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The projective version of feedback cyclization

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Abstract. In this paper we define a projective version of the feedback cyclization over commutative rings. This definition generalizes the free case, implies coefficient assignability, is stable under the feedback group and permits us to prove a projective version of the Emre-Khargonekar theorem.

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Introduction

It is possible to exhibit elementary examples of rings which do not satisfy feedback cyclization, coefficient assignability or pole assignability. Thus, $\mathbb{R}[z]$ does not satisfy the feedback cyclization property, $\mathbb{Z}$ does not have the coefficient assignability property and $\mathbb{R}[x, y]$ does not have the pole assignability property. Dynamic versions of these properties have been defined for such rings (see [1], [2], [3], [4] and [5]). The dynamic properties were defined initially for free systems ([1]). Later the dynamic versions of the coefficient and pole assignment for projective systems were introduced (see [3] and [4]). In [8] arose the question about the projective version of the feedback cyclization (static and dynamic). In this paper we present a projective theory of feedback cyclization which generalizes the free case, implies coefficient assignability, is stable under the feedback group and allows us to prove a projective version of the Emre-Khargonekar theorem.

1 Static feedback cyclization

Let R be a commutative ring. If X is a finitely generated module over R, we write $\mu(X)$ for the number of elements of any set of generators of X with

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minimal cardinality. A set S of generators of X is *minimal* if S has $\mu(X)$ elements. Let X be projective of constant rank n and $F : X \longrightarrow X$ an endomorphism of X. We say that X has an *F-semi-cyclic set of generators* if there exist elements $x_1, \dots, x_t \in X$ and positive integers $n_1, \dots, n_t$ such that $\{x_1, F(x_1), \dots, F^{n_1-1}(x_1), \dots, x_t, F(x_t), \dots, F^{n_t-1}(x_t)\}$ is a minimal set of generators of X. By the Cayley-Hamilton Theorem, $n_i \leq n$ for $i = 1, 2, \dots, t$. We also can assume that $n_1 \geq \cdots \geq n_t$. Let (U, X, F, G) be a projective linear system over R of rank n. We say that (U, X, F, G) is *FC (feedback cyclizable)* if there exists an R-homomorphism $K : X \longrightarrow U$ and elements $u_1, \dots, u_t \in U$ such that $RG(u_1) + \cdots + RG(u_t)$ is rank one projective and $\{G(u_1), \dots, G(u_t)\}$ is an $(F + GK)$-semi-cyclic set of generators of X. Finally, we say that R has the *feedback cyclization property (FC)* if every reachable projective linear system over R is FC ((U, X, F, G) is reachable if $G(U) + FG(U) + \cdots + F^{n-1}G(U) = X$, see [4]).

Let FFC be the free version of the feedback cyclization property (see [1], [5] and [8]). Then we have the following result.

Theorem 1. *For every ring R, FC implies FFC.*

Proof. Let $R^m \xrightarrow{G} R^n \xrightarrow{F} R^n$ be a reachable free system over R; there exists a homomorphism $K \colon R^n \to R^m$ and elements $u_1, \dots, u_t \in R^m$ such that $\{G(u_1), \dots, G(u_t)\}$ is a $(F + GK)$-semi-cyclic set of generators of R^n. Since R^n can be generated by n elements, then $n_1 + \cdots + n_t = n$. This means that

$$\{G(u_1), (F + GK)G(u_1), \dots, (F + GK)^{n_1-1}G(u_1), \dots,\\ G(u_t), (F + GK)G(u_t), \dots, (F + GK)^{n_t-1}G(u_t)\}$$

is a basis of R^n. By Theorem 3 in [8], the system (F, G) is FFC. $\boxed{QED}$

We denote by FCA the coefficient assignment for free systems; it is well known that FFC implies FCA. In the projective case we have the following theorem.

Theorem 2. *$FC \Rightarrow CA$.*

Proof. Let (U, X, F, G) be a reachable system over R; there exists an R-homomorphism $K : X \longrightarrow U$ and elements $u_1, \dots, u_t \in U$ such that

$$\{G(u_1), (F + GK)G(u_1), \dots, (F + GK)^{n_1-1}G(u_1), \dots,\\ G(u_t), (F + GK)G(u_t), \dots, (F + GK)^{n_t-1}G(u_t)\}$$

is a minimal set of generators of X and $U' = RG(u_1) + \cdots + RG(u_t)$ is a rank one projective submodule of X. Note that (U', X', F', G') is a single input reachable system where $X' = X, F' = F + GK$ and G' is the inclusion of U'

into X. But for single input systems, reachability is equivalent to coefficient assignability (see [6] and [7]). Thus, given a monic polynomial $b(z) \in R[z]$ there exists an homomorphism $L : X' \longrightarrow U'$ such that the characteristic polynomial of $F' + G'L$ is $b(z)$. Moreover, there exists an homomorphism $H : U' \longrightarrow Ru_1 + \cdots + Ru_t \subseteq U$ such that $G \circ H = 1_{U'}$. The homomorphism $H \circ L$ satisfies

$$\text{char. pol.}\,(F + G(K + HL)) = \text{char.pol.}(F' + G'L) = b(z).$$

This proves that (U, X, F, G) is CA. $\boxed{QED}$

Remark 1. Since coefficient assignability implies pole assignability which in turn implies reachability, it follows that reachability is a consequence of feedback cyclization and hence is required in the definition.

Next we prove that FC is stable under the feedback group. We shall say that two systems (U, X, F, G) and $(\widetilde{U}, \widetilde{X}, \widetilde{F}, \widetilde{G})$ are *equivalent* if one can be obtained from the other after a finite sequence of elementary operations of the following type:

(i) Change of basis in the state space: $\widetilde{G} = PG, \widetilde{F} = PFP^{-1}, \widetilde{U} = U$, where $P : X \longrightarrow \widetilde{X}$ is an isomorphism.

(ii) Change of basis in the input space: $\widetilde{G} = GP^{-1}, \widetilde{F} = F, \widetilde{X} = X$, where $P : U \longrightarrow \widetilde{U}$ is an isomorphism.

(iii) Feedback: $\widetilde{G} = G, \widetilde{F} = F + GL, \widetilde{U} = U, \widetilde{X} = X$, where $L : X \longrightarrow U$ is a homomorphism.

Theorem 3. *Let (U, X, F, G) and $(\widetilde{U}, \widetilde{X}, \widetilde{F}, \widetilde{G})$ be equivalent systems. Then, (U, X, F, G) is FC if and only if $(\widetilde{U}, \widetilde{X}, \widetilde{F}, \widetilde{G})$ is FC.*

Proof. By the symmetry of the problem we only need to assume that (U, X, F, G) is FC. Moreover, we can consider each operation separately. We will use the above notation. (i) Change of basis in the state space: There exists a homomorphism $K : X \longrightarrow U$ and vectors $u_1, \ldots, u_t \in U$ such that $RG(u_1) + \cdots + RG(u_t)$ is rank one projective and $\{G(u_1), \ldots, G(u_t)\}$ is an $(F+GK)$-semi-cyclic set of generators of X. Since $\widetilde{U} = U$ and P is an isomorphism we only need to exhibit an homomorphism $\widetilde{K} : \widetilde{X} \longrightarrow \widetilde{U}$ such that $\{PG(u_1), \ldots, PG(u_t)\}$ is a $(\widetilde{F} + \widetilde{G}\widetilde{K})$-semi-cyclic set of generators of $\widetilde{X}$. We take $\widetilde{K} = KP^{-1}$ and then

$$\begin{aligned}
X =& RG(u_1) + \cdots + R(F + GK)^{n_1-1}G(u_1) + \cdots + \\
& RG(u_t) + \cdots + R(F + GK)^{n_t-1}G(u_t), \\
\widetilde{X} =& PX = RPG(u_1) + \cdots + RP(F + GK)^{n_1-1}P^{-1}PG(u_1) + \cdots + \\
& RPG(u_t) + \cdots + RP(F + GK)^{n_t-1}P^{-1}PG(u_t) \\
=& R\widetilde{G}(u_1) + \cdots + R(\widetilde{F} + \widetilde{G}\widetilde{K})^{n_1-1}\widetilde{G}(u_1) + \cdots + \\
& R\widetilde{G}(u_t) + \cdots + R(\widetilde{F} + \widetilde{G}\widetilde{K})^{n_t-1}\widetilde{G}(u_t)
\end{aligned}$$

Since $\widetilde{X} \cong X$ and

$$\{G(u_1), \ldots, (F+GK)^{n_1-1}G(u_1), \ldots, G(u_t), \ldots, (F+GK)^{n_t-1}G(u_t)\}$$

is a minimal set of generators of X, then

$$\{\widetilde{G}(u_1), \ldots, (\widetilde{F}+\widetilde{G}\widetilde{K})^{n_1-1}\widetilde{G}(u_1), \ldots, \widetilde{G}(u_t), \ldots, (\widetilde{F}+\widetilde{G}\widetilde{K})^{n_t-1}\widetilde{G}(u_t)\}$$

is a minimal set of generators of $\widetilde{X}$.

(ii) Change of basis in the input space: Taking $\widetilde{K} = PK$ the proof is similar to the previous.

(iii) Feedback: With $\widetilde{K} = K - L$, the proof follows directly from the definition. $\boxed{QED}$

2 Dynamic feedback: the projective case

For rings not satisfying CA or PA, dynamic versions of these properties have been introduced. (See [4]; the dynamic version of FFC can be find in [8]). We next define the dynamic version of FC.

Let R be a commutative ring and r a non negative integer; R has the *FC-r* property if for every reachable system (U, X, F, G) over R of rank n there exists a projective module P of rank r (r depending on n) such that the enlarged system $(U \oplus P, X \oplus P, \widetilde{F}, \widetilde{G})$ with

$$\widetilde{F} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}, \ \widetilde{G} = \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix}$$

has the FC property (I denotes the identity homomorphism of P). Note that the static case FC corresponds to $r = 0$ for all n. The free version of dynamic feedback consists in enlarging the free system (R^m, R^n, F, G) with a free summand $P = R^r$.

The classical theorem of Emre and Khargonekar establishes that for n-dimensional free systems over arbitrary commutative rings, the augmentation $r = n^2 - n$ is enough (of course for some special rings r can be reduced, e.g. for principal ideals domains, $r = 1$, see [5]). Following the geometric approach of [8] we next present the projective version of the Emre-Khargonekar theorem.

For the system (U, X, F, G) we define the *control process*

$$x_0 = 0, \ \ x_k = Fx_{k-1} + Gu_k, \ \ k \geq 1 \tag{1}$$

where the sequence of entries $\{u_k\}$ is given. For $r \geq 0$ an integer, we shall say that (U, X, F, G) is *r-cyclizable* if there exists a control process (1) such that the sequence of states $\{x_1, x_2, \ldots, x_{\mu(X)+r}\}$ contains a minimal system of generators of X. We shall say that R is *r-cyclizable* if every reachable system (U, X, F, G) of rank n is r-cyclizable (r depending on n).

Theorem 4. *Let (U, X, F, G) be a reachable system of rank n. Then the system (U, X, F, G) is $\mu(X)(n-1)$-cyclizable.*

Proof. In order to simplify the notation we write $t = \mu(X)$. Let $e_1, \ldots, e_t$ be a minimal system of generators of X. By reachability there exist elements $v_i^j \in U$ such that

$$\begin{aligned} e_1 &= F^{n-1}G(v_1^1) + \cdots + FG(v_{n-1}^1) + G(v_n^1) \\ e_j - F^n(e_{j-1}) &= F^{n-1}G(v_1^j) + \cdots + FG(v_{n-1}^j) + G(v_n^j) \end{aligned}$$

for $2 \leq j \leq t$. We define

$$\begin{aligned} u_1 &= v_1^1, \ldots, u_n = v_n^1 \\ u_{n+1} &= v_1^2, \ldots, u_{2n} = v_n^2 \\ &\vdots \\ u_{tn-n+1} &= v_1^t, \ldots, u_{tn} = v_n^t \end{aligned}$$

and the control process

$$x_0 = 0, \;\; x_k = Fx_{k-1} + Gu_k, \quad 1 \leq k \leq tn.$$

Then $x_n = e_1, x_{2n} = e_2, \ldots, x_{tn} = e_t$ and the system (U, X, F, G) is $t(n-1)$-cyclizable. $\boxed{QED}$

If (U, X, F, G) is free, $\mu(X) = n$; moreover, for free systems r-cyclizable implies FFC-r ([8]). Thus, in the free case, our Theorem 4 produces the classical theorem of Emre and Khargonekar. On the other hand, it is well known that in the free case FFC-0 is equivalent to 0-cyclizable. What happens in the projective case? We will answer this question partially in the next section.

3 Stratified Dynamic Feedback

Let X be a projective module of rank n and $F : X \longrightarrow X$ an endomorphism of X. We shall say that X has a *stratified F-semi-cyclic* set of generators if X contains an F-semi-cyclic set of generators $x_1, \ldots, x_t$ such that

$$\begin{aligned} F^s(x_i) \in & RF^{n_i-1}(x_i) + \cdots + RF(x_i) + Rx_i + \cdots + \\ & RF^{n_t-1}(x_t) + \cdots + RF(x_t) + Rx_t \end{aligned}$$

for $s \geq 0$ and $1 \leq i \leq t$. We shall say that (U, X, F, G) has the *stratified feedback cyclization* property, SFC, if there exists a homomorphism $K : X \longrightarrow U$ and elements $u_1, \ldots, u_t \in U$ such that $RG(u_1) + \cdots + RG(u_t)$ is rank one projective and $\{G(u_1), \ldots, G(u_t)\}$ is a stratified $(F+GK)$-semi-cyclic set of generators of X. Finally, we say that R is an SFC ring if every reachable system over R is SFC. Of course $SFC \Rightarrow FC$.

Theorem 5. *$SFC \Rightarrow 0-cyclizable$.*

Proof. Let (U, X, F, G) be a reachable system, then there exists a homomorphism $K : X \longrightarrow U$ and elements $u_1, \ldots, u_t$ such that $\{z_i = G(u_i)\}_{i=1}^{t}$ is an $(F+GK)$-semi-cyclic set of generators of X. This means that

$$\begin{aligned} S = \{&z_1, (F+GK)z_1, \ldots, (F+GK)^{n_1-1}z_1, \ldots, \\ &z_t, (F+GK)z_t, \ldots, (F+GK)^{n_t-1}z_t\} \end{aligned}$$

is a minimal system of generators of X and

$$\begin{aligned} (F+GK)^s z_i \in &R(F+GK)^{n_i-1}z_i + \cdots + R(F+GK)z_i + Rz_i + \cdots + \\ &R(F+GK)^{n_t-1}z_t + \cdots + R(F+GK)z_t + Rz_t \end{aligned} \tag{2}$$

for $s \geq 0$ and $1 \leq i \leq t$. We define the control process

$$\begin{aligned} x_1 &= z_t \\ x_2 &= Fx_1 + Gu_2, \quad u_2 = Kx_1 \\ &= (F+GK)z_t \\ x_3 &= Fx_2 + Gu_3, \quad u_3 = Kx_2 \\ &= (F+GK)^2 z_t \\ &\vdots \\ x_{n_t} &= Fx_{n_t-1} + Gu_{n_t}, \quad u_{n_t} = Kx_{n_t-1} \\ &= (F+GK)^{n_t-1}z_t \\ x_{n_t+1} &= Fx_{n_t} + Gu_{n_t+1}, \quad u_{n_t+1} = Kx_{n_t} + u_{t-1} \\ x_{n_t+1} &= (F+GK)^{n_t}z_t + z_{t-1} \\ x_{n_t+2} &= Fx_{n_t+1} + Gu_{n_t+2}, \quad u_{n_t+2} = Kx_{n_t+1} \\ x_{n_t+2} &= (F+GK)^{n_t+1}z_t + (F+GK)z_{t-1} \\ x_{n_t+3} &= Fx_{n_t+2} + Gu_{n_t+3}, \quad u_{n_t+3} = Kx_{n_t+2} \\ x_{n_t+3} &= (F+GK)^{n_t+2}z_t + (F+GK)^2 z_{t-1} \\ &\vdots \end{aligned}$$

$$x_{n_t+n_{t-1}} = Fx_{n_t+n_{t-1}-1} + Gu_{n_t+n_{t-1}}, \quad u_{n_t+n_{t-1}} = Kx_{n_t+n_{t-1}-1}$$
$$x_{n_t+n_{t-1}} = (F+GK)^{n_t+n_{t-1}-1}z_t + (F+GK)^{n_{t-1}-1}z_{t-1}$$

From (2) and the above equalities follow that

$$z_{t-1}, (F+GK)z_{t-1}, \ldots, (F+GK)^{n_{t-1}-1}z_{t-1} \in < x_i | 1 \leq i \leq n_t + n_{t-1} > .$$

Continuing the above control process we get all elements $x_1, \ldots, x_{\mu(X)}$ and, as we just saw, $S \subseteq < x_i | 1 \leq i \leq \mu(X) >$. This implies that $< x_i | 1 \leq i \leq \mu(X) > = X$ and hence, (U, X, F, G) is 0-cyclizable. $\boxed{QED}$

References

[1] J. W. Brewer, J. W. Bunce, F. S. Van Vleck: Linear Systems over Commutative Rings, Marcel Dekker, New York, 1986.

[2] J. W. Brewer, L. Klingler *Pole shifting for families of systems: the dimension one case*, Math. Control Signals Systems **1**, 1988, 285–292.

[3] J. W. Brewer, L. Klingler *Dynamic feedback over rings whose homomorphic images are local-global*, Comm. Algebra **17** (7) 1989, 1759–1773.

[4] J. W. Brewer, L. Klingler *Dynamic feedback over commutative rings*, Linear Algebra and its Applications **98** 1988, 137–168.

[5] J. W. Brewer, L. Klingler, W. Schmale *The dynamic feedback cyclization problem for principal ideal domains*, J. Pure and Appl. Algebra **92** (3) 1994, 213–225.

[6] *O. Lezama, A. Rodado Sistemas lineales proyectivos*, Lecturas Matemáticas **17** (1) 1996, 15–35.

[7] F. Minnaar, C. G. Naudé, G. Naudé, F. Wiid *Pole assignability of rings of low dimension*, J. Pure and Appl. Algebra **51** 1988, 197–203.

[8] W. Schmale *A geometric approach to dynamic feedback cyclization over commutative rings*, Int. J. Control **57** (4) 1993, 991–997.