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(ω)topological connectedness and hyperconnectedness

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Abstract. In this paper, the notions of connectedness and hyperconnectedness in (ω)topological spaces are introduced and studied.

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1 Introduction

A set X equipped with an increasing sequence $\{\mathcal{J}_n\}$ of topologies is called an (ω)topological space [2]. It is denoted by $(X, \{\mathcal{J}_n\})$ or, simply, by X if there is no scope for confusion. Separation axioms, compactness and paracompactness of (ω)topological spaces were studied in [2]. In [3], we proved Michael's theorem (Theorem 1, [10]) and Stone's theorem [13] on paracompactness in (ω)topological spaces. In [14], Thomas proved some results on maximal connected spaces. Mathew [9] studied hyperconnected topological spaces (Steen and Seebach [12]).

In this paper, we introduce (ω)connected and (ω)hyperconnected spaces. We also introduce (ω)semiopen sets as an analogue of semiopen sets (Levine [7]). Along with other results, we prove (i) a set lying between an (ω)connected set and its (ω)closure is (ω)connected, (ii) if an (ω)topological space X is maximal (ω)hyperconnected, then the class of all nonempty (ω)open sets is an ultrafilter.

The aim of studying (ω)topological spaces is to develop a framework for studying an increasing (evolving) sequence of topologies on a set. In literature, we get occurrence of infinite sequence of evolving topologies and topological spaces, a few examples are given below.

In (Datta and Roy Choudhuri [4]) and (Raut and Datta [11]) authors studied a non-archimedean extension of the real number system R involving nontrivial infinitesimally small elements which are modelled as hierarchically structured local p-adic fields. This approach aims at offering a natural framework for dealing with an infinite sequence of topologically distinct spaces in a complex evolutionary process.

In digital topology and evolving infinite networks (Fan, Chen and Ko [5]) change of topologies with dynamical consequences are considered. Such topological notions have applications in computer science, infinite graphs and related areas. In dynamical system theory, emergence of chaos in a deterministic system relates to an interplay of finite or infinite number of different topologies in the underlying set.

Another motivation of (ω)topology is that if $\{\mathcal{J}_n\}$ be an increasing sequence of topological spaces on X and $\mathcal{J} = \cup_n \mathcal{J}_n$ then $(X, \mathcal{J})$ is not a topological space and even it is not an Alexandroff space [1] which is a generalization of a topological space requiring only countable union of open sets to be open. In fact, an arbitrary (or countable) union of sets $\in \mathcal{J}$ may not belong to $\mathcal{J}$. But taking advantage of the topologies $\mathcal{J}_n$ we can, however, get many properties of $(X, \{\mathcal{J}_n\})$([2], [3]), close to that of a topological space which is not necessarily possessed by an Alexandroff space.

2 Preliminaries

For an (ω)topological space $(X, \{\mathcal{J}_n\})$, a set $G \in \cup_n \mathcal{J}_n$ is called an (ω)open set. A set F is (ω)closed if its complement $F^c = X - F$ is (ω)open. The union and intersection of a finite number of (ω)open sets is (ω)open. However, the countable union of (ω)open sets may not be (ω)open. These sets are called $(\sigma\omega)$open sets. Since the arbitrary union of $(\mathcal{J}_n)$open sets is $(\mathcal{J}_n)$open, the union of an arbitrary number of (ω)open sets is also $(\sigma\omega)$open. Similarly, $(\delta\omega)$closed sets are defined as the intersection of a countable number of (ω)closed sets. The intersection of all (ω)closed sets containing a set A is called the (ω)closure of A and is denoted by $(\omega)clA$. Obviously, it is a $(\delta\omega)$closed set.

It is clear that the class $\mathcal{T}$ of all $(\sigma\omega)$open sets in X forms a topology.

3 (ω)connectedness

Definition 1. An (ω)topological space $(X, \{\mathcal{J}_n\})$ is said to be (ω)connected, if X cannot be expressed as the union of two disjoint nonempty (ω)open sets.

Obviously, when $\mathcal{J}_n = \mathcal{J}$ for all n, the space $(X, \{\mathcal{J}_n\})$ is an (ω)connected space iff the topological space $(X, \mathcal{J})$ is connected. Further, an (ω)topological space $(X, \{\mathcal{J}_n\})$ is (ω)connected iff $(X, \mathcal{J}_n)$ is connected for all n.

Definition 2. A subset Y of an (ω)topological space $(X, \{\mathcal{J}_n\})$ is said to be (ω)connected, if the (ω)topological space $(Y, \{\mathcal{J}_n|Y\})$ is (ω)connected.

Theorem 1. *If the space $(X, \mathcal{T})$ is connected, then the (ω)topological space $(X, \{\mathcal{J}_n\})$ is (ω)connected.*

Proof. Since every (ω)open set is ($\sigma\omega$)open, the result follows. QED

We now give an example to show that the converse of the theorem is not true.

Example 1. Let $P\{1, 2, 3, \ldots, n\}$ denote the power set of the set $\{1, 2, 3, \ldots, n\}$. We define an increasing sequence $\{\mathcal{T}_n\}$ of topologies on N as follows:

$$\mathcal{T}_n = \{N\} \cup P\{1, 2, 3, \ldots, n\}.$$

Then the (ω)topological space $(N, \{\mathcal{T}_n\})$ is (ω)connected. However, the topology of all ($\sigma\omega$)open sets of the above (ω)topology is not connected. Since, the set of all even positive integers and odd positive integers are two disjoint ($\sigma\omega$)open sets whose union is N.

If X is not (ω)connected, then there exist two disjoint nonempty (ω)open sets A and B such that $X = A \cup B$. In this case, X is said to be (ω)disconnected and we write $X = A|B$. We call it an (ω)separation of X. Since the two (ω)open sets A and B belong to some $\mathcal{J}_n$, it is clear that if X is not (ω)connected then for some n, the topological space $(X, \mathcal{J}_n)$ is not connected. As a consequence we get the following theorem.

Theorem 2. *If C is an (ω)connected subset of an (ω)topological space X which has the (ω)separation $X = A|B$, then either $C \subset A$ or $C \subset B$.*

Corollary 1. *If any two points of $Y \subset X$ are contained in some (ω)connected subset of Y, then Y is (ω)connected.*

Corollary 2. *The union of a family of (ω)connected sets having nonempty intersection is (ω)connected.*

Corollary 3. *If C is an (ω)connected set in X and $C \subset E \subset (\omega)clC$, then E is (ω)connected.*

Proof. If E is not (ω)connected, then it has an (ω)separation $E = A|B$. By Theorem 2, $C \subset A$ or $C \subset B$. Let us assume $C \subset A$. Suppose $A, B \in \mathcal{J}_n|E$.

Then

$$\begin{aligned} B &= B \cap (\omega)clC \\ &\subset B \cap (\mathcal{J}_n|E)clA \\ &= \phi \text{ (since } A \cap B = \phi). \end{aligned}$$

This is a contradiction and so E is (ω)connected. QED

Definition 3. X is said to be an $(\omega)T_0$- space if for every pair of distinct points x and y of X, there exists an (ω)open set G such that $x \in G$ and $y \notin G$.

Definition 4. X is said to be an $(\omega)T_1$- space if for every pair of distinct points x and y of X, there exists a n such that for some $U, V \in \mathcal{J}_n$, we have $x \in U$, $y \in V$, $y \notin U$ and $x \notin V$.

Definition 5. An (ω)topology $\{\mathcal{J}'_n\}$ on X is said to be stronger (resp. weaker) than an (ω)topology $\{\mathcal{J}_n\}$ on X if $\cup_n \mathcal{J}_n \subset \cup_n \mathcal{J}'_n$(resp. $\cup_n \mathcal{J}'_n \subset \cup_n \mathcal{J}_n$). If, in addition, $\cup_n \mathcal{J}_n \neq \cup_n \mathcal{J}'_n$, then $\{\mathcal{J}'_n\}$ is said to be strictly stronger(resp. strictly weaker) than $\{\mathcal{J}_n\}$.

Definition 6. An (ω)topological space $(X, \{\mathcal{J}_n\})$ with property P is said to be maximal(resp. minimal) if for any other (ω)topology $\{\mathcal{J}'_n\}$ strictly stronger (resp. strictly weaker) than $\{\mathcal{J}_n\}$, the space $(X, \{\mathcal{J}'_n\})$ cannot have this property.

Theorem 3. *If X is maximal (ω)connected, then X is $(\omega)T_0$.*

Proof. Suppose, if possible X is not $(\omega)T_0$. Then there exist $x, y \in X, x \neq y$ such that $x \in (\omega)cl\{y\}$ and $y \in (\omega)cl\{x\}$. Let $\mathcal{J}'_n$ be the topology generated by $\mathcal{J}_n \cup \{y\}$. Then the (ω)topological space $(X, \{\mathcal{J}'_n\})$ is not (ω)connected. Let $X = A|B$ be an (ω)separation of $(X, \{\mathcal{J}'_n\})$. Then either $\{x, y\} \subset A$ or $\{x, y\} \subset B$. Suppose $\{x, y\} \subset A$. Then there exists a set $U \in \mathcal{J}_n$ with $x \in U$. But U also contains y. Since for any point $z \in A$ with $z \neq y$, there is a $(\mathcal{J}_n)$open neighborhood $G \subset A$ of z, it follows that $A \in \mathcal{J}_n$. Clearly $B \in \mathcal{J}_n$. Thus $A|B$ is an (ω)separation of $(X, \{\mathcal{J}_n\})$ which is a contradiction. QED

Below we provide examples to show that a maximal (ω)connected space may or may not be $(\omega)T_1$.

Example 2. Let us consider the (ω)topological space defined in Example 1. Clearly $(N, \{\mathcal{T}_n\})$ is $(\omega)T_1$. Also this space is (ω)connected and hence, can be extended to a maximal (ω)connected space.

Example 3. Let us define an (ω)topological space $(N, \{\mathcal{J}_n\})$ as follows:

$$\mathcal{J}_n = \{\phi\} \cup \{E \subset N \mid 1 \in E\} \text{ for all } n.$$

Then clearly $(N, \{\mathcal{J}_n\})$ is a maximal (ω)connected space. However, $(N, \{\mathcal{J}_n\})$ is not $(\omega)T_1$.

Theorem 4. *Let $(X, \{\mathcal{J}_n\})$ be maximal (ω)connected and G be an (ω)open (ω)connected subset of X. Then $(G, \{\mathcal{J}_n|G\})$ is maximal (ω)connected.*

Proof. If possible, suppose $(G, \{\mathcal{J}_n|G\})$ is not maximal (ω)connected. Let $\{\mathcal{P}_n\}$ be an (ω)topology on G strictly stronger than $\{\mathcal{J}_n|G\}$ and $(G, \{\mathcal{P}_n\})$ is (ω)connected. Let $H \subset G$ be such that $H \in \mathcal{P}_{n_0} - \mathcal{J}_{n_0}|G$ for some n_0. If $\mathcal{Q}_n$ is a topology on G generated by $(\mathcal{J}_n|G) \cup \{H\}$, then $\{\mathcal{Q}_n\}$ is an (ω)connected (ω)topology on G. Also if $\mathcal{S}_n$ is the topology on X generated by $\mathcal{J}_n \cup \{H\}$, then the (ω)topology $\{\mathcal{S}_n\}$ on X is strictly stronger than $\{\mathcal{J}_n\}$ and so $(X, \{\mathcal{S}_n\})$ is not (ω)connected. Let $X = A|B$ be an (ω)separation of $(X, \{\mathcal{S}_n\})$. Then either $G \subset A$ or $G \subset B$, since, otherwise, $(G \cap A)|(G \cap B)$ is an (ω)separation of $(G, \{\mathcal{Q}_n\})$. Suppose $G \subset A$. Since $G \in \cup_n \mathcal{J}_n$, it follows that $A \in \cup_n \mathcal{J}_n$. But obviously $B \in \cup_n \mathcal{J}_n$. Therefore $X = A|B$ is an (ω)separation of $(X, \{\mathcal{J}_n\})$ which is a contradiction. QED

Definition 7. Let $x \in X$. The component $C(x)$ of x in X is the union of all (ω)connected subsets of X containing x.

From Corollary 2, it follows that $C(x)$ is (ω)connected.

Theorem 5. *In an (ω)topological space $(X, \{\mathcal{J}_n\})$, (i) each component $C(x)$ is a maximal (ω)connected set in X, (ii) the set of all distinct components in X forms a partition of X, (iii) each $C(x)$ is $(\delta\omega)$closed in X.*

Proof. Straightforward. QED

4 (ω)hyperconnectedness

Definition 8. X is said to be (ω)hyperconnected if for any two nonempty (ω)open sets U and V, $U \cap V \neq \phi$.

Therefore for any nonempty (ω)open set U, $(\omega)clU = X$, since otherwise $V_1 = X - (\omega)clU$ is a nonempty $(\sigma\omega)$open set and $U \cap V_1 = \phi$ which implies that for any nonempty (ω)open set $V \subset V_1$, we have $U \cap V = \phi$.

Theorem 6. *$(X, \{\mathcal{J}_n\})$ is (ω)hyperconnected iff the topological space $(X, \mathcal{T})$ is hyperconnected.*

Proof. Suppose $(X, \{\mathcal{J}_n\})$ is (ω)hyperconnected. Let A and B be two nonempty $(\mathcal{T})$open sets. Then $A = \cup_{i=1}^{\infty} A_i$, $B = \cup_{j=1}^{\infty} B_j$ where A_i and B_j are nonempty (ω)open sets. Now $A \cap B \supset A_i \cap B_j \neq \phi$. Thus $(X, \mathcal{T})$ is hyperconnected.

Conversely, since each (ω)open set is $(\sigma\omega)$open set. The hyperconnectedness of the space $(X, \mathcal{T})$, implies that the (ω)topological space $(X, \{\mathcal{J}_n\})$ is (ω)hyperconnected. QED

Definition 9. A set $A \subset X$ is said to be (ω)semiopen if there exists an n such that for some $U \in \mathcal{J}_n$, we have

$$U \subset A \subset (\omega)clU.$$

Let $SO_\omega(X, \{\mathcal{J}_n\})$ or, simply, $SO_\omega(X)$ denote the set of all (ω)semiopen sets. If some set A satisfies the above relation for some set $U \in \mathcal{J}_n$, we say that A is $(\mathcal{J}_n - \omega)$semiopen. The set of all $(\mathcal{J}_n - \omega)$semiopen sets is denoted by $(\mathcal{J}_n)SO_\omega(X)$. Thus

$$SO_\omega(X) = \cup_n (\mathcal{J}_n) SO_\omega(X).$$

Theorem 7. *X is (ω)hyperconnected iff $SO_\omega(X) - \{\phi\}$ is a filter.*

PROOF. Suppose X is (ω)hyperconnected. Let $A, B \in SO_\omega(X) - \{\phi\}$. Then there exists a $k \in N$ such that for some U and V with $U, V \in \mathcal{J}_k$, we have

$$\begin{aligned} U &\subset A \subset (\omega)clU, \\ V &\subset B \subset (\omega)clV. \end{aligned}$$

Since X is (ω)hyperconnected, $U \cap V \neq \phi$ and $(\omega)cl(U \cap V) = X$. Therefore it follows that $A \cap B \neq \phi$ and

$$U \cap V \subset A \cap B \subset (\omega)cl(U \cap V).$$

Thus $A \cap B \in SO_\omega(X) - \{\phi\}$. Again if $B \supset A \in SO_\omega(X) - \{\phi\}$, there exists, for some k, a $U \in \mathcal{J}_k$ such that

$$\begin{aligned} U &\subset A \subset (\omega)clU \text{ and so} \\ U &\subset B \subset (\omega)clU \quad (\text{since } (\omega)clU = X). \end{aligned}$$

Hence $B \in SO_\omega(X) - \{\phi\}$. Therefore $SO_\omega(X) - \{\phi\}$ is a filter.

Since every (ω)open set is (ω)semiopen, the converse follows. QED

It is easy to see that the union of an arbitrary number of $(\mathcal{J}_n - \omega)$semiopen sets is $(\mathcal{J}_n - \omega)$semiopen. Also if X is (ω)hyperconnected, then the intersection of a finite number of $(\mathcal{J}_n - \omega)$semiopen sets is $(\mathcal{J}_n - \omega)$semiopen. Thus if X is (ω)hyperconnected, then the class $(\mathcal{J}_n)SO_\omega(X) = S_n$ forms a topology on X and $S_n \subset S_{n+1}$. Hence $\{S_n\}$ is an (ω)topology on X.

From Theorem 7, we get the following result.

Theorem 8. *If $(X, \{\mathcal{J}_n\})$ is (ω)hyperconnected, then so is $(X, \{S_n\})$.*

Corollary 4. *If $(X, \{\mathcal{J}_n\})$ is maximal (ω)hyperconnected, then $\cup_n \mathcal{J}_n = \cup_n S_n$.*

For any set $A \notin \cup_n \mathcal{J}_n$, let $\mathcal{J}_n(A)$ denote the simple extension (Levine [8]) of $\mathcal{J}_n$. Then $(X, \{\mathcal{J}_n(A)\})$ forms an (ω)topology on X and $\mathcal{J}_n \subset \mathcal{J}_n(A)$ for all n. We call $\{\mathcal{J}_n(A)\}$, a simple extension of $\{\mathcal{J}_n\}$.

Theorem 9. *If* $(X, \{\mathcal{J}_n\})$ *is maximal* (ω)*hyperconnected, then* $SO_\omega(X) - \{\phi\}$ *is an ultrafilter.*

PROOF. Suppose $(X, \{\mathcal{J}_n\})$ is maximal (ω)hyperconnected. For $E \subset X$, suppose $E \notin SO_\omega(X, \{\mathcal{J}_n\}) - \{\phi\}$. Then $E \notin \cup_n \mathcal{J}_n$. Let us consider the simple extension $\{\mathcal{J}_n(E)\}$ of $\{\mathcal{J}_n\}$. Since $(X, \{\mathcal{J}_n\})$ is maximal (ω)hyperconnected, $(X, \{\mathcal{J}_n(E)\})$ is not (ω)hyperconnected. Therefore for some n, there exist two nonempty sets $G, H \in \mathcal{J}_n(E)$ such that $G \cap H = \phi$. Let $G = G_1 \cup (G_2 \cap E)$ and $H = H_1 \cup (H_2 \cap E)$ where $G_1, G_2, H_1, H_2 \in \mathcal{J}_n$. Then $G_1 \cap H_1 = \phi$. Since $(X, \{\mathcal{J}_n\})$ is (ω)hyperconnected, either $G_1 = \phi$ or $H_1 = \phi$. Suppose $G_1 = \phi$. If $H_1 = \phi$, then $G_2 \neq \phi$ and $H_2 \neq \phi$, since $G \neq \phi$ and $H \neq \phi$. Thus by (ω)hyperconnectivity of $(X, \{\mathcal{J}_n\})$, $G_2 \cap H_2 \neq \phi$. Again since $G \cap H = \phi$, we have $G_2 \cap H_2 \cap E = \phi$. Hence $G_2 \cap H_2 \subset E^c$, and therefore by Theorem 7, $E^c \in SO_\omega(X, \{\mathcal{J}_n\}) - \{\phi\}$. Now consider the case $H_1 \neq \phi$. Since $G \neq \phi$, we have $G_2 \neq \phi$. Therefore $G_2 \cap H_1 \neq \phi$. From the relation $G \cap H = \phi$, it follows that $(G_2 \cap E) \cap H_1 = \phi$. Hence $G_2 \cap H_1 \subset E^c$, and so $E^c \in SO_\omega(X, \{\mathcal{J}_n\}) - \{\phi\}$. Thus $SO_\omega(X) - \{\phi\}$ is an ultrafilter. QED

Using Corollary 4, we get the following result.

Corollary 5. *If* $(X, \{\mathcal{J}_n\})$ *is maximal* (ω)*hyperconnected, then the class of all nonempty* (ω)*open sets is an ultrafilter.*

Definition 10. $(X, \{\mathcal{J}_n\})$ is said to be an (ω)door space if for every subset E of X, $E \in \mathcal{J}_n$ or $E^c \in \mathcal{J}_n$ for some n.

We now show that for an (ω)door space $(X, \{\mathcal{J}_n\})$, the topological spaces $(X, \mathcal{J}_n)$ need not be door (Kelley [6]).

Example 4. Let us define an (ω)topological space $(N, \{\mathcal{J}_n\})$ as follows:

$$\mathcal{J}_n = \{\phi, N\} \cup \{E \subset \{1, 2, \ldots, n\} \mid 1 \in E\} \text{ for all } n < 10, \text{ and}$$
$$\mathcal{J}_n = \{\phi\} \cup \{E \subset N \mid 1 \in E\} \text{ for all } n \geq 10.$$

Then clearly $(N, \{\mathcal{J}_n\})$ is an (ω)door space. But for any $n < 9$, the topological space $(N, \mathcal{J}_n)$ is not a door space.

Example 5. Taking $X = [0, 1)$, let us define an (ω)topological space $(X, \{\mathcal{J}_n\})$ as follows:

$\mathcal{J}_n$ is the topology generated by the subbase $\{E \mid E \subset [0, 1 - \frac{1}{n+1})$ and $0 \in E\} \cup \{\phi$, all the sets $\subset X$ containing 0 and having 1 as a limit point$\}$.

Then it is easy to see that $(X, \{\mathcal{J}_n\})$ forms an (ω)door space. However, $(X, \mathcal{J}_n)$ is not a door space for any n.

Definition 11. A property P of an (ω)topological space $(X, \{\mathcal{J}_n\})$ is said to be contractive(resp. expansive) if it is possessed by (ω)topological spaces $(X, \{\mathcal{J}'_n\})$ whenever it is possessed by $(X, \{\mathcal{J}_n\})$, where the (ω)topologies $\{\mathcal{J}'_n\}$ are weaker(resp. stronger) than $\{\mathcal{J}_n\}$.

It is clear that (ω)connectedness and (ω)hyperconnectedness are contractive properties while (ω)doorness is an expansive property.

Theorem 10. *$(X, \{\mathcal{J}_n\})$ is an (ω)hyperconnected (ω)door space iff $\mathcal{F} = (\cup_n \mathcal{J}_n) - \{\phi\}$ is an ultrafilter.*

Proof. Suppose $(X, \{\mathcal{J}_n\})$ is an (ω)hyperconnected (ω)door space. Then for $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$. Now let $B \supset A \in \mathcal{F}$. If $B = X$, then $B \in \mathcal{F}$. If $B \neq X$, then $B^c \notin \mathcal{F}$, since otherwise $A \cap B^c \neq \phi$. Therefore $B \in \mathcal{F}$. Hence $\mathcal{F}$ is a filter and so an ultrafilter.

The converse part is obvious. QED

Theorem 11. *If $(X, \{\mathcal{J}_n\})$ is (ω)hyperconnected and (ω)door, then $(X, \{\mathcal{J}_n\})$ is maximal (ω)hyperconnected and minimal (ω)door.*

Proof. Let $\{\mathcal{J}'_n\}$ be an (ω)topology on X stronger than $\{\mathcal{J}_n\}$ such that $(X, \{\mathcal{J}'_n\})$ is (ω)hyperconnected. If possible, suppose G be a nonempty set with $G \in \mathcal{J}'_m$ for some m and $G \notin \cup_n \mathcal{J}_n$. Since $(X, \{\mathcal{J}_n\})$ is (ω)door, $X - G \in \mathcal{J}_l$ for some l. Hence $X - G \in \cup_n \mathcal{J}'_n$. This contradicts the fact that $(X, \{\mathcal{J}'_n\})$ is (ω)hyperconnected. Thus $G \in \cup_n \mathcal{J}_n$. Therefore $\cup_n \mathcal{J}'_n = \cup_n \mathcal{J}_n$.

Again let $(X, \{\mathcal{J}'_n\})$ be an (ω)door space such that $\cup_n \mathcal{J}'_n \subset \cup_n \mathcal{J}_n$. Suppose, if possible, G is a nonempty set with $G \in \cup_n \mathcal{J}_n$ and $G \notin \cup_n \mathcal{J}'_n$. But then $X - G \in \cup_n \mathcal{J}'_n$. So $X - G \in \cup_n \mathcal{J}_n$ which contradicts the (ω)hyperconnectedness of $(X, \{\mathcal{J}_n\})$. Therefore $\cup_n \mathcal{J}'_n = \cup_n \mathcal{J}_n$. QED

Definition 12. A set $E \subset X$ is said to be (ω)dense if $(\omega)clE = X$.

Definition 13. X is said to be submaximal if every (ω)dense subset of X is (ω)open.

Theorem 12. *$(X, \{\mathcal{J}_n\})$ is maximal (ω)hyperconnected iff it is submaximal and (ω)hyperconnected.*

Proof. Suppose $(X, \{\mathcal{J}_n\})$ is maximal (ω)hyperconnected. Let $E \subset X$ be (ω)dense. By Corollary 5, $(\cup_n \mathcal{J}_n) - \{\phi\}$ is an ultrafilter. Therefore E must be (ω)open. For, if E is not (ω)open, then E^c must be (ω)open, since $(\cup_n \mathcal{J}_n) - \{\phi\}$ is an ultrafilter. Therefore E is (ω)closed and hence $(\omega)clE = E$. Again since E is (ω)dense, $(\omega)clE = X$. Therefore $E = X$. Thus X is submaximal.

Conversely, suppose $(X, \{\mathcal{J}_n\})$ is submaximal and (ω)hyperconnected. Let $(X, \{\mathcal{J}'_n\})$ be (ω)hyperconnected with $\cup_n \mathcal{J}'_n \supset \cup_n \mathcal{J}_n$. If $G \in \cup_n \mathcal{J}'_n$ be a nonempty set, then, since $(X, \{\mathcal{J}'_n\})$ is (ω)hyperconnected, $(\omega)clG$(the (ω)closure of G in $(X, \{\mathcal{J}'_n\})$) coincides with X. This implies that $(\omega)clG$(the (ω)closure of G in $(X, \{\mathcal{J}_n\})$)$= X$(since $(\omega)clG$(w.r.t $(X, \{\mathcal{J}_n\})$)$\supset$ $(\omega)clG$(w.r.t $(X, \{\mathcal{J}'_n\})$)), and so it follows that G is (ω)dense in $(X, \{\mathcal{J}_n\})$. Hence $G \in \cup_n \mathcal{J}_n$. Thus $\cup_n \mathcal{J}'_n = \cup_n \mathcal{J}_n$. QED

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