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A mathematical model for environment evaluations in landscape ecology

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Abstract. The present paper proposes a mathematical model for ecological evaluations in environmental systems. An application to a relevant area in the district of Viterbo is then given through numerical tests.

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Introduction

Landscape ecology [1],[2],[3] is a rather new field of research needing integration between theoretical development, empirical testing and mathematical modelling. Landscape is a complex system characterized by flows of energy and materials between its components, called here *Landscape Units* (LU). In general, a landscape responds stably only to a limited range of small perturbations, but more frequently can show significant environmental modifications [4]. In this context mathematical models of evolution may be useful tools to give information about the trend towards future scenarios of the environment under investigation [5],[6].

The state of an environment is well represented by a spatial model called *ecological graph* [7], determinable by the *Geographic Information System* (GIS), which furnishes all the parameters to be inserted in the evolution model. In fact an ecological graph provides data relative to the production of biological energy, due to the biomass present in each LU, and to transmission of such an energy to the neighbor LUs.

A first attempt of an evolution model has been proposed in paper [8], with applications to a district of Cremona (north Italy). The behavior of this model was not completely satisfactory, so that a modified version [9] was proposed with applications in the region of Cuneo (north Italy) and in that of Viterbo (central Italy) [10].

The state variables of these models are given by two quantities, the former, M, proportional to the *Biological Territorial Capacity* (BTC) [3],[7], evaluating at the same time production and diffusivity of bio-energy, the latter, V, determined by the percentage of soil surface characterized by green with high value of BTC. In the present paper we study a new model that takes into account not only the evolution of these state variables in the whole environmental system but also those in each LU. In the last section a numerical test for several LUs in the southwest of the district of Viterbo is presented and discussed.

Let us observe that a correct use of such simulations consists in comparing different choices of landscape planning in order to estimate possible future decisions.

1 Determination of the model parameters

In this section the main parameters characterizing an environmental system will be defined. Such parameters can be directly obtained using a suitable software applied to the GIS.

An environmental system is represented as a territory subdivided in a given number n of LUs, separated from each other by *barriers*. Examples of barriers are railroads, freeways, local roads, compact edified grounds, urban sprawl, rivers, ridges,... According to the book [7] each barrier is classified by an index of permeability $p \in [0,1]$, $p = 0$ as complete impermeability and $p = 1$ as complete permeability. Each LU is then divided into other patches, called *biotopes*, classified, again by the above book, according to the use of its soil; each biotope is characterized by its vegetation and each kind of vegetation by bio-energy production, defined by the BTC index $B^b \in [0, B^b_{max}]$, measured in $Mcal/m^2/year$ with $B^b_{max} = 6.5$ [7],[11], which corresponds to oak or coniferous forests.

The mean value of BTC of the i-th LU, $i = 1, \dots, n$, is given by the following formula

$$B_{i0} = \frac{1}{m_i} \sum_{j=1}^{m_i} B^b_{ji}\, s_{ji} \tag{1}$$

where s_{ji} is the area of the j-biotope, $j = 1, \dots, m_i$, belonging to the i-th LU, and having BTC index B^b_{ji}.

In [7], instead of using simply the values of BTC, it is suggested to consider a generalized bio-energy in order to include in each LU, beside the energy production, also its capability to be transmitted into the other neighbor LUs.

Thus a new quantity $\mathcal{M}_{i0}$ can be defined depending on several morphological and physical characters of the LUs, i.e.

$$\mathcal{M}_{i0} = (1 + \mathcal{K}_i) B_{i0}, \tag{2}$$

where $\mathcal{K}_i \in [0, 1]$ is a dimensionless environmental parameter which may augment the actual value B_{i0} of BTC. In particular $\mathcal{K}_i$ depends upon the border shape and permeability, the bio-diversity, the sun exposition and the relative humidity of the soil. For the actual formulas defining the parameter $\mathcal{K}_i$ the reader is addressed to paper [10].

Moreover the *maximum producible bio-energy* of each i-LU can be defined by

$$\mathcal{M}_i^{max} = 2B_{max}^b\, A_i,$$

where A_i is the area of the i-th LU.

Another quantity which can be recovered directly from the GIS is the area $\mathcal{V}_{i0}$ of soil surface characterized, in each LU, by a vegetation with high value of BTC, say $3.5 \leq B^b \leq 6.5$.

Next section considers the mathematical model object of this piece of research. It results to be represented by a system of $2n$ ordinary differential equations in the unknowns $\mathcal{M}_i$ and $\mathcal{V}_i$ for which $\mathcal{M}_{i0}$ and $\mathcal{V}_{i0}$ are the initial data.

2 The dynamical model

As already said, in paper [9] a mathematical model, considering as state variables the generalized bio-energy and the percentage of BTC high value areas for the whole environmental system, has been derived. Conversely, in this paper the state variables will be the same of the above model but defined at the level of each LU. Thus, the model we are proposing has almost the same mathematical structure of that of [9] but the time-dependent variables are now $\mathcal{M}_i(t)$ and $\mathcal{V}_i(t)$.

According to such a generalization, we write here the model equations for $i = 1, \ldots, n$

$$\begin{cases} \mathcal{M}_i'(t) = c_i(t)\mathcal{M}_i(t)\left(1 - \dfrac{\mathcal{M}_i(t)}{\mathcal{M}_i^{max}}\right) - \nu_i\left(1 - \dfrac{\mathcal{V}_i(t)}{A_i}\right)\mathcal{M}_i(t) \\ \\ \mathcal{V}_i'(t) = \dfrac{\mathcal{M}_i(t)}{\mathcal{M}_i^{max}}\,\mathcal{V}_i(t)\left(1 - \dfrac{\mathcal{V}_i(t)}{A_i}\right) - \mu_i U_i \mathcal{V}_i(t), \end{cases} \tag{3}$$

where ν_i is the ratio between the length of the impermeable barriers inside the i-th LU and that of the perimeter P_i of the whole LU; μ_i is the ratio between the perimeter of the edified areas (with $B^b = 0$) and P_i. Finally $U_i \in [0,1]$ is the ratio between the edified areas present in the i-th LU and its total surface.

Beside the fact that the equations on $\mathcal{V}_i$ and $\mathcal{M}_i$ are fully coupled, the main modification with respect to the model [9] regards the presence of the *connectivity index* $c_i(t)$ for which it is necessary now to state new definitions.

First of all let us define the flux between two neighbor LUs, say i and k. Such a flux will be given by

$$F_{ik} = \frac{\mathcal{M}_i + \mathcal{M}_k}{2(P_i + P_k)} \sum_{r=1}^{s} L_{ik}^r \, p^r, \tag{4}$$

where L_{ik}^r are the lengths of the LUs borders characterized by the permeability index $p^r \in [0,1]$. Indeed, the border itself is divided into s tracts which may obviously present different permeability. As already said, P_i and P_k are the perimeters of the two LUs. Moreover, it is necessary to define the absolute maximum flux F_{ik}^{max} between two LUs, i.e.

$$F_{ik}^{max} = \frac{\mathcal{M}_i^{max} + \mathcal{M}_k^{max}}{2(P_i + P_k)} \, L_{ik}, \tag{5}$$

L_{ik} being the length of their border, i.e. $L_{ik} = \sum_{r=1}^{s} L_{ik}^r$.

After these definitions the connectivity index between two LUs i and k, as well as the total connectivity index c_i between the i-th LU and all its neighbors, are defined by the following formulas

$$c_{ik} = \frac{F_{ik}}{F_{ik}^{max}} = \frac{\mathcal{M}_i + \mathcal{M}_k}{(\mathcal{M}_i^{max} + \mathcal{M}_k^{max}) L_{ik}} \sum_{r=1}^{s} L_{ik}^r \, p^r, \tag{6}$$

$$c_i = \frac{1}{n_i} \sum_{k \in I_i} c_{ik}, \tag{7}$$

where I_i is the set of the neighbors of the i-th LU and n_i their number. Last expression can be written in a more explicit form by introducing the quantity

$$H_{ik} = \frac{1}{L_{ik}} \sum_{r=1}^{s} L_{ik}^r p^r, \tag{8}$$

that can be computed once for all from the characteristic parameters of each LU given by the GIS; thus the total connectivity index of the i-th LU can be finally written as

$$c_i = \frac{1}{n_i} \sum_{k \in I_i} \frac{\mathcal{M}_i + \mathcal{M}_k}{\mathcal{M}_i^{max} + \mathcal{M}_k^{max}} H_{ik}, \tag{9}$$

where the quantities H_{ik} e $\mathcal{M}_k^{max}$ are computed for all the n_i neighbors of i. Moreover, since $\mathcal{M}_i = \mathcal{M}_i(t)$ and $\mathcal{M}_k = \mathcal{M}_k(t)$, the connectivity index (9) results to be time-dependent, i.e. $c_i = c_i(t)$, and, through it, all the equations on $\mathcal{M}_i$ are coupled.

The model will be re-written in terms of the normalized variables, defined as

$$M_i = \frac{\mathcal{M}_i}{\mathcal{M}_i^{max}} \leq 1, \qquad V_i = \frac{\mathcal{V}_i}{A_i} \leq 1. \tag{10}$$

Therefore, dividing the first equation of (3) by $\mathcal{M}_i^{max}$ and the second by A_i, and taking into account the expression of $c_i(t)$ given by (9), the following final version of the model is obtained

$$\begin{cases} M_i' = \left[\dfrac{1}{n_i} \displaystyle\sum_{k \in I_i} \frac{M_i \mathcal{M}_i^{max} + M_k \mathcal{M}_k^{max}}{\mathcal{M}_i^{max} + \mathcal{M}_k^{max}} H_{ik} \right] M_i(1 - M_i) - \nu_i (1 - V_i) M_i \\ \\ V_i' = M_i\, V_i (1 - V_i) - \mu_i U_i V_i. \end{cases} \tag{11}$$

System (11) is equipped with the following initial data

$$M_i(t=0) = M_{i0} = \mathcal{M}_{i0}/\mathcal{M}_i^{max}, \qquad V_i(t=0) = V_{i0} = \mathcal{V}_i/A_i.$$

Once the variables $M_i(t)$ and $V_i(t)$ are determined from equations (11), one can recover, at each time t, the corresponding variables at the level of the whole environmental system; in particular the non-dimensionless variables $\mathcal{M}$ and $\mathcal{V}$ can be computed by

$$\mathcal{M}(t) = \frac{1}{n} \sum_{i=1}^{n} M_i(t) \mathcal{M}_i^{max}, \qquad \mathcal{V}(t) = \sum_{i=1}^{n} V_i(t) A_i \tag{12}$$

whereas the dimensionless ones M and V are given by

$$M(t) = \mathcal{M}(t)/\mathcal{M}^{max}, \qquad V(t) = \mathcal{V}(t)/A, \tag{13}$$

where A is the total area of the environment and $\mathcal{M}^{max}$ the *absolute maximum value producible* by the generalized bio-energy in the whole system; this last quantity is defined by

$$\mathcal{M}^{max} = 2 B_{max}^{b}\, A.$$

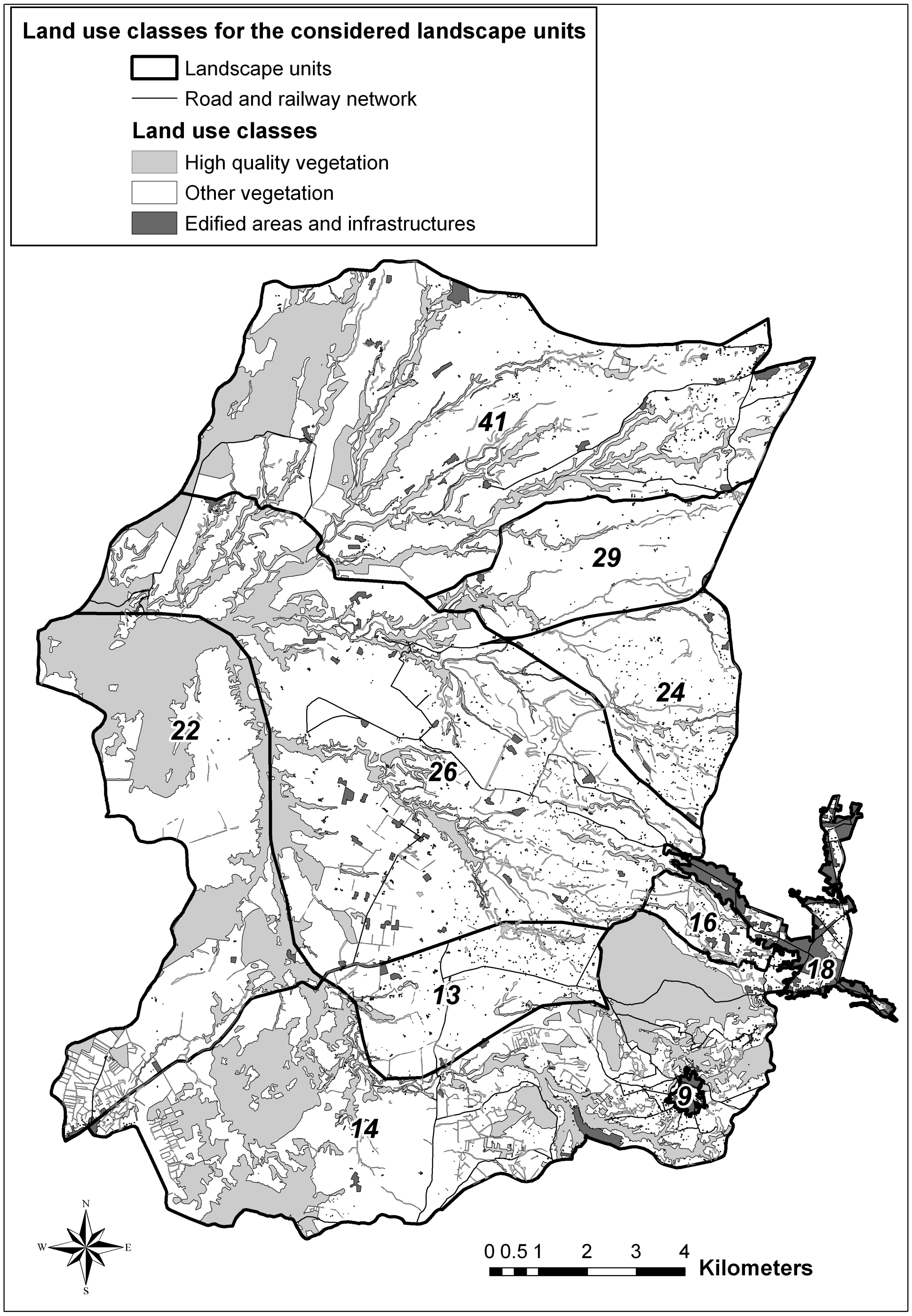

Figure 1. Map of the area under investigation

3 Numerical simulations

Some tests have been performed, solving system (11) with the well assessed LSODE solver [12] built in the Octave software, and using data concerning an environment in central Italy. Indeed, in paper [10] with the aim of proposing an evaluation on almost the whole district of Viterbo, this environmental system was divided into 46 LUs. Here the numerical experiment for the model given by equations (11) will be performed only in a relative small portion of the district, in particular in the one characterized by ten LUs, indicated in Fig.1 by the numbers 9, 13, 14, 16, 18, 22, 24, 26, 29, 41. This area, confined in the south-west part of the district, seems to be relevant since it should include in future the extension of a freeway connecting two important motor roads, the one running on the side of the Tirrenian sea (under construction) with the well-known Autostrada del Sole.

Except for the LUs n.9 and n.18, characterized by an important presence of edified areas (BTC almost 0), the other LUs do not include many buildings or infrastructures. The LUs n.14 and n.22 present, respectively, percentages of 41.3% and 44.8% of soil of high ecological quality, whereas the other ones are characterized by low values of BTC (V_0 between 0.06 and 0.18). On the side of barriers, in general the borders of the LUs consist in roads characterized by strong impermeability ($p = 0.05$) due to the presence of heavy traffic. The only barriers with high permeability are the ones between the LUs n.13/n.14 and n.22/n.26. Moreover barriers with rather strong impermeability are also present inside the LUs n.14 and n.26.

According to these data the environmental quality of the territory seems to be rather critical. In fact this general impression is well documented by the numerical simulations which show (Fig.2 and Fig.3) that only the LUs n.16 and n.22 exhibit an increase of V and M, whereas all the other LU are characterized by a strong decay of these quantities, with behaviors like the ones of Fig.4 and Fig.5, related, respectively, to the LUs n.24 and n.29.

It is interesting to note that the curves are not always monotone and that decay or growth may be slower in some LU. In particular, this is the case of Lu n.16 with respect to LU n.22. In fact this last one is characterized by a fast growth of M and V, due to a very high percentage of BTC and low percentage of edified soil (0.6%). Conversely the LU n.16 does not present a high value of BTC ($V_0 = 0.16$), but thanks to a complete absence of barriers inside its territory exhibits a growth which, however, is rather slow in comparison with that of the LU n.22.

According to the environmental criticality of almost all the considered LUs the results given by equations (13), for the whole environmental system, present

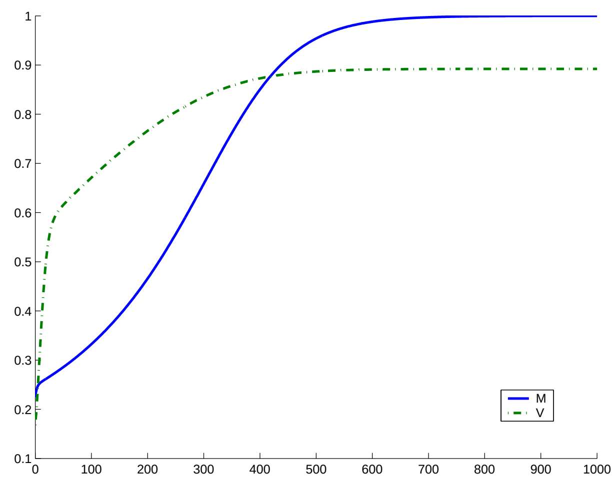

Figure 2. Trend of $V_i(t)$ and $M_i(t)$ with $i = 16$

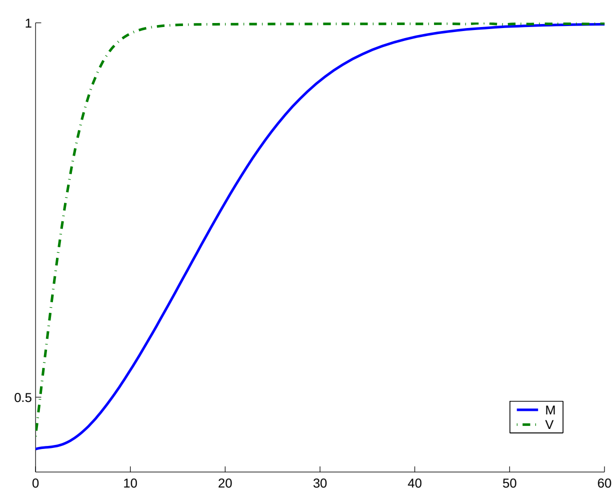

Figure 3. Trend of $V_i(t)$ and $M_i(t)$ with $i = 22$

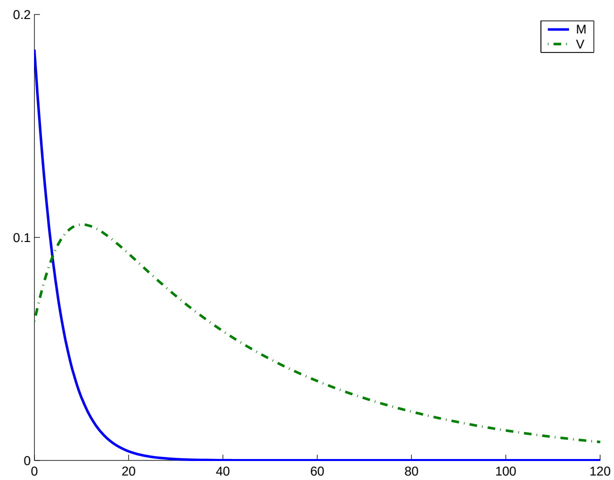

Figure 4. Trend of $V_i(t)$ and $M_i(t)$ with $i = 24$

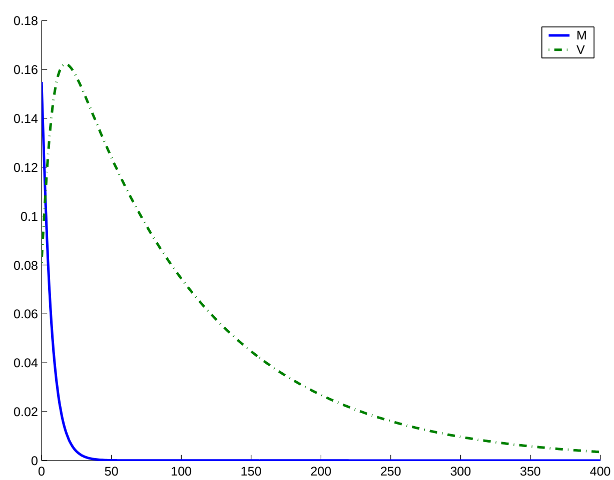

Figure 5. Trend of $V_i(t)$ and $M_i(t)$ with $i = 29$

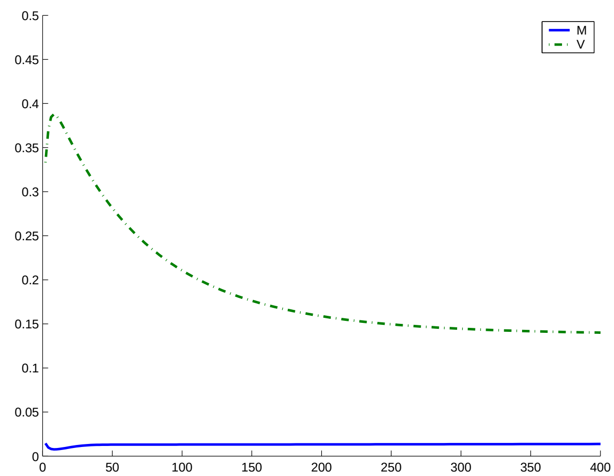

Figure 6. Trend of $V(t)$ and $M(t)$ in the whole environment

a negative trend for both $V(t)$ and $M(t)$. In fact, as shown by Fig.6, the generalized bio-energy M reaches asymptotically a very low level together with that of V, which, after a transitory growth, decays to a value corresponding to a percentage of about 15% of soil characterized by high ecological green: in conclusion, one can argue that the considered territory presents high fragmentation, including only some “islands” of moderate production of BTC, which may be even reduced by the presence of new anthrop barriers given, for example, by freeway viaducts.

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