Totally umbilical CMC hypersurfaces of a conformally recurrent manifold

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Abstract. It has been shown that a non-degenerate totally umbilical constant mean curvature hypersurface of a conformally recurrent pseudo-Riemannian manifold is conformally recurrent.

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Introduction

By a conformally recurrent manifold, we mean (see Adati and Miyazawa [1]) a pseudo-Riemannian manifold with a non-degenerate metric $g$ satisfying $\nabla C = p \otimes C$, where $p$ is a 1-form, $\nabla$ the Levi-Civita connection and $C$ the Weyl conformal curvature tensor of $g$. Such manifolds of dimension 4 and with Lorentzian metric $g$ were completely classified within the framework of Lorentzian geometry by McLenaghan and Leroy [6] as a subclass of the class of complex recurrent spacetimes (defined by the condition that the self-dual part of $C$ is recurrent with a complex recurrent vector). Complex recurrent (in particular, conformally recurrent) manifolds also seem to be important in the study of Huygens’ principle [5]. A conformally recurrent manifold with $p = 0$ is known as a conformally symmetric manifold and was studied by Chaki and Gupta [3] and also by Sharma [7] assuming the existence of a 1-parameter group of proper conformal motions. The purpose of this paper is to prove the following result and state its consequences.

Theorem 1. Let $M$ (dim. $> 3$) be a totally umbilical hypersurface of a pseudo- Riemannian conformally recurrent manifold. If $M$ has constant mean curvature (CMC), then it is conformally recurrent with recurrence vector as the tangential component of the recurrence vector of the ambient space.
An odd-dimensional Riemannian manifold $M$ is said to be a Sasakian manifold if it admits a global unit Killing vector field satisfying

$$ R(x, y)\xi = g(y, \xi)x - g(x, \xi)y $$

where $x, y$ denote arbitrary vector fields on $M$ and $R$ the curvature tensor of the Riemannian metric $g$ (see Blair [2] for details).

**Corollary 1.** Under the hypothesis of the above theorem, if $M$ is a Sasakian manifold, then it is locally isometric to a unit sphere.

**Corollary 2.** Let $M$ (dim. $> 3$) be a closed orientable hypersurface of an orientable Riemannian conformally recurrent manifold $\overline{M}$ with a homothetic vector field $V$ which is nowhere tangential to $M$. If $M$ has CMC and the Ricci curvature of $\overline{M}$ along the normal vector field is non-negative on $M$, then $M$ is conformally recurrent and the tangential component of $V$ is Killing on $M$.

## 1 Preliminaries

We assume both $M$ and $\overline{M}$ orientable and denote their inner product by $<, >$. Let $N$ denote the unit normal vector field such that $< N, N > = \epsilon = \pm 1$. Then the Gauss’ and Weingarten’s formulas are

$$ \nabla_x y = \nabla_x y + B(x, y)N, \quad \nabla_x N = -\epsilon Ax $$

where $x, y$ are arbitrary vector fields tangent to $M$; $\nabla, \overline{\nabla}$ denote Levi-Civita connections of $M$ and $\overline{M}$ respectively, $B$ the second fundamental form and $A$ the Weingarten operator such that $B(x, y) = \epsilon < Ax, y >$. Let $R, S, Q, r$ and $C$ denote the curvature tensor, Ricci tensor, Ricci operator, scalar curvature and Weyl conformal curvature tensor of $M$ and the same letters with overbars denote the corresponding objects of $\overline{M}$. By hypothesis, $B(x, y) = k < x, y >$, for a constant $k$. The Gauss’ and Codazzi equations are therefore

$$ \overline{R}(x, y, z, w) = R(x, y, z, w) - \epsilon k^2(\langle y, z \rangle < x, w \rangle - \langle x, z \rangle < y, w \rangle) $$

$$ \overline{R}(x, y)N = 0, \quad \overline{S}(x, N) = 0 $$

$$ \overline{S}(x, y) - \epsilon < \overline{R}(N, x)y, N > = S(x, y) + \epsilon k(k - n + 1) < x, y > $$

## 2 Proof of the Theorem

Differentiating (1) along the hypersurface gives

$$ (\nabla_v \overline{R})(x, y, z, w) = (\nabla_v R)(x, y, z, w) $$

(1)
for an arbitrary vector field $v$ tangent to $M$. Next, differentiating the expression for Weyl tensor $\overline{C}$ of $\overline{M}$ along $v$ and using the conformal recurrence hypothesis, we find

\[
(\nabla_v \mathcal{R})(X,Y,Z,W) = p(v)\overline{C}(X,Y,Z,W) + \frac{1}{n-2}[(\nabla_v \overline{S})(Y,Z) < X,W > \\
- (\nabla_v \overline{S})(X,Z) < Y,W > + (\nabla_v \overline{S})(X,W) < Y,Z > \\
- (\nabla_v \overline{S})(Y,W) < X,Z > - \frac{v^2}{n-1}(< Y,Z > < X,W > \\
- < X,Z > < Y,W >)]
\]  

(2)

where $n-1$ is the dimension of $M$ and $X,Y,Z,W$ denote arbitrary vector fields on $\overline{M}$. From (4) and (5) one obtains

\[
(\nabla_v \mathcal{R})(x,y,z,w) = p(v)\mathcal{R}(x,y,z,w) + \frac{1}{n-2}T(y,z) < x,w > - \\
T(x,z) < y,w > + T(x,w) < y,z > - T(y,w) < x,z > - \\
(f + \epsilon k^2 p(v))(< y,z > < x,w > - < x,z > < y,w >)
\]

(3)

where $T = \nabla_v \overline{S} - p(v)\overline{S}$ and $f = \frac{v^2 - p(v)^2}{(n-1)(n-2)}$. Now equations (1) and (4) yield

\[
(\nabla_v \mathcal{R})(x,y,z,w) = p(v)\mathcal{R}(x,y,z,w) + (\nabla_v \mathcal{R})(x,y,z,w) - p(v)(\mathcal{R}(x,y,z,w) \\
+ \epsilon k^2(< y,z > < x,w > - < x,z > < y,w >)).
\]

(4)

At this point we let $(e_i)$ denote a local orthonormal basis of the tangent space of $M$. Hence $(e_i, N)$ is a local orthonormal basis of the tangent space of $\overline{M}$ at points of $M$. Substituting $y = z = e_i$, in (7), multiplying by $(e_i, e_i) = \epsilon_i$, summing over $i$, using (5) with $X = x, W = w, Y = Z = N$, and (3) gives

\[
(n-2)t(x,w) = (n-3)T(x,w) - \epsilon T(N,N) < x,w > + \\
\left[ (n-2)f - \epsilon (n-2)^2 p(V)k^2 \right] < x,w >
\]

(5)

where $t = \nabla_v S - p(v)S$. Substituting $x = w = e_i$, multiplying by $\epsilon_i$, summing over $i$, gives

\[
2\epsilon T(N,N) = p(v)r - vr + (n-1)(n-2)(f - \epsilon p(v)k^2)
\]

(6)

using this in (8) provides

\[
(n-3)T(x,w) = (n-2)t(x,w) + \frac{1}{2} < x,w > \left[ (n-2)(n-3)f \\
- vr + p(v)r + \epsilon (n-2)(n-3)p(v)k^2 \right]
\]

(7)
Finally, using this in (6) we obtain $\nabla C = p \otimes C$, completing the proof.

Corollary 1 follows from the following result of Ghosh and Sharma [4]: A conformally recurrent Sasakian manifold is locally isometric to a unit sphere. Corollary 2 follows from the following result of Yano [9]: Let $M$ be a closed orientable hypersurface of an orientable Riemannian manifold $\overline{M}$ with a homothetic vector field which is nowhere tangent to $M$. If $M$ has constant mean curvature and the Ricci curvature of $\overline{M}$ along the normal $N$ is non-negative on $M$, then $M$ is totally umbilical, and the well-known fact that a homothetic vector field on a compact orientable manifold without boundary, is necessarily Killing (see Yano [8]).

References