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Postulation of double lines and associated objects in the range of quadrics

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Abstract. Fix a closed subscheme $U \subset \mathbb{P}^n$. Here we study the integer $h^0(\mathcal{I}_U(2)) - h^0(\mathcal{I}_{U \cup Y}(2))$ when Y is a general double line, a general reducible conic, a general chain of lines or some unreduced structure associated to double structures on linear subspaces.

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Introduction and Notation

Starting from [6] many papers studied the postulation of general disjoint unions inside $\mathbb{P}^n$ of certain natural objects, e.g. linear spaces ([2], [3], [4]). As a tool even in [6], Example 2.1.1, and in the papers influenced by [6] some unreduced schemes were used (sundials in [4]). Here we propose the study of other related unreduced schemes. Following [2] we work out the cohomological properties of their general disjoint unions in the range of quadrics, i.e. for the linear system $|\mathcal{O}_{\mathbb{P}^n}(2)|$. We work over an algebraically closed base field $\mathbb{K}$ such that char($\mathbb{K}$) = 0. We use very much this assumption, not only to quote [2], but at several places. Call n the dimension of the ambient projective space. The interested reader may extend the proofs to the case char($\mathbb{K}$) $\gg n$ (it is sufficient to assume char($\mathbb{K}$) $> 2^n$).

For any scheme M and any $P \in M_{reg}$ let $\chi_M(P)$ denote the first infinitesimal neighborhood of P in M, i.e., the closed subscheme of M with $(\mathcal{I}_P)^2$ as its ideal sheaf. The scheme $\chi_M(P)$ is zero-dimensional, $(\chi_M(P))_{red} = \{P\}$ and $\deg(\chi_M(P)) = m + 1$, where m is the dimension of M at its smooth point P.

We only consider very particular double structures on a line. For general double structures on a line, see [5], pp. 32–34, [7] and [1]. Let $C \subset \mathbb{P}^n$, $n \geq 3$,

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be a closed subscheme. We will say that C is a *double line* if there is a 3-dimensional linear subspace $M \subseteq \mathbb{P}^n$ and a smooth quadric surface $Q \subset M$ such that C is a divisor of type $(2,0)$ or $(0,2)$ on Q and it is not reduced. Notice that C is a flat degeneration inside Q and hence inside $\mathbb{P}^n$ of a family of pairs of disjoint lines. Hence if C is a double line, then its Hilbert polynomial p_C satisfies $p_C(t) = 2t+2$ for all $t \in \mathbb{Z}$. We will say that C is an *unreduced conic* if C_{red} is a line and there is a plane $N \subset \mathbb{P}^n$ such that $C \subset N$ and C is a degree 2 Cartier divisor of C. In this case we have $p_C(t) = 2t+1$ for all $t \in \mathbb{Z}$. We will say that C is a *pointed unreduced conic* if there are a plane N, a tridimensional linear subspace $M \subset \mathbb{P}^n$ containing N, an unreduced conic $B \subset N$ and $P \in B_{red}$ such that $C = B \cup \chi_M(P)$. In this case we have $p_C(t) = 2t+2$ for all t. Notice that $C_{red} = B_{red}$ is a line, $C \subset M$ and C is uniquely determined by the flag $P \in B_{red} \subset N \subset M$. Hence any two pointed unreduced conics of $\mathbb{P}^n$ are projectively equivalent. It is easy to see that any pointed unreduced conic $C \subset M$ is the flat limit inside M and hence inside $\mathbb{P}^n$ of a family of disjoint unions of 2 lines. Here we will prove the stronger statement that any pointed unreduced conic is a flat limit of a flat family of double lines (see Lemma 1). Let $C \subset \mathbb{P}^n$ be a double line. Take a general plane $N \subset \mathbb{P}^n$ containing the line $A := C_{red}$. The scheme $N \cap C$ will be called a *pointed line* (see Lemmas 2 and 4 and Remark 2 for more).

We generalize the notion of pointed line in $\mathbb{P}^n$ in the following way. Fix an integer t such that $1 \le t \le n-2$. A *pointed t-plane* or a *pointed linear subspace of dimension t* of $\mathbb{P}^n$ is a scheme $T \cup \chi_N(P)$ with T a t-dimensional linear subspace of $\mathbb{P}^n$, N a $(t+1)$-dimensional linear subspace of $\mathbb{P}^n$ and $P \in T \subset N$. Any pointed t-plane is uniquely determined by the flag (P, T, N) and the converse holds. A *pointed linear subspace* of $\mathbb{P}^n$ is a pointed t-plane for some (uniquely determined) integer $t \in \{1, \dots, n-2\}$. In section 1 we will prove the following result, which generalize [2], theorem 4.3. Its proof will easily follow from the statement of [2] , theorem 4.3. We prove the needed reduction in an abstract setting (see Proposition 4).

Theorem 1. *Fix integers $n \ge 3$, $a \ge 0$, $b \ge 0$, $m_1 \ge \cdots \ge m_a > 0$, $t_1 \ge \cdots \ge t_b > 0$. Assume $m_1 + m_2 < n$ (if $a \ge 2$) $m_1 + t_1 < n$ (if $a > 0$ and $b > 0$) and $t_1 + t_2 < n$ (if $b \ge 2$). Let $X \subset \mathbb{P}^n$ be a general union of a linear subspaces of dimension $m_1, \dots, m_a$ and b pointed linear subspaces of dimension $t_1, \dots, t_b$. Then $h^0(\mathcal{I}_X(2)) = \max\{0, \binom{n+2}{2} - \sum_{i=1}^{a} \binom{m_i+2}{2} - \sum_{i=1}^{b} \binom{t_i+2}{2} - b\}$ and $h^1(\mathcal{I}_X(2)) = \max\{0, \sum_{i=1}^{a} \binom{m_i+2}{2} + \sum_{i=1}^{b} \binom{t_i+2}{2} + b - \binom{n+2}{2}\}$.*

Remark 1. In the set-up of Theorem 1 set $Y := X_{red}$. Notice that $Y \subset \mathbb{P}^r$ is a general union of $a+b$ linear subspaces of dimension $m_1, \dots, m_a, t_1, \dots, t_b$. Our assumptions on m_i and t_j imply that Y has $a+b$ connected components of dimension $m_1, \dots, m_a, t_1, \dots, t_b$. The case $b = 0$ of Theorem 1 is just [2], theorem

4.3. The explicit computation in [2], theorem 4.3, gives the corresponding one in the case $b > 0$, because $h^0(X, \mathcal{O}_X(2)) = h^0(Y, \mathcal{O}_Y(2)) + b$ and $h^1(X, \mathcal{O}_X(2)) = 0$ (see Lemma 4).

We also prove several results on the Hilbert function in the range of quadrics of the union of an arbitrary closed subscheme and a general pointed t-linear subspace (Proposition 1), a general line (Propositions 2 and 4), a plane unreduced conic (Proposition 5) and some other reducible union of lines (Propositions 7 and 8).

Let Π be a linear subspace of the projective space $|\mathcal{O}_{\mathbb{P}^n}(2)|$ of all quadric hypersurfaces of $\mathbb{P}^n$. For any subscheme $Z \subset \mathbb{P}^n$ set $\Pi(-Z) := \{Q \in \Pi : Z \subset Q\}$. If $\Pi \neq \emptyset$ and B is its base-locus, then Π induces a morphism $\psi : \mathbb{P}^n \setminus B \to \mathbb{P}^m$, $m := \dim(\Pi)$. Set $\rho(\Pi) := \dim(\mathrm{Im}(\psi(\mathbb{P}^n \setminus B)))$. Since $\mathrm{char}(\mathbb{K}) = 0$, ψ is separable and hence $\rho(\Pi)$ is the rank of the differential $d\psi(P)$ of ψ at a general $P \in \mathbb{P}^n \setminus B$.

As a consequence of the results proved in section 2 we get the following result.

Theorem 2. *Fix a closed subscheme $U \subset \mathbb{P}^n$, $n \geq 3$. Assume that the base locus of $|\mathcal{I}_U(2)|$ contains no hyperplane and that $\rho(|\mathcal{I}_U(2)|) \geq 4$. Let $C \subset \mathbb{P}^n$ be a general double line and $E \subset \mathbb{P}^n$ a general pointed unreduced conic. Then $h^0(\mathcal{I}_{U \cup C}(2)) = h^0(\mathcal{I}_{U \cup E}(2)) = h^0(\mathcal{I}_U(2)) - 6$ and $h^1(\mathcal{I}_{U \cup C}(2)) = h^1(\mathcal{I}_{U \cup E}(2)) = h^1(\mathcal{I}_U(2))$.*

In the set-up of Theorem 2 notice that $h^0(\mathcal{O}_C(2)) = h^0(\mathcal{O}_E(2)) = 6$.

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1 Preliminary Lemmas and the proof of Theorem 1

For any sheaf $\mathcal{F}$ on $\mathbb{P}^n$ we write $H^i(\mathcal{F})$ or $h^i(\mathcal{F})$ instead of $H^i(\mathbb{P}^n, \mathcal{F})$ or $h^i(\mathbb{P}^n, \mathcal{F})$. Let $G(x, n)$ denote the Grassmannian of all x-dimensional linear subspaces of $\mathbb{P}^n$.

Lemma 1. *Let $C \subset \mathbb{P}^n$, $n \geq 3$, be a pointed unreduced conic. Fix a 3-dimensional linear subspace $M \subset \mathbb{P}^n$ such that $C \subset M$. Then C is in the closure $\overline{E}$ (in the Hilbert scheme Hilb(M) of M) of the set of all double lines of M.*

Proof. Since any two pointed unreduced conics of M are projectively equivalent, it is sufficient to prove that $\overline{E}$ contains at least one pointed unreduced conic. Assume that $M = \mathrm{Proj}(\mathbb{K}[x, y, z, w])$. Take the family $\{Z_t\}_{t \in \mathbb{K} \setminus \{0\}}$ of double lines defined by the homogeneous ideal $(x^2, xy, y^2, yz + tzw)$, $t \in \mathbb{K} \setminus \{0\}$. The flat limit for $t \to 0$ of this family is the pointed unreduced conic defined by $(x^2, y) \cap (x, y, z)^2$. QED

We recall that for any projective scheme W, every closed subscheme Z of W and every effective divisor H of W the residual scheme $\mathrm{Res}_H(Z)$ of Z with respect to H is the closed subscheme of W with $\mathcal{I}_Z : \mathcal{I}_H$ as its ideal sheaf. To apply Horace method as in [6] it is important to control the residual schemes of our unreduced objects with respect to hyperplanes.

Remark 2. We call any scheme $T \cup \chi_N(P)$ with T a line, N a plane and $P \in T \subset N$ a *pointed line* or *the pointed line associated to the flag* (P, T, N). Indeed, $T \cup \chi_N(P)$ is uniquely determined by the flag (P, T, N), while any such flag gives a unique pointed scheme. The scheme $T \cup \chi_N(P)$ is a flat degeneration inside N of a family $\{T \cup \{P_\lambda\}\}$ with $P_\lambda \in N \setminus T$.

Lemma 2. *Let $H \subset \mathbb{P}^n$, $n \geq 4$, be a hyperplane and $B \subset \mathbb{P}^n$ a double line such that $T := B_{red} \subset H$ and $B \nsubseteq H$. Then $Res_H(B) = T$ and $B \cap H = T \cup \chi_N(P)$ for some $P \in T$ and some plane $N \subset H$ (scheme-theoretic intersection). Conversely, for any hyperplane H, any line $T \subset H$, any $P \in T$ and any plane N such that $T \subset N \subset H$ there is a double line D such that $D_{red} = T$ and $D \cap H = T \cup \chi_N(P)$.*

Proof. Let $M := \langle B \rangle \subset \mathbb{P}^n$ be the 3-dimensional linear subspace defining B. Let $Q \subset M$ be a smooth quadric such that $B \subset Q$. Choose homogeneous coordinates $x_0, \ldots, x_n$ such that M has equations $x_i = 0$ for all $i \geq 4$, Q has equations $x_0x_1 + x_2x_3 = 0$, $x_i = 0$ for all $i \geq 4$, and T has equations $x_1 = x_2 = 0$ inside M. Hence $N := H \cap M$ has the equation $ax_1 + bx_2 = 0$ for some $(a, b) \neq (0, 0)$, say $a \neq 0$, inside M. Hence N has the equation $x_1 = cx_2$, $c := -b/a$, inside M. The scheme B has equations $x_1^2 = x_1x_2 = x_2^2 = x_0x_1 + x_2x_3 = 0$ inside M. Hence $B \cap H = B \cap N$ has equations $c^2x_2^2 = cx_2^2 = x_2^2 = cx_0x_2 + x_2x_3 = x_1 - cx_2 = 0$ inside M, i.e. it has equations $x_2^2 = x_2(cx_0 + x_3) = x_1 - cx_2 = 0$. Write $z_i = x_i$, for all $i \notin \{1, 3\}$, $z_1 = x_1 - cx_2$ and $z_3 = cx_0 + x_3$. Hence $\mathrm{Res}_H(B) = T$ and $B \cap H = B \cap N$ has equations $z_2^2 = z_2z_3 = z_1 = 0$, i.e. the equations of $T \cup \chi_N(P)$.

The converse part follows from the following observation. For a fixed pair (T, H), any two planes N, N' containing T and contained in H and any automorphism $h : H \to H$ such that $h(P) = P$, $h(T) = T$ and $h(N) = N'$ the morphism h maps $T \cup \chi_N(P)$ isomorphically onto $T \cup \chi_{N'}(P)$. QED

The proof of Lemma 2 gives the following result, which justifies the introduction of t-pointed linear subspaces.

Lemma 3. *Fix integers n, t such that $1 \leq t \leq n - 2$, $N \in G(t, n)$ and $H \in G(n - 1, n)$ such that $H \supset N$. Let M be a general $(t + 1)$-dimensional linear subspace of $\mathbb{P}^n$ containing N. Let $Z \subset M$ the degree 2 Cartier divisor with N as its support. Then $Res_H(Z) = N$ and $Z \cap H$ is a general t-pointed linear subspace of H with N as its support.*

We may change the previous statement first fixing M containing N and then taking H general among the hyperplanes containing N.

Lemma 4. *Let $Z \subset \mathbb{P}^n$ be any pointed linear subspace. Set $T := Z_{red}$. We have $h^0(Z, \mathcal{O}_Z(y)) = h^0(T, \mathcal{O}_T(y)) + 1$ and $h^i(Z, \mathcal{O}_Z(y)) = 0$ for every integer $y \geq 0$ and every integer $i > 0$.*

Proof. Since $\chi(\mathcal{O}_Z(x)) = \chi(\mathcal{O}_T(x)) + 1$ for all $x \in \mathbb{Z}$, it is sufficient to prove $h^i(Z, \mathcal{O}_Z(y)) = 0$ for all $i \geq 1$ and all $y \geq 0$. Let η be the kernel of the surjection $\mathcal{O}_Z(y) \to \mathcal{O}_T(y)$ induced by the closed embedding $T \hookrightarrow Z$ (for $y = 0$ it is the nilradical of $\mathcal{O}_Z$). The sheaf η is a 1-dimensional vector space supported by the non-reduced point of Z. Hence $h^j(Z, \eta) = 0$ for all $j \geq 1$. Use the exact sequence

$$0 \to \eta \to \mathcal{O}_Z(y) \to \mathcal{O}_T(y) \to 0 \tag{1}$$

and that $h^i(T, \mathcal{O}_T(y)) = 0$ for all $y \geq 0$, and $i \geq 1$. QED

Proposition 1. *Fix integers $n \geq t+2 \geq 3$, a closed subscheme $U \subset \mathbb{P}^n$ and $T \in G(t, n)$ such that $T \nsubseteq U$. Let $Z \subset \mathbb{P}^n$ be a general pointed t-linear subspace such that $Z_{red} = T$.*

(i) If every quadric hypersurface $Q \in |\mathcal{I}_{U \cup T}(2)|$ is a cone with vertex containing T, then $h^0(\mathcal{I}_{U \cup Z}(2)) = h^0(\mathcal{I}_{U \cup T}(2))$.

(ii) If some quadric hypersurface $Q \in |\mathcal{I}_{U \cup T}(2)|$ is not a cone with vertex containing T, then $h^0(\mathcal{I}_{U \cup Z}(2)) = h^0(\mathcal{I}_{U \cup T}(2)) - 1$.

Proof. If $h^0(\mathcal{I}_{U \cup T}(2)) = 0$, then part (i) is obvious, while the assumption of part (ii) is not satisfied. Hence we may assume $h^0(\mathcal{I}_{U \cup T}(2)) > 0$. Since T is not contained in the scheme U, a general $P \in T$ is not contained in U_{red}. Fix a pointed t-linear space Z' with T as its reduction and associated to a point $P' \in T \setminus T \cap U_{red}$. As in (1) call η the kernel of the surjection $\mathcal{O}_{U \cup Z'}(2) \to \mathcal{O}_{U \cup T}(2)$. As in the proof of Lemma 1 we get $h^0(U \cup Z', \mathcal{O}_{U \cup Z'}(2)) = h^0(U \cup T, \mathcal{O}_{U \cup T}(2)) + 1$. The coherent sheaf $\mathcal{I}_{U \cup T}(2)/\mathcal{I}_{U \cup Z'}(2)$ is supported by P' and $h^0(\mathbb{P}^n, \mathcal{I}_{U \cup T}(2)/\mathcal{I}_{U \cup Z'}(2)) = 1$. Hence either $h^0(\mathcal{I}_{U \cup Z'}(2)) = h^0(\mathcal{I}_{U \cup T}(2))$ or $h^0(\mathcal{I}_{U \cup Z'}(2)) = h^0(\mathcal{I}_{U \cup T}(2)) - 1$. Fix any $Q \in |\mathcal{I}_{U \cup T}(2)|$. The quadric Q is a cone with vertex containing $O \in \mathbb{P}^n$ if and only if $\chi_{\mathbb{P}^n}(O) \subset Q$. Notice that $\chi_{\mathbb{P}^n}(O) \cup T$ contains any pointed linear subspace with T as its reduction and O as the support of its nilpotent subsheaf. Hence we get (i). Now assume that Q is not a cone with vertex containing T and fix $P \in T \setminus U_{red}$ such that Q is smooth at P. Since the tangent space $T_P Q$ of Q at P is a hyperplane of $\mathbb{P}^n$, there is a $(t+1)$-dimensional linear subspace N of $\mathbb{P}^n$ containing T, but not contained in $T_P Q$. Set $Z_1 := T \cup \chi_N(P)$. Since $Q \notin |\mathcal{I}_{U \cup Z_1}(2)|$, we get $h^0(\mathcal{I}_{U \cup Z_1}(2)) = h^0(\mathcal{I}_{U \cup T}(2)) - 1$. The set of all schemes Z_1 as above covers a non-empty open subset of the set of all pointed linear spaces with T as their reduction. Hence we get part (ii) for a general Z. QED

Lemma 5. *Let Π be a linear subspace of the projective space $|\mathcal{O}_{\mathbb{P}^n}(2)|$ of all quadric hypersurfaces of $\mathbb{P}^n$. Set $m := \dim(\Pi)$.*

(i) There is no integer x such that $x \geq 0$, $\binom{x+2}{2} \leq m-1$ and every $Q \in \Pi(-T)$ is a cone with vertex containing T for a general $T \in G(x,n)$.

(ii) Assume $m \leq \binom{y+2}{2}$ for some integer $y > 0$ and that $\Pi(-A) \neq \emptyset$ for a general $A \in G(y,n)$. Then for a general $A \in G(y,n)$ the general $Q \in \Pi(-A)$ is not a cone with vertex containing A.

PROOF. Take the set-up of (i). Assume the existence of such an integer x. Fix a general x-dimensional linear subspace $T \subset \mathbb{P}^n$ and a general $Q \in \Pi(-T)$. By assumption Q is a cone with vertex containing T. First assume that Q is integral. Let Q_x be the set of all x-dimensional linear subspaces of Q. Let Γ be any irreducible component of Q_x containing T. Fix a general $T' \in \Gamma$. Since T' (as a deformation of T) may be seen as a general x-dimensional linear subspace of $\mathbb{P}^n$ and $Q \in \Pi(-T')$, Q is a cone with vertex containing T'. Since we may find $T' \in \Gamma$ passing through a general point of Q, we get that Q is the double of a hyperplane, contradiction. Now assume that Q is reduced, but not integral, say $Q = H \cup M$ with H and M hyperplanes and $H \neq M$. Since $H \cap M \subseteq \mathrm{Sing}(Q)$, Q is general and we work in characteristic zero, $H \cap M$ is contained in the base locus of Π. Since $T \subseteq \mathrm{Sing}(Q)$, we have $T \subseteq H \cap M$. Take as Γ the family of all x-dimensional linear subspaces of H. Since Γ is irreducible, a general $T' \in \Gamma$ is not contained in $H \cap M$ and $H \cap M$ is the vertex of Q, we get a contradiction. Now assume that Q is the double of a hyperplane. Since we work in characteristic $\neq 2$, this implies $m = 0$, contradicting the inequality $\binom{x+2}{2} \leq m-1$ even when $x = 0$.

In the set-up of (ii) we use that we may move $A \in G(y,n)$ preserving the non-emptiness of $\Pi(-A)$. QED

Proof of Theorem 1. Our assumptions on n, m_i, t_j imply the existence inside $\mathbb{P}^n$ of a disjoint union of a disjoint linear spaces of dimension $m_1, \dots, m_a$ and b pointed linear spaces of dimension $t_1, \dots, t_b$.

Fix integers $u_1, \dots, u_x$ for which $\mathbb{P}^n$ contains a disjoint union of x linear spaces of dimension $u_1, \dots, u_x$, i.e., assume $u_i + u_j < n$ for all $i \neq j$ and $u_1 \leq n$. Let $E \subset \mathbb{P}^n$ be a general union of x linear spaces of dimension $u_1, \dots, u_x$. Hence E has x connected components. Since $\chi(\mathcal{O}_E(2)) = h^0(E, \mathcal{O}_E(2)) = \sum_{i=1}^{x} \binom{u_i+2}{2}$, [2], Theorem 4.3, says that either $h^0(\mathcal{I}_E(2)) = 0$ (case $\sum_{i=1}^{x} \binom{u_i+2}{2} \geq \binom{n+2}{2}$) or $h^1(\mathcal{I}_E(2)) = 0$ (case $\sum_{i=1}^{x} \binom{u_i+2}{2} \leq \binom{n+2}{2}$).

Let $F \subset \mathbb{P}^n$ be any disjoint union of linear subspaces and of pointed linear spaces. Call y the number of the unreduced components of F. We have $h^i(F, \mathcal{O}_F(2)) = 0$ for all $i > 0$ and $\chi(\mathcal{O}_F(2)) = h^0(F, \mathcal{O}_F(2)) = h^0(F_{red}, \mathcal{O}_{F_{red}}(2)) + y$ (Lemma 4). Hence $h^0(\mathcal{I}_F(2)) - h^1(\mathcal{I}_F(2)) = \binom{n+2}{2} - \chi(\mathcal{O}_F(2))$. Hence if we know y and the dimension of the connected components of F_{red}, then knowing $h^0(\mathcal{I}_F(2))$ is equivalent to knowing $h^1(\mathcal{I}_F(2))$.

Since the case $b = 0$ is true ([2], Theorem 4.3) we may prove Theorem 1 by induction on b. Assume $b > 0$ and that the result is true for the integer $b' := b - 1$ and all integers $a' \geq 0$. Let $U \subset \mathbb{P}^n$ be a general union of a linear spaces and $b-1$ pointed linear spaces of dimension $t_1, \dots, t_{b-1}$. Let $T \subset \mathbb{P}^n$ be a general t_b-dimensional linear subspace. Notice that both U and $U \cup T$ have $b-1$ unreduced components. By the inductive assumption we know the integers $h^0(\mathcal{I}_U(2))$ and $h^0(\mathcal{I}_{U\cup T}(2))$. If $h^0(\mathcal{I}_{U\cup T}(2)) = 0$, then $h^0(\mathcal{I}_{U\cup Z}(2)) = 0$ for any pointed t_b-linear subspace with T as its reduction. Since $U \cup Z$ is a general union X of a linear subspaces of dimension $m_1, \dots, m_a$ and b pointed linear subspaces of dimension $t_1, \dots, t_b$, we get $h^0(\mathcal{I}_X(2)) = 0$.

Now assume $h^0(\mathcal{I}_{U\cup T}(2)) > 0$. Since T and Z are general, the support of the nilpotent sheaf of U does not intersect T and the support of the nilpotent sheaf of Z is not a point of U_{red}. Hence the proof of Lemma 4 gives that the coherent sheaf $\mathcal{I}_{U\cup T}(2)/\mathcal{I}_{U\cup Z}(2)$ is supported by a point, that $h^0(\mathbb{P}^m, \mathcal{I}_{U\cup T}/\mathcal{I}_{U\cup Z}(2)) = 1$, and that $h^0(U\cup Z, \mathcal{O}_{U\cup Z}(2)) = h^0(U\cup T, \mathcal{O}_{U\cup T}(2)) + 1$. Hence either $h^0(\mathcal{I}_{U\cup Z}(2)) = h^0(\mathcal{I}_{U\cup T}(2))$ or $h^0(\mathcal{I}_{U\cup Z}(2)) = h^0(\mathcal{I}_{U\cup T}(2)) - 1$. To prove Theorem 1 for the integer b it is sufficient to prove $h^0(\mathcal{I}_{U\cup Z}(2)) < h^0(\mathcal{I}_{U\cup T}(2))$. By Proposition 1 it is sufficient to prove that not every element of $|\mathcal{I}_{U\cup T}(2)|$ is a cone with vertex containing T. Since T is taken general after fixing U, we may apply Lemma 5.

Since $h^0(U \cup Z, \mathcal{O}_{U\cup Z}(2)) = h^0((U \cup Z)_{red}, \mathcal{O}_{(U\cup Z)_{red}}(2)) + b$, we get the explicit values of $h^i(\mathcal{I}_X(2))$, $i = 0, 1$. QED

2 The other results

Proposition 2. *Fix a closed subscheme $U \subset \mathbb{P}^n$. Let E be the base locus of the linear system $|\mathcal{I}_U(2)|$. Let $A \subset \mathbb{P}^n$ be a general line.*

(i) If E contains a hyperplane, then $h^0(\mathcal{I}_{U\cup A}(2)) = \max\{h^0(\mathcal{I}_U(2)) - 2, 0\}$.

(ii) Assume that E does not contain a hyperplane. Then $h^0(\mathcal{I}_{U\cup A}(2)) = \max\{h^0(\mathcal{I}_U(2)) - 3, 0\}$ and $h^1(\mathcal{I}_{U\cup A}(2)) = h^1(\mathcal{I}_U(2)) + \max\{0, 3 - h^0(\mathcal{I}_U(2))\}$.

Proof. We have $h^0(\mathcal{I}_U(2)) = h^0(\mathcal{I}_E(2))$ and $h^0(\mathcal{I}_{U\cup A}(2)) = h^0(\mathcal{I}_{E\cup A}(2))$.

Since any 2 points of $\mathbb{P}^n$ are contained in a line, the inequality $h^0(\mathcal{I}_{U\cup A}(2)) \leq \max\{h^0\ (\mathcal{I}_U(2)) - 2, 0\}$ is obvious. Hence we may assume $h^0(\mathcal{I}_U(2)) \geq 3$. Since any quadric hypersurface containing 3 points of A contains A, we have $h^0(\mathcal{I}_{E\cup A}(2)) \geq h^0(\mathcal{I}_{E\cup A}(2)) - 3$.

(a) Here we assume $\dim(E) = n - 1$. In this case the $(n-1)$-dimensional part of the scheme E must be a hyperplane, because $h^0(\mathcal{I}_E(2)) \geq 2$. Since

$\dim(E) \ge n-1$, we have $E \cap A \neq \emptyset$. Taking a point of $E_{red} \cap A$ and two general points of A we see that $h^0(\mathcal{I}_{E\cup A}(2)) \ge h^0(\mathcal{I}_E(2)) - 2$. Hence (i) is true.

(b) Here we assume $\dim(E) \le n-2$. We may assume $h^0(\mathcal{I}_{E\cup A}(2)) \ge 3$ and we need to prove $h^0(\mathcal{I}_{E\cup A}(2)) \le h^0(\mathcal{I}_E(2)) - 3$. Since A is general, we have $E \cap A = \emptyset$. Hence $h^0(E \cup A, \mathcal{O}_{E\cup A}(2)) = h^0(E, \mathcal{O}_E(2)) + 3$. Hence $h^0(\mathcal{I}_{U\cup A}(2)) - h^0(\mathcal{I}_U(2)) + 3 = h^1(\mathcal{I}_{U\cup A}(2)) - h^1(\mathcal{I}_U(2))$. Since $h^0(\mathcal{I}_{E\cup A}(2)) \ge h^0(\mathcal{I}_E(2)) - 3$, it is sufficient to prove $h^0(\mathcal{I}_{U\cup A}(2)) \le h^0(\mathcal{I}_U(2)) - 3$. Fix a general $P \in \mathbb{P}^n$. Hence $P \notin E_{red}$. Hence $h^0(\mathcal{I}_{E\cup\{P\}}(2)) = h^0(\mathcal{I}_E(2)) - 1$. Fix a general $Q \in |\mathcal{I}_{E\cup\{P\}}(2)|$. The case $x = 0$ of Lemma 5 gives that Q is not a cone with vertex containing P. It is also obvious that Q is irreducible. Hence a general $P' \in Q$ is not in the base locus of $|\mathcal{I}_{U\cup\{P\}}(2)|$. Hence $h^0(\mathcal{I}_{U\cup\{P,P'\}}(2)) = h^0(\mathcal{I}_U(2)) - 2$. Let A be the line spanned by P and P'. Since Q is not a cone with vertex P and P' is general in Q, $A \nsubseteq Q$. Hence $h^0(\mathcal{I}_{U\cup A}(2)) \le h^0(\mathcal{I}_{U\cup\{P,P'\}}(2)) - 1 = h^0(\mathcal{I}_U(2)) - 3$. Since $H^0(U \cup A, \mathcal{O}_{U\cup A}(2)) \cong H^0(U, \mathcal{O}_U(2)) \oplus H^0(A, \mathcal{O}_A(2))$, we also get $h^1(\mathcal{I}_{U\cup A}(2)) = h^1(\mathcal{I}_U(2)) + \max\{0, 3 - h^0(\mathcal{I}_U(2))\}$.

QED

Proposition 3. *Fix a closed subscheme $U \subset \mathbb{P}^n$ such that the base locus E of the linear system $|\mathcal{I}_U(2)|$ does not contain a hyperplane. Let $T \subset \mathbb{P}^n$ be a general reducible conic. If $h^0(\mathcal{I}_U(2)) \le 5$, then $h^0(\mathcal{I}_{U\cup T}(2)) = 0$. If $h^0(\mathcal{I}_U(2)) \ge 5$, then $h^0(\mathcal{I}_{U\cup T}(2)) = h^0(\mathcal{I}_U(2)) - 5$ and $h^1(\mathcal{I}_{U\cup T}(2)) = h^1(\mathcal{I}_U(2))$.*

Proof. We have $h^0(T, \mathcal{O}_T(2)) = 5$. We may assume $h^0(\mathcal{I}_E(2)) = h^0(\mathcal{I}_U(2)) \ge 2$. Hence the assumption on E implies $\dim(E) \le n-2$. Hence for a general T we have $T \cap E = \emptyset$. Hence $T \cap U = \emptyset$. Hence $H^0(U \cup T, \mathcal{O}_{U\cup T}(2)) \cong H^0(U, \mathcal{O}_U(2)) \oplus H^0(T, \mathcal{O}_T(2))$. Hence $h^0(\mathcal{I}_{U\cup T}(2)) \ge h^0(\mathcal{I}_U(2)) - 5$ and $h^1(\mathcal{I}_{U\cup T}(2)) = h^1(\mathcal{I}_U(2)) + 5 - h^0(\mathcal{I}_U(2)) + h^0(\mathcal{I}_{U\cup T}(2))$.

Let $A \subset \mathbb{P}^n$ be a general line. Proposition 2 implies $h^0(\mathcal{I}_{U\cup A}(2)) = \max\{h^0(\mathcal{I}_U(2)) - 3, 0\}$. We may take as T the union of A and a general line A' intersecting A. We may take such a line with the additional condition that A' contains a general point of $\mathbb{P}^n$. Hence $h^0(\mathcal{I}_{U\cup A\cup A'}(2)) \le \max\{0, h^0(\mathcal{I}_{U\cup A}(2)) - 1\}$. Hence the lemma is true if $h^0(\mathcal{I}_U(2)) \le 4$. Now assume $h^0(\mathcal{I}_U(2)) \ge 5$. Hence $h^0(\mathcal{I}_{U\cup A}(2)) = h^0(\mathcal{I}_U(2)) - 3 \ge 2$. Fix a general $Q \in |\mathcal{I}_{U\cup A}(2)|$. First assume that Q is not a cone with vertex containing A. Since $h^0(\mathcal{I}_{U\cup A}(2)) \ge 2$, a general $P \in Q$ is not in the base locus of $|\mathcal{I}_{U\cup A'}(2)|$. Hence $h^0(\mathcal{I}_{U\cup A\cup\{P\}}(2)) = h^0(\mathcal{I}_{U\cup A}(2)) - 1 > 0$. Since Q is not a cone with vertex containing A and P is general in Q, there is a line $A' \subset \mathbb{P}^n$ such that $P \in A'$, $A' \cap A \neq \emptyset$ and $A' \nsubseteq Q$. Since $Q \in |\mathcal{I}_{U\cup A\cup\{P\}}(2)|$, but $Q \notin |\mathcal{I}_{U\cup A}(2)|$, we have $h^0(\mathcal{I}_{U\cup A\cup A'}(2)) \le h^0(\mathcal{I}_U(2)) - 5$. We may take $A \cup A'$ as T.

Now assume that Q is a cone with vertex containing A. The case $x = 1$ of Lemma 5 gives a contradiction. $\boxed{QED}$

Remark 3. For any closed subscheme $M \subset \mathbb{P}^n$ let $M^{(1)}$ denote the first infinitesimal neighborhood of M in $\mathbb{P}^n$, i.e. the closed subscheme of $\mathbb{P}^n$ with $(\mathcal{I}_M)^2$ as its ideal sheaf. If M is smooth, then for all integers $d \geq 2$ the projective space $|\mathcal{I}_{M^{(1)}}(d))|$ parametrizes all degree d hypersurfaces whose singular locus contains M. Hence $|\mathcal{I}_{M^{(1)}}(2)|$ parametrizes all quadric hypersurfaces whose vertex contains the linear space $\langle M\rangle$ spanned by M. Assume $M \neq \emptyset$ and set $x := \dim(\langle M\rangle)$. We get $h^0(\mathcal{I}_{M^{(1)}}(2)) = \binom{n-x+2}{2}$. Hence if $n - x = 2$ (resp. $n - x \leq 1$), then $h^0(\mathcal{I}_{M^{(1)}}(2)) = 6$ (resp. $h^0(\mathcal{I}_{M^{(1)}}(2)) \leq 3$).

Example 1. Fix an integer $n \geq 3$ and an $(n-3)$-dimensional linear subspace M of $\mathbb{P}^n$. Remark 3 gives $h^0(\mathcal{I}_{M^{(1)}}(2)) = 6$. Let $A \subset \mathbb{P}^n$ be a general line. Fix any $Q \in |\mathcal{I}_{M^{(1)}}(2)|$. Since Q has rank at most 3, every line contained in Q intersects the vertex of Q. Hence for a general line $A \subset \mathbb{P}^n$ every element of $|\mathcal{I}_{M^{(1)}\cup A}(2)|$ is singular at some point of A.

Proposition 3, Remark 3 and Remark 1 give the following improvement of the case $x = 1$ of Lemma 4.

Proposition 4. *Fix a closed subscheme $U \subset \mathbb{P}^n$. Let B (resp. D) denote the set-theoretic (resp. scheme-theoretic) base locus of the linear system $|\mathcal{I}_U(2)|$. Assume $h^0(\mathcal{I}_U(2)) \geq 5$ and that B does not contains a hyperplane. Let $A \subset \mathbb{P}^n$ be a general line. A general element of $|\mathcal{I}_{U\cup A}(2)|$ is smooth at every point of A if and only if there is no $(n-3)$-dimensional linear subspace $M \subset \mathbb{P}^n$ such that either $D = M^{(1)}$ (case $h^0(\mathcal{I}_U(2)) = 6$) or $D = M^{(1)} \cup L$ for a uniquely determined line $L \subset \mathbb{P}^n$ such that $L \cap M \neq \emptyset$, $L \nsubseteq M$ (case $h^0(\mathcal{I}_U(2)) = 5$).*

Proof. Notice that $U_{red} \subseteq B$, $U \subseteq D$ and that the inclusion $U \hookrightarrow D$ induces an isomorphism $H^0(\mathcal{I}_D(2)) \to H^0(\mathcal{I}_U(2))$. Set $\Gamma := |\mathcal{I}_U(2)|$ and $\gamma := \dim(\Gamma)$. Proposition 2 gives $\dim(\Gamma(-A)) = \gamma - 3$ for a general $A \in G(1,n)$. Set $\Delta := \{A \in G(1,n) : dim(\Gamma(-A)) = \gamma - 3\}$. By semicontinuity Δ is a non-empty open subset of the Grassmannian $G(1,n)$. Let Σ_1 be the closure of $\cup_{A\in\Delta}\Gamma(-A)$. Assume that for general A the general element of $\Gamma(-A)$ is singular at some point of A. Since $\gamma \geq 3$, a general $F \in G(1,n)$ is contained in some $Q \in \Gamma$. Fix any such Q and call Θ any irreducible component of the set of all lines in Q. A general $F' \in \Theta$ may be seen as a deformation of F and hence $F' \in \Delta$. Hence Q must be singular at some point of F'. Hence every element of Θ intersects the vertex of Q. This is true if and only if Q has rank at most 3. Hence every element of Γ has rank at most 3, i.e., its singular locus has dimension at least $n - 3$. By Bertini's theorem the intersection M of all the vertices of $Q \in \Gamma$ is contained in B. Notice that M is a linear space. We got $\dim(M) \geq n - 3$. Since $h^0(\mathcal{I}_U(2)) \geq 5$, Remark 3 gives $\dim(M) = n - 3$

and that either $H^0(\mathcal{I}_U(2))$ is a hyperplane of $H^0(\mathcal{I}_{M^{(1)}}(2))$ and $h^0(\mathcal{I}_U(2)) = 5$ or $H^0(\mathcal{I}_U(2)) = H^0(\mathcal{I}_{M^{(1)}}(2))$ and $h^0(\mathcal{I}_U(2)) = 6$. Hence if $h^0(\mathcal{I}_U(2)) \geq 6$, then $D = M^{(1)}$ and hence $h^0(\mathcal{I}_U(2)) = 6$. Now assume $h^0(\mathcal{I}_U(2)) = 5$, i.e. $h^0(\mathcal{I}_D(2)) = 5$. Since $M^{(1)} \subseteq D$ and $h^0(\mathcal{I}_{M^{(1)}}(2)) = 6$, we have $M^{(1)} \subsetneq D$. Since $|\mathcal{I}_{M^{(1)}}(2)|$ is the set of all quadric cones with vertex containing M, $M^{(1)} \subsetneq D$ and D is the scheme-theoretic base locus of $|\mathcal{I}_D(2)|$, there is a unique line $L \subset \mathbb{P}^n$ such that $L \cap M \neq \emptyset$, $L \nsubseteq M$ and $D = M^{(1)} \cup L$. QED

In the last sentence of the statement of Proposition 4 we cannot use U instead of D, because too many schemes U have the same scheme-theoretic base locus (3 collinear points or a line have the same scheme-theoretic base locus; the union of 4 general lines of a 3-dimensional subspace N or N give the same scheme-theoretic base locus, and so on).

Proposition 5. *Fix a closed subscheme $U \subset \mathbb{P}^n$. Let B denote the set-theoretic base locus of the linear system $|\mathcal{I}_U(2)|$. Assume $h^0(\mathcal{I}_U(2)) \geq 5$ and that B does not contain a hyperplane. Let $C \subset \mathbb{P}^n$ be a general unreduced conic. Then $h^0(\mathcal{I}_{U\cup C}(2)) = h^0(\mathcal{I}_U(2)) - 5$.*

Proof. Set $\Gamma := |\mathcal{I}_U(2)|$ and $\dim(\Gamma) = \gamma$. It is sufficient to prove the inequality $\dim(\Gamma\ (-C)) \leq \gamma - 5$. Set $A := C_{red}$. Proposition 2 gives $\dim(\Gamma(-A)) = \gamma - 3$. Let $A' \subset \mathbb{P}^n$ be a general line intersecting A. Proposition 4 gives dim $(\Gamma(-(A \cup A')) = \gamma - 5$. Set $M := \langle A \cup A' \rangle$. Since M is a plane containing $A \cup A'$, either $\Gamma(-M) = \Gamma(-(A \cup A'))$ or $\dim(\Gamma(-M)) = \gamma - 6$. If $\dim(\Gamma(-M)) = \gamma - 6$, then $\dim(\Gamma(-E)) = \gamma - 5$ for any conic $E \subset M$. In this case we may take as C any unreduced conic of M. Now assume $\Gamma(-M) = \Gamma(-(A \cup A'))$, i.e., assume that Γ induces a 4-dimensional linear subspace Π of $|\mathcal{O}_M(2)|$. Since $\mathrm{char}(\mathbb{K}) \neq 2$, $|\mathcal{O}_M(2)|$ is spanned by the unreduced conics of M. Hence there is $C \in |\mathcal{O}_M(2)| \setminus \Pi$. Fix any such unreduced conic C. Since C imposes 5 independent conditions to Π, we have $\dim(\Gamma(-C)) \leq \gamma - 5$. QED

Proposition 6. *Fix any closed subscheme $U \subset \mathbb{P}^n$ and any unreduced conic T. Let Z be a general pointed conic containing T.*

(i) If every $Q \in |\mathcal{I}_{U\cup T}(2)|$ is a cone with vertex containing the line T_{red}, then $h^0(\mathcal{I}_{U\cup Z}(2)) = h^0(\mathcal{I}_{U\cup T}(2))$.

(ii) If some $Q \in |\mathcal{I}_{U\cup T}(2)|$ is not a cone with vertex containing the line T_{red}, then $h^0(\mathcal{I}_{U\cup Z}(2)\ ) = h^0(\mathcal{I}_{U\cup T}(2)) - 1$.

Proof. Since the singular locus of a quadric hypersurface is its vertex, some $Q \in |\mathcal{I}_{U\cup T}(2)|$ is not a cone with vertex containing the line T_{red} if and only if

$$h^0(\mathcal{I}_{U\cup T\cup \chi_{\mathbb{P}^n}(P)}(2)) < h^0(\mathcal{I}_{U\cup T}(2))$$

for a general $P \in T_{red}$. Since $Z = T \cup \chi_M(P)$ with P general in T_{red} and M a general 3-dimensional linear subspace of $\mathbb{P}^n$ containing the plane $\langle T \rangle$ and $h^0(Z, \mathcal{O}_Z(2)) = h^0(T, \mathcal{O}_T(2)) + 1$, we get both parts of Proposition 6. QED

Definition 1. Fix an integer $x \geq 1$. A *chain of x lines* in $\mathbb{P}^n$ is a connected and nodal curve $Y \subset \mathbb{P}^n$ such that Y has x irreducible components, each irreducible component is a line, and there is an ordering $A_1, \dots, A_x$ of the irreducible components of Y such that $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Any such ordering of the irreducible components of Y is called a *good ordering.* Since Y is assumed to be nodal, its Hilbert polynomial satisfies $p_Y(t) = xt + 1$ for all $t \in \mathbb{Z}$. Now assume $2 \leq x \leq n$. A *brush of x lines* in $\mathbb{P}^n$ is a reduced and connected curve $X \subset \mathbb{P}^n$ which is the union of x lines, it has a unique singular point and it spans a linear space of dimension x. The latter assumption implies that the singular point of X is a seminormal singularity, that $p_a(X) = 0$ and that $p_X(t) = xt + 1$ for all $t \in \mathbb{Z}$. A *nodal tree of z lines*, $z \geq 1$, is a nodal and connected curve $E \subset \mathbb{P}^n$ with z irreducible components, each of them being a line, and with arithmetic genus 0.

Notice that for a fixed x the set of all chains of x lines (resp. brushes of x lines) in $\mathbb{P}^n$ is parametrized by an integral variety. Hence it makes sense to use the words “ general chain of x lines ” (resp. “ general brush of x lines ”) in $\mathbb{P}^n$. If $z \geq 4$, then the set of all nodal trees of z lines in $\mathbb{P}^n$ is parametrized by an equidimensional variety of dimension $2nz$ with several irreducible components. To specify each irreducible component it is sufficient to consider instead of nodal trees triples $(E, <, \tau)$, where $E \subset \mathbb{P}^n$ is a nodal tree of z lines, $<$ is a total ordering of the irreducible components of E and $\tau : \{2, \dots, z\} \to \{1, \dots, z-1\}$ is a map such that $\tau(i) < i$ for all $i \in \{2, \dots, z\}$. Indeed, if we use the total ordering $<$ to call $A_1, \dots, A_z$ the irreducible components of E, then associated to τ there is the set of nodal trees of z lines $A_1 \cup \cdots \cup A_z \subset \mathbb{P}^n$ such that A_i, $i \in \{2, \dots, z\}$, intersects A_j, $j < i$, if and only if $j = \tau(i)$. With any such ordering $A_1, \dots, A_z$ and any integer $x \in \{1, \dots, z\}$ the curve $A_1 \cup \cdots \cup A_x$ is connected and hence it is a degree x nodal tree.

Proposition 7. *Fix an integer $x \geq 2$ and a closed subscheme $U \subset \mathbb{P}^n$ such that the base locus of the linear system $|\mathcal{I}_U(2)|$ does not contain a hyperplane and $h^0(\mathcal{I}_U(2)) \geq 2x + 1$. Let $Y \subset \mathbb{P}^n$ be a general chain of x lines. Then $h^0(\mathcal{I}_{U \cup Y}(2)) = h^0(\mathcal{I}_U(2)) - 2x - 1$ and $h^1(\mathcal{I}_{U \cup Y}(2)) = h^1(\mathcal{I}_U(2))$.*

Proof. Since $h^0(Y, \mathcal{O}_Y(2)) = 2x + 1$ and $Y \cap U = \emptyset$, the two assertions (on h^0 and on h^1) are equivalent. Hence it is sufficient to prove the first one. We use induction on x, the case $x = 2$, being true by Proposition 3. Now assume $x \geq 3$. Take a good ordering $A_1, \dots, A_x$ of the irreducible components of Y. Hence $A_1 \cup \cdots \cup A_{x-1}$ is a chain of $x - 1$ lines. The inductive assumption gives $h^0(\mathcal{I}_{U\cup A_1\cup\cdots A_{x-1}}(2)) = h^0(\mathcal{I}_U(2)) - 2x + 1$. Then we repeat the proof of Proposition 3 taking A_{x-1} instead of A and A_x instead of A', except that we need to do again the case in which a general $Q \in |\mathcal{I}_{U\cup A_1\cup\cdots A_{x-1}}(2)|$ is a cone with vertex containing A_{x-1}. To get from $A_1 \cup \cdots A_{x-1}$ a chain of x lines we may also take a general line intersecting A_1. Hence we conclude, unless Q is a cone with vertex containing $\langle A_1 \cup A_{x-1}\rangle$. The proof below is a variation of the case $x = 1$ of Lemma 5, because the line A_{x-1} which we are adding to $A_1 \cup \cdots \cup A_{x-2}$ is not general, but it is general among the lines which intersects A_{x-2}. Set $\Pi := |\mathcal{I}_{U\cup A_1\cup\cdots\cup A_{x-2}}(2)|$. Notice that every $Q \in \Pi$ contains A_{x-2}. Fix a general line $T \subset \mathbb{P}^n$ intersecting A_{x-2} and a general $Q \in \Pi(-T)$. By assumption Q is a cone with vertex containing T. First assume that the vertex of Q has dimension at most $n-4$ (it may be empty). In this case a general line contained in Q is contained in Q_{reg}. Hence we may deform A_{x-2} inside Q until it is contained in Q_{reg}. We may simultaneously deform A_{x-3} in a family preserving the condition $A_{x-2} \cap A_{x-3} \neq \emptyset$. If $x \geq 5$ we do that simultaneously for all lines of the chain $A_1 \cup \cdots \cup A_{x-2}$ preserving the condition that two components with consecutive indices meet. Since $T \cap A_{x-2} \neq \emptyset$, $A_{x-2} \subset Q_{reg}$ and $T \subseteq \mathrm{Sing}(Q)$, we get a contradiction. Now assume that the vertex V_Q of Q has dimension $n-3$. Let $R, R' \subset Q$ be lines such that $R \neq R'$ and $R \cap R' \neq \emptyset$. Then every chain $A_1 \cup \cdots \cup A_{x-1}$ of $x - 1 \geq 3$ lines has the property that $A_2 \cup \cdots \cup A_{x-3} \subset V_Q$. We assumed $A_1 \cup A_{x-1} \subset V_Q$. Hence $A_1 \cup \cdots \cup A_{x-1} \subset V_Q$. We may deform $A_1 \cup \cdots \cup A_{x-1}$ to a chain $A'_1 \cup \cdots \cup A'_{x-1}$ with $A'_i = A_i$ if $i \in \{2, \ldots, x-2\}$, while A'_1 and A'_{x-1} intersects V_Q only at one point. With the new chain $A'_1 \cup \cdots \cup A'_{x-1}$ some $Q \in |\mathcal{I}_{U\cup A'_1\cup\cdots A'_{x-1}}(2)|$ is not singular at a general point of A'_{x-1}, contradiction. Now assume that the vertex V_Q of Q has dimension $n-2$, i.e. assume that Q is not integral. By assumption $A_1 \cup A_{x-1} \subset V_Q$. But again we may move A_{x-1} outside V_Q moving $A_1 \cup \cdots \cup A_{x-2}$ inside the codimension two linear space V_Q, contradiction. QED

Remark 4. A nodal tree $Y \subset \mathbb{P}^n$, $n \geq 3$, of x lines is a flat degeneration of a family of smooth degree x rational curves. Hence Proposition 7 gives the corresponding result taking as Y a general smooth degree x rational curve. However, if reducible curves are only used as a tool to prove something concerning smooth rational curves, then it is often easier to work with arbitrary nodal trees of lines, instead of using only chains of lines.

Proposition 8. *Let* Π *be a linear subspace of* $|\mathcal{O}_{\mathbb{P}^n}(2)|$. *Fix an integer* x *such that* $2 \leq x \leq n$ *and assume* $m := \dim(\Pi) \geq 2x + 1$. *Let* $Y \subset \mathbb{P}^n$ *a general brush of* x *lines.*

(i) If $\rho(\Pi) < x$, *then* $\dim(\Pi(-Y)) \geq m - x - 1 - \rho(\Pi) \geq m - 2x$.

(ii) Assume $\Pi = |\mathcal{I}_U(2)|$ *with* U *a closed subscheme of* $\mathbb{P}^n$. *Then* $\dim(\Pi(-Y))$

$= m - 2x - 1$ *if and only if* $\rho(\Pi) \geq x$ *and the base locus of* Π *does not contain a hyperplane.*

PROOF. Let B the base locus of Π and $\psi : \mathbb{P}^n \setminus B \to \mathbb{P}^m$ the morphism associated to Π. Set $\{P\} := \mathrm{Sing}(Y)$. Let $M := \langle Y \rangle$ be the x-dimensional linear subspace of $\mathbb{P}^n$ spanned by Y. Let $A_1, \dots, A_x$ be the irreducible components of Y. Notice that $\chi_M(P) \subset Y$ and that from the point of view of the postulation with respect to any linear system of quadrics we may substitute Y with $\chi_M(P) \cup \{P_1, \dots, P_x\}$, where each P_i is an arbitrary point of $A_i \setminus \{P\}$. For general Y the point P is a general point of $\mathbb{P}^n$ and M is a general x-dimensional linear subspace containing it. Since $P \notin B$, the integer $m - \dim(\Pi(-\chi_M(P))) - 1$ is (up to the addendum -1 coming from $\dim(\Pi(-P)) - m$) the rank of the restriction to M of the differential $d\psi(P)$ of ψ at P. The generality of P and M gives $\dim(\Pi(-\chi_M(P))) = m - 1 - \min\{x, \rho(\Pi)\}$. Hence we get part (i) and one half of the " only if " part of (ii). Assume $\Pi = |\mathcal{I}_U(2)|$. If B contains a hyperplane, then it is easy to check that $\dim(\Pi(-Y)) = m - \min\{\rho(\Pi), x\}$. Hence we get the other half of the " only if " part of (ii). Now assume $\rho(\Pi) \geq x$ and that B does not contain a hyperplane. We saw that $\dim(\Pi(-\chi_M(P))) = m - x - 1$. Hence it is sufficient to prove $\dim(\Pi(-(\chi_M(P) \cup \{P_1, \dots, P_x\}))) = \dim(\Pi(-\chi_M(P))) - x$. Now we fix P and M, but take as Y a general brush spanning M and with P as its singular point. Hence $A_1, \dots, A_x$ are x general lines of M passing through P. Since $\dim(M) = x$, for general Y the set $\{P_1, \dots, P_x\}$ is a general subset of M with cardinality x. Hence it is sufficient to prove $\dim(\Pi(-M)) \leq m - 2x - 1$. This is true for instance because we may take as M the linear span of a general chain of x lines and we may apply Proposition 7. QED

PROOF OF THEOREM 2. By semicontinuity and Lemma 2 it is sufficient to prove Theorem 2 for a general pointed unreduced conic E. Let T be the unreduced conic associated to E. Proposition 5 gives $h^0(\mathcal{I}_{U \cup T}(2)) = h^0(\mathcal{I}_U(2)) - 5$. By Proposition 6 to prove Theorem 2 it is sufficient to prove that a general element of $|\mathcal{I}_{U \cup T}(2)|$ is not a cone with vertex containing the line $A := T_{red}$. Assume that this is the case. We have $h^0(\mathcal{I}_{U \cup A}(2)) = h^0(\mathcal{I}_U(2)) - 3$. We get $h^0(\mathcal{I}_{U \cup A^{(1)}}(2)) = h^0(\mathcal{I}_U(2)) - 5$. Let B denote the base locus of $\Pi := |\mathcal{I}_U(2)|$ and $\psi : \mathbb{P}^n \setminus B \to \mathbb{P}^m$, $m := h^0(\mathcal{I}_U(2)) - 1$, the morphism associated to Π. Set $m := \dim(\Pi)$. Fix a general $P \in \mathbb{P}^n$. Hence $P \notin B$ and $d\psi(P)$ has rank $\rho(\Pi) \geq 4$. Fix a general 3-dimensional linear subspace M of $\mathbb{P}^n$ containing P. Since M is general, as in the proof of Proposition 7 we get $\dim(\Pi(-\chi_M(P))) = m - 4$. Since $m \geq 5$, $\Pi(-\chi_M(P))$ is a non-constant linear system and hence $\rho(\Pi(-\chi_M(P))) \geq 1$. Hence for a general $P_1 \in \mathbb{P}^n$ and a general tangent vector ν to $\mathbb{P}^n$ at P_1 we have $\dim(\Pi(-\chi_M(P))(-\nu)) = \dim(\Pi(-\chi_M(P))) - 2$, i.e., $\dim(\Pi(-(\chi_M(P) \cup \nu)) = m - 6$. Since the pair (P, P_1) is general, the line A spanned by $\{P, P_1\}$ is general. Since $\chi_M(P) \cup \nu \subseteq A^{(1)}$, we obtained a contradiction. QED

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