
\theta_n(0) &= \Phi_{1,n}(0) = \Phi_{2,n}(0), \quad i = 1, 2, n \in \{1, \ldots, m\}. \tag{46}
\end{align*}
\]

\( \forall m \in \mathbb{N}, \) is guaranteed by (23). On the other hand, introducing the evolution system governing the \( n \)-th component of the of the perturbation \( (\theta, \Phi_1, \Phi_2) \)

\[
\begin{cases}
\frac{\partial}{\partial t} \theta_n = a_{1n}\theta_n + a_{2n}\Phi_{1,n} + a_{3n}\Phi_{2,n} - U_m \cdot \nabla \theta_n, \\
\frac{\partial}{\partial t} \Phi_{1,n} = b_{1n}\theta_n + b_{2n}\Phi_{1,n} + b_{3n}\Phi_{2,n} - U_m \cdot \nabla \Phi_{1,n}, \\
\frac{\partial}{\partial t} \Phi_{2,n} = c_{1n}\theta_n + c_{2n}\Phi_{1,n} + c_{3n}\Phi_{2,n} - U_m \cdot \nabla \Phi_{2,n},
\end{cases} \tag{47}
\]
\[
U_m = \sum_{n=1}^{m} u_n, u_n = \frac{1}{a^2} \left( \frac{\partial^2 \omega_n}{\partial x \partial z} i + \frac{\partial^2 \omega_n}{\partial y \partial z} j + \omega_n k \right),
\]

\[
\omega_n = \tilde{\omega}_n(x, y, t) \sin(n \pi z), \theta_n = \tilde{\theta}_n(x, y, t) \sin(n \pi z),
\]

\[
\Phi_{in} = \tilde{\Phi}_{in}(x, y, t) \sin(n \pi z), i = 1, 2, n \in \{1, \ldots, m\} \tag{48}
\]

(44) are immediately obtained by adding with respect to \( n \) from \( n = 1 \) to \( n = m \) each equations of (47). Therefore the nonexistence of subcritical instabilities and the global stability is guaranteed by showing that (23) imply the asymptotic stability of the null solution of (45)-(48) \( \forall m \in \mathbb{N} \).

**Theorem 4.** Let (23) hold. Then, for any \( m \in \mathbb{N} \), the zero solution of (45)-(48) is asymptotically stable for any initial data

**Proof.** Denoting by \( \lambda_{ni} \), \( i = 1, 2, 3 \), the roots of (21), (23) guarantee that \( \lambda_{ni} \) have negative real part and at least one - say \( \lambda_{n1} \) - be a negative real number. Then denoting by \( \tilde{U}_n = (1, \tilde{U}_{n2}, \tilde{U}_{n3}) \) an eigenvector associated to \( \lambda_{n1} \), Lemma 1 can be applied by setting

\[
\left\{
\begin{aligned}
Z_{n1} &= \Phi_{n1} - \tilde{U}_{n2} \theta_n, \\
Z_{n2} &= \Phi_{n2} - \tilde{U}_{n3} \theta_n, \\
Z_{n3} &= \Phi_{n3} - \tilde{U}_{n4} \theta_n,
\end{aligned}
\right.
\]

\[
F_1 = U_m \cdot \nabla \theta_n, \\
F_2 = U_m \cdot \nabla \Phi_{n1}, \\
F_3 = U_m \cdot \nabla \Phi_{n2},
\]

\[
\tilde{F}_1 = F_1, \\
\tilde{F}_2 = F_2 - \tilde{U}_{n2} F_1, \\
\tilde{F}_3 = F_3 - \tilde{U}_{n3} F_1
\]

Let

\[
\tilde{V}_n = \int_{\Omega} \tilde{W}_n d\Omega, \tag{50}
\]

with \( \tilde{W}_n \) given by (39) with \( \lambda_{ni} \) at the place of \( \lambda_i \). Then - instead of (41) - one has to show that

\[
\tilde{\Psi} = \langle \tilde{A}_{1n} Z_{n2} - \tilde{A}_{3n} Z_{n3}, \tilde{F}_2 \rangle + \langle \tilde{A}_{2n} Z_{n3} - \tilde{A}_{3n} Z_{n2}, \tilde{F}_3 \rangle + \langle Z_{n1}, \tilde{F}_1 \rangle = 0,
\]

with \( \tilde{A}_{1n}, \tilde{A}_{2n}, \tilde{A}_{3n} \) constants.

In view of (49) it follows that \( Z_{ni}, (i = 1, 2, 3) \), is of the kind

\[
Z_{ni} = \tilde{Z}_{ni}(x, y, t) \sin(n \pi z), \tag{52}
\]

\[
U_m \cdot \nabla \theta_n = \sum_{p=1}^{m} \left[ \frac{p \pi}{a^2} \left( \frac{\partial \tilde{\omega}_p}{\partial x} \frac{\partial \tilde{\theta}_n}{\partial x} + \frac{\partial \tilde{\omega}_p}{\partial y} \frac{\partial \tilde{\theta}_n}{\partial y} \right) \cos(p \pi z) \sin(n \pi z) + n \pi \tilde{\omega}_p \tilde{\theta}_p \sin(p \pi z) \cos(n \pi z) \right]
\]
Onset of convection in porous layers salted from above and below

and hence

\[ < Z_m, \tilde{F}_1 > = \frac{\tilde{\theta}_n \sin(n\pi z)}{U_{n1}}, U_m \cdot \nabla \theta_n >. \] \hspace{1cm} (54)

Since it easily turns out that

\[ \int_0^1 \sin(q\pi z) \cos(p\pi z) \sin(n\pi z)dz = 0, \text{for } p + q \neq n. \] \hspace{1cm} (55)

But one easily verifies that all the other scalar products appearing in \( \tilde{\Psi} \) are linear combinations of terms of kind (54) (with \( \Phi_m \) at the place of \( \theta_n \)) and hence by virtue of (55) it turn out that \( \tilde{\Psi} = 0. \)

6 Proof of (7)

In the case \( P_1 = 1 \), (4) reduces to

\[
\begin{align*}
\nabla p &= -u + (R\theta - R\Phi_1 - R\Phi_2)k, \\
\theta_t &= R\omega + \Delta \theta - u \cdot \nabla \theta, \\
\Phi_{1t} &= R\omega + \Delta \Phi_1 - u \cdot \Phi_1, \\
\Phi_{2t} &= -\frac{R_2}{P_2} \omega + \frac{1}{P_2} \Delta \Phi_1 - \frac{u}{P_2} \cdot \Phi_1.
\end{align*}
\] \hspace{1cm} (56)

Setting

\[ \varphi = R_1\theta - R\Phi_1 = 0 \Leftrightarrow \Phi_1 = \frac{1}{R_1}(R_1\theta - \varphi), \] \hspace{1cm} (57)

it follows that

\[
\begin{align*}
\nabla p &= -u + \left( \frac{R_2^2 - R_1^2}{R} \theta + \frac{R_1}{R} \varphi - R_2\Phi_2 \right) k, \\
\varphi_t &= \Delta \varphi - u \cdot \nabla \varphi, \\
\theta_t &= R\omega + \Delta \theta - u \cdot \nabla \theta, \\
\Phi_{2t} &= -\frac{R_2}{P_2} \omega + \frac{1}{P_2} \Delta \Phi_2 - \frac{u}{P_2} \cdot \Phi_2.
\end{align*}
\] \hspace{1cm} (58)

In view of theorem 1, one obtains

\[ \tilde{\omega}_n = \eta_n \left( \frac{R_2^2 - R_1^2}{R} \theta_n + \frac{R_1}{R} \varphi_n - R_2\Phi_{2n} \right), \] \hspace{1cm} (60)
and hence

\[
\begin{align*}
\varphi_t &= \sum_{n=1}^{\infty} \left( \bar{a}_{1n} \varphi_n + \bar{a}_{2n} \theta_n + \bar{a}_{3n} \Phi_{2n} \right) - \mathbf{u} \cdot \nabla \varphi, \\
\theta_t &= \sum_{n=1}^{\infty} \left( \bar{b}_{1n} \varphi_n + \bar{b}_{2n} \theta_n + \bar{b}_{3n} \Phi_{2n} \right) - \mathbf{u} \cdot \nabla \theta, \\
\Phi_{2t} &= \sum_{n=1}^{\infty} \left( \bar{c}_{1n} \varphi_n + \bar{c}_{2n} \theta_n + \bar{c}_{3n} \Phi_{2n} \right) - \mathbf{u} \cdot \nabla \Phi_2,
\end{align*}
\]

with

\[
\begin{align*}
\bar{a}_{1n} &= -\xi_n, \\
\bar{a}_{2n} &= \bar{a}_{3n} = 0, \\
\bar{b}_{1n} &= R_1 \eta_n, \\
\bar{b}_{2n} &= (R^2 - R_1^2) \eta_n - \xi_n, \\
\bar{b}_{3n} &= -RR_2 \eta_n, \\
\bar{c}_{1n} &= -\frac{R_1 R_2}{RP_2} \eta_n, \\
\bar{c}_{2n} &= -\frac{R_2 (R^2 - R_1^2)}{RP_2} \eta_n, \\
\bar{c}_{3n} &= \frac{R_2^2 \eta_n - \xi_n}{P_2}.
\end{align*}
\]

The auxiliary system, governing the \(n\)–th component of the fields \((\varphi, \theta, \Phi_{2n})\) analogous to (47), can easily found to be

\[
\begin{align*}
\frac{\partial}{\partial t} \varphi_n &= -\xi_n \varphi_n + 0 + 0 - \mathbf{U}_m \cdot \nabla \varphi_n, \\
\frac{\partial}{\partial t} \theta_n &= \bar{b}_{1n} \varphi_n + \bar{b}_{2n} \theta_n + \bar{b}_{3n} \Phi_{2n} - \mathbf{U}_m \cdot \nabla \theta_n, \\
\frac{\partial}{\partial t} \Phi_{2n} &= \bar{c}_{1n} \varphi_n + \bar{c}_{2n} \theta_n + \bar{c}_{3n} \Phi_{2n} - \mathbf{U}_m \cdot \nabla \Phi_{2n}.
\end{align*}
\]

Setting

\[
\begin{align*}
\mathcal{L}_n &= \bar{b}_{2n} + \bar{c}_{3n} = \eta_n \left[ R^2 - R_1^2 + \frac{R_2^2}{P_2} - \left( 1 + \frac{1}{P_2} \right) \frac{\xi_n^2}{a^2} \right], \\
\mathcal{A}_n &= \bar{b}_{2n} \bar{c}_{3n} + \bar{b}_{3n} \bar{c}_{2n} = \eta_n \xi_n \left[ -(R^2 - R_1^2) - \frac{R_2^2}{P_2} - \frac{\xi_n^2}{a^2} \right], \\
\tilde{V}_n &= \frac{1}{2} \int_{\Omega} \left\{ \varphi_n^2 + \mathcal{A}_n (\theta_n^2 + \Phi_{2n}^2) + (\bar{b}_{2n} \Phi_{2n} - \bar{c}_{2n} \theta_n)^2 + (\bar{c}_{3n} \Phi_{2n} - \bar{c}_{2n} \theta_n)^2 \right\} ds
\end{align*}
\]
Onset of convection in porous layers salted from above and below

it follows that

\[
\frac{d\tilde{V}_n}{dt} \leq \frac{1}{2} \int_{\Omega} \begin{bmatrix}
-\xi_n \varphi^2_n + I_n A_n (\theta^2_n + \Phi^2_n)
\end{bmatrix} d\Omega
\]  
(66)

with

\[
\xi_n = a^2 + n^2 \pi^2 \geq \pi^2.
\]  
(67)

Therefore

\[
\begin{cases}
I_n < 0 \iff R^2 < R_1^2 - \frac{R_2^2}{P_2} + 4\pi^2 \left(1 + \frac{1}{P_2}\right) \\
A_n > 0 \iff R^2 < R_1^2 - R_2^2 + 4\pi^2
\end{cases}
\]  
(68)

i.e. (7) guarantee the global stability.

\textbf{Remark 3.} We remark that:

1) The same procedure can be applied for obtaining (8)-(9) \cite{29};

2) in the cases \(P_2 \geq 1, P_1 \leq 1\), (7)-(8) reduce to

\[
R_C^2 = R_1^2 - R_2^2 + 4\pi^2.
\]  
(69)

\textbf{Acknowledgements.} This paper has been performed under the auspices of G.N.F.M. of I.N.D.A.M. and was supported in part from the Leverhulm Trust, “Tipping points: mathematics metaphors and meanings”.

\textbf{References}


Onset of convection in porous layers salted from above and below


[27] B. Straughan, J. Tracey, *Multi-component convection-diffusion with internal heating or cooling*. Acta Mechanica vol.133, pp. 219-230 (1999);


