Strong convergence of the new modified composite iterative method for strict pseudo-contractions in Hilbert spaces

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Abstract. In this paper, we introduce and analyze a new modified Mann iterative scheme for strict pseudo-contractive mappings in Hilbert spaces. The results presented in this paper improve and extend the main results in [1] and many others.

Keywords: Strong convergence, Strict pseudo-contractions, Composite iterative scheme, Hilbert space

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1 Introduction

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let $I$ be the identity mapping on $H$ and $C$ a closed convex subset of $H$.

A mapping $T$ of $H$ into itself is called a $k$-strict pseudo-contraction mapping, if $\forall x, y \in K, \|Tx - Ty\|^2 \leq \|x - y\|^2 + k \| (I - T)x - (I - T)y \|^2$, here $0 \leq k < 1$. We use $F(T)$ to denote the set of fixed points of $T$ (i.e. $F(T) = \{ x \in K : Tx = x \}$).

In Hilbert spaces, it is clear that a $k$-strict pseudo-contraction mappings is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \| x - y \|^2 - \frac{1 - k}{2} \| (I - T)x - (I - T)y \|^2,$$  \hspace{1cm} (1)

i.e.

$$\frac{1 - k}{2} \| (I - T)x - (I - T)y \|^2 \leq \langle (I - T)x - (I - T)y, x - y \rangle.$$  \hspace{1cm} (2)

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Remark. Notice that a mapping $T : H \to H$ is called non-expansive mappings, if for all $x, y \in H$, $\|Tx - Ty\| \leq \|x - y\|$. Therefore, a non-expansive mapping $T$ is a $0$-strict pseudo-contractive mapping.

A linear bounded operator $B$ is strongly positive if there exists a constant $\gamma > 0$ with property $\langle Bx, x \rangle \geq \gamma \|x\|^2$, $\forall x \in H$.

Marino and Xu [2] introduced a new iterative scheme by the viscosity approximation method:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0,$$

where, $S : H \to H$ is a non-expansive mapping. They proved that the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle \gamma f - Bq, p - q \rangle \leq 0, \quad \forall p \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{p \in F(S)} \frac{1}{2} \langle Bp, p \rangle - h(p), \quad \forall p \in F(S),$$

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

The normal Mann’s iterative process was introduced by Mann [3] in 1953 as follows:

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0,$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$.

If $T$ is a non-expansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the normal Mann’s iterative process (1.4) weakly converges to a fixed point of $T$ (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [4], or more generally, in a uniformly convex Banach space such that its dual has the KK property as proved by Garcia Falset, Kaczor, Kuczumow and Reich in [5]). However, this scheme has only weak convergence even in a Hilbert space [6]. Therefore, many authors try to modify normal Mann’s iteration process to have strong convergence; see, e.g., [7-12, 13, 14] and the references therein.

Yao et al. [14] considered the following iteration process.

$$\begin{align*}
\{ y_n &= \beta_n x_n + (1 - \beta_n)Tx_n, \\
\alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad n \geq 0, \}
\end{align*}$$

(5)
where $T$ is a non-expansive mapping of $C$ into itself and $f$ is an $\alpha-$contraction (i.e. $\|f(x) - f(y)\| \leq \alpha \|x - y\|, 0 \leq \alpha < 1$). They proved the sequence $\{x_n\}$ defined by (5) strongly converges to a fixed point of $T$ provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Motivated by Marino and Xu[2,9] and Yao et al. [14], Marino et al. [1] introduced a composite iteration scheme as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
y_n = \beta_n x_n + (1 - \beta_n) \sum_{i=1}^{N} \eta_i T_i x_n, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n,
\end{array} \right. \quad n \geq 0,
\end{align*}
\]

where $f$ is an $\alpha-$contraction, $\gamma$ is a suitable coefficient and $B$ is a linear bounded strongly positive operator, $T_i$ is a $k_i-$pseudo-contraction on $H$ for some $0 \leq k_i < 1$ with $\Omega = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $f$ be an $\alpha-$contraction. Let $B$ be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in (0,1), satisfying the following conditions

(M1) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$

(M2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

(M3) $0 \leq \max_i \{k_i\} \leq \beta_n \leq \beta < 1$ for all $n \geq 0.$

Then $\{x_n\}$ defined by (6) converges strongly to some common fixed point $q$ of $\{T_1, T_2, \ldots, T_N\}$, which solves the following variational inequality:

$$\langle \gamma f q - B q, p - q \rangle \leq 0, \quad \forall \ p \in \Omega.$$ 

**Theorem M1.** [1]. Let $H$ be a Hilbert space and let for $i = 1, 2, \ldots, N$, $T_i$ be a $k_i-$strict pseudo-contraction on $H$ for some $0 \leq k_i < 1$ with $\Omega = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $f$ be an $\alpha-$contraction. Let $B$ be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\eta_i^{(n)}\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in (0,1), satisfying the following conditions

(M1) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$

(M2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$
(M3') for every fixed $n$, $\sum_{i=1}^{N} \eta_i^{(n)} = 1$ and $\inf_n \eta_i^{(n)} > 0$;

(M4') $0 \leq \max \{ b_i \} \leq \beta_n \leq \beta < 1$ for all $n \geq 0$;

(M5') $\sum_{n=0}^{\infty} |\eta_i^{(n+1)} - \eta_i^{(n)}| < \infty$ (for $i = 1, 2, \cdots, N$).

Let $\{x_n\}$ be defined by

\[
\begin{aligned}
y_n &= \beta_n x_n + (1 - \beta_n) \sum_{i=1}^{N} \eta_i^{(n)} T_i x_n, \\
x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n, \quad n \geq 0,
\end{aligned}
\]

then $\{x_n\}$ converges strongly to the common fixed point $q$ of $\{T_1, T_2, \cdots, T_N\}$, which solves the following variational inequality:

\[
\langle \gamma f q - B q, p - q \rangle \leq 0, \quad \forall p \in \Omega.
\]

Inspired by Marino et al. [1], in this paper, our purpose is to introduce a modified composite iterative algorithm (given in next section 3) to approximate a common fixed point of finite family of strict pseudo-contraction mappings, which solves some variational inequality. Our results improve and extend the results of Marino et al. [1], Kim and Xu [8], Marino and Xu [2], Yao et al. [14].

2 Preliminaries

**Lemma 1.** ([15]). Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition:

\[
a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq n_0,
\]

where $n_0$ is some nonnegative integer and $\{\lambda_n\} \subset (0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.** ([16]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \lim \inf \beta_n \leq \lim \sup \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \geq 0$ and $\lim \sup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then, $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

**Lemma 3.** ([17]). Let $E$ be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ the following inequality holds:

\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).
\]

Especially, when $E = H$, then $J = I$, so from Lemma 3 we have that

\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle, \quad \forall x, y \in H.
\]
Lemma 4. ([2]). Assume that $A$ is a strongly positive linear bounded operator on a Hilbert space $H$ with \( \bar{\gamma} > 0 \) and \( 0 < \rho < \| A \|^{-1} \). Then \( \| I - \rho A \| \leq 1 - \rho \bar{\gamma} \).

Lemma 5. (Marino and Xu [2]). Let $H$ be a real Hilbert space. Let $A$ be a strongly positive linear bounded self-adjoint operator with \( \bar{\gamma} > 0 \). Let $f$ be an $\alpha$-contraction. Assume that \( 0 < \gamma < \bar{\gamma} / \alpha \). Let $T : H \rightarrow H$ be a non-expansive mapping. For $t \leq \| A \|^{-1}$, let $x_t$ be the fixed point of the contraction $x \mapsto \gamma f(x) + (I - tA)x$. Then $\{ x_t \}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of $T$, which solves the variational inequality

\[
(\gamma f \tilde{x} - B\tilde{x}, z - \tilde{x}) \leq 0, \quad \forall z \in F(T).
\]

Lemma 6. (Acedo and Xu [7]). Let $H$ be a real Hilbert space, $K$ a closed convex subset of $H$. Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : K \to K$ is a $k_i$-strict pseudo-contraction for some $0 \leq k_i < 1$. Assume \( \{ \eta_i \}_{i=1}^N \) is a positive sequence such that \( \sum_{i=1}^n \eta_i = 1 \). Then \( \sum_{i=1}^n \eta_i T_i \) is a $k$-strict pseudo-contraction, with $k = \max \{ k_i : 1 \leq i \leq N \}$.

Lemma 7. (Acedo and Xu [7]). Let $\{ T_i \}$ and $\{ \eta_i \}$ be given as in Lemma 6. Suppose that $\{ T_i \}$ has a common fixed point. Then $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.

Lemma 8. (Zhou [18]). Let $T : H \rightarrow H$ be a $k$-strict pseudo-contraction. Define $S : H \rightarrow H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in H$. Then, as $\lambda \in [k, 1)$, $S$ is non-expansive such that $F(S) = F(T)$.

A Banach space $E$ is said to satisfy Opial’s condition [19] if, for any $\{ x_n \} \subset E$ with $x_n \rightharpoonup x$, the following inequality holds:

\[
\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \|,
\]

for all $y \in E$ with $y \neq x$. It is well-known that Hilbert spaces satisfies Opial’s condition.

3 Main results

Theorem 1. Let $H$ be a real Hilbert space and let $T$ be a $k$-strict pseudo-contraction on $H$ for some $0 \leq k < 1$ with $F(T) \neq \emptyset$ and $f$ be an $\alpha$-contraction. Let $B$ be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that \( 0 < \gamma < \bar{\gamma} / \alpha \). Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{ \alpha_n \}_{n=0}^\infty$ and $\{ \beta_n \}_{n=0}^\infty$ in $(0,1)$, satisfying the following conditions

(i) \( \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^\infty \alpha_n = \infty \).
(ii) $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 0$.

Let $\{x_n\}$ be defined by

$$
\begin{cases}
  y_n = \beta_n x_n + (1 - \beta_n) S x_n, \\
  x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n,
\end{cases}
$$

where $S = \sigma I + (1 - \sigma)T$, $k \leq \sigma < 1$, then $\{x_n\}$ defined by Theorem 1 converges strongly to a fixed point $q$ of $T$, which solves the following variational inequality:

$$
\langle \gamma f - Bq, p - q \rangle \leq 0, \quad \forall p \in F(T).
$$

**Proof.**

**Step 1.** Prove $\{x_n\}$ is bounded. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n < \|B\|^{-1}$ for all $n \geq 0$. From Lemma 4, we know that $\|I - \alpha_n B\| \leq 1 - \alpha_n \bar{\gamma}$. For all $p \in F(T)$, since $S$ is a non-expansive mapping and $F(S) = F(T)$ by Lemma 8, we have

$$
\|y_n - p\|^2 = \|\beta_n (x_n - p) + (1 - \beta_n) (S x_n - p)\|^2 \leq \|x_n - p\|^2,
$$

and

$$
\|x_{n+1} - p\| = \|\alpha_n (\gamma f(x_n) - B p) + (I - \alpha_n B)(y_n - p)\|
\leq \alpha_n \|\gamma f(x_n) - B p\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\|
\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - B p\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\|
\leq (1 - \alpha_n (\bar{\gamma} - \gamma \alpha)) \|x_n - p\| + \alpha_n \|\gamma f(p) - B p\|
\leq \max \{\|x_0 - p\|, \|\gamma f(p) - B p\| / (\bar{\gamma} - \gamma \alpha)\}.
$$

Therefore, $\{x_n\}$ is bounded, so is $\{y_n\}$.

**Step 2.** Prove $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Set $\gamma_n = 1 - \beta_n$ and $v_n = \frac{x_n + 1 - \gamma_n - \alpha_n x_n}{\gamma_n} = \alpha_n (\gamma f(x_n) - B y_n) + S x_n$, then

$$
\|v_{n+1} - v_n\| \leq \frac{\alpha_n}{1 - \beta_n} M + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} M + \|S x_{n+1} - S x_n\|
\leq \frac{\alpha_n}{1 - \beta_n} M + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} M + \|x_{n+1} - x_n\| \quad (9)
$$

where $M$ is a constant satisfying $\|\gamma f(x) - B y\| \leq M$ for $n \geq 1$. It follows from (9), $\limsup_{n \to \infty} \{\|v_{n+1} - v_n\| - \|v_{n+1} - x_n\|\} = 0$, which implies that $\lim_{n \to \infty} \|v_n - x_n\| = 0$ by Lemma 2. From the definition of $v_n$ we obtain

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \quad (10)
$$
Step 3. Prove $\|Tx_n - x_n\| \to 0$ as $n \to \infty$. In fact, from (8) and (10) we have
\[
\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0, \quad \lim_{n \to \infty} \|x_n - y_n\| = 0.
\]
Hence
\[
\lim_{n \to \infty} \|Sx_n - x_n\| = \lim_{n \to \infty} \frac{1}{1 - \beta_n} \|x_n - y_n\| = 0,
\]
which implies that
\[
\lim_{n \to \infty} \|Sx_n - x_n\| = \lim_{n \to \infty} (1 - \sigma) \|Tx_n - x_n\| = 0. \tag{11}
\]

Step 4. Let $q = \lim_{t \to 0^+} x_t$, where $x_t$ is the fixed point (for $t \in (0, \|B\|^{-1})$) of contraction $x \mapsto tx + (I - tB)Sx$. From Lemma 5 and Lemma 8, $q \in F(S) = F(T)$, and
\[
\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad \forall p \in F(T) \tag{12}
\]
Claim $\limsup_{n \to \infty} \langle \gamma f(q) - Bq, x_n - q \rangle \leq 0$. Take the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that
\[
\limsup_{n \to \infty} \langle \gamma f(q) - Bq, x_n - q \rangle = \lim_{j \to \infty} \langle \gamma f(q) - Bq, x_{n_j} - q \rangle. \tag{13}
\]
Since $\{x_{n_j}\}$ is bounded, so there exists a subsequence of $\{x_{n_j}\}$ such that it converges weakly to a point $p \in K$. Without loss generality, let $\{x_{n_j}\}$ denote it and $x_{n_j} \rightharpoonup p$. By (11) and Opial’s condition, $p \in F(S) = F(T)$. Thus from (12) and (13) we have that
\[
\limsup_{n \to \infty} \langle \gamma f(q) - Bq, x_n - q \rangle \leq 0. \tag{14}
\]

Step 5. Prove that $\{x_n\}$ converges strongly to $q$. It follows from Lemma 3 and (8) that
\[
\|x_{n+1} - q\|^2 = \|\alpha_n (\gamma f(x_n) - Bq) + (I - \alpha_n B)(y_n - q)\|^2 \\
\leq (1 - \gamma \alpha_n)^2 \|y_n - q\|^2 + 2 \alpha_n \langle \gamma f(x_n) - Bp, x_{n+1} - q \rangle \\
= (1 - \gamma \alpha_n)^2 \|x_n - q\|^2 \\
+ 2 \alpha_n \langle \gamma f(x_n) - \gamma f(q) + \gamma f(q) - Bq, x_{n+1} - q \rangle \\
\leq (1 - \gamma \alpha_n)^2 \|x_n - q\|^2 + 2 \alpha_n \alpha \|x_n - q\| \|x_{n+1} - q\| \\
+ 2 \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
\leq (1 - \gamma \alpha_n)^2 \|x_n - q\|^2 + \alpha \|x_n - q\|^2 + \|x_{n+1} - q\|^2 + 2 \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle,
\]
which yields that
\[
\|x_{n+1} - q\|^2 \leq (1 - \alpha_n \frac{2\hat{\gamma} - 2\alpha\gamma}{1 - \alpha_n \alpha\gamma}) \|x_n - q\|^2 + \frac{\alpha_n^2 \gamma^2}{1 - \alpha_n \alpha\gamma} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \alpha\gamma} \langle \gamma f(q) - Bq, x_{n+1} - q \rangle.
\] (15)

By boundness of \(\{x_n\}\) and the condition (i) and Lemma 1, \(\{x_n\}\) converges strongly to \(q\). This completes the proof of Theorem 1.

Next, we give strong convergence theorems for a finite family of strict pseudo-contractions.

**Theorem 2.** Let \(H\) be a real Hilbert space and for \(i = 1, 2, \ldots, N\), let \(T_i\) be a \(k_i\)-strict pseudo-contraction on \(H\) for some \(0 \leq k_i < 1\) with \(\bigcap_{i=1}^N F(T_i) \neq \emptyset\) and \(f\) be an \(\alpha\)-contraction. Let \(B\) be a strongly positive linear bounded self-adjoint operator with coefficient \(\hat{\gamma} > 0\). Assume that \(0 < \gamma < \hat{\gamma}/\alpha\), \(\eta_i \in (0, 1)\) and \(\sum_{i=1}^N \eta_i = 1\). Given the initial guess \(x_0 \in H\) chosen arbitrarily and given sequences \(\{\alpha_n\}_{n=0}^\infty\) and \(\{\beta_n\}_{n=0}^\infty\) in \((0,1)\), satisfying the following conditions

(i) \(\lim_{n \to \infty} \alpha_n = 0\), \(\sum_{n=1}^\infty \alpha_n = \infty\);

(ii) \(0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1\).

Let \(\{x_n\}\) be defined by
\[
\left\{ \begin{array}{l}
y_n = \beta_n x_n + (1 - \beta_n) S x_n, \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n,
\end{array} \right. \quad n \geq 0,
\] (16)

where \(S = \sigma I + (1 - \sigma) \sum_{i=1}^N \eta_i T_i\), \(k = \max\{k_i : 1 \leq i \leq N\} \leq \sigma < 1\), then \(\{x_n\}\) converges strongly to a common fixed point \(q\) of \(\{T_1, T_2, \ldots, T_N\}\), which solves the following variational inequality:

\[
\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).
\]

**Proof.** Since \(\sum_{i=1}^N \eta_i T_i\) is a \(k\)-strict pseudo-contraction mapping, by Theorem 1 we know Theorem 2 is true. This completes the proof of Theorem 2.

**Theorem 3.** Let \(H\) be a real Hilbert space and for \(i = 1, 2, \ldots, N\), let \(T_i\) be a \(k_i\)-strict pseudo-contraction on \(H\) for some \(0 \leq k_i < 1\) with \(\bigcap_{i=1}^N F(T_i) \neq \emptyset\) and \(f\) be an \(\alpha\)-contraction. Let \(B\) be a strongly positive linear bounded self-adjoint operator with coefficient \(\hat{\gamma} > 0\). Assume that \(0 < \gamma < \hat{\gamma}/\alpha\). Given the initial guess \(x_0 \in H\) chosen arbitrarily and given sequences \(\{\alpha_n\}_{n=0}^\infty\), \(\{\eta_i^{(n)}\}_{n=0}^\infty\) and \(\{\beta_n\}_{n=0}^\infty\) in \((0,1)\), satisfying the following conditions
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(i) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty; \)

(ii) \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 0; \)

(iii) \( \sum_{i=1}^{N} \eta_i^{(n)} = 1, \inf_{n} \eta_i^{(n)} > 0, \lim_{n \to \infty} |\eta_i^{(n+1)} - \eta_i^{(n)}| = 0 \) for \( i = 1, 2, \ldots, N. \)

Let \( \{x_n\} \) be defined by

\[
\begin{align*}
\{y_n &= \beta_n x_n + (1 - \beta_n) S_n x_n, \\
x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \quad n \geq 0, \tag{17}
\end{align*}
\]

where \( S_n = \sigma I + (1 - \sigma) \sum_{i=1}^{N} \eta_i^{(n)} T_i \), \( k = \max\{k_i : 1 \leq i \leq N\} \leq \sigma < 1 \), then \( \{x_n\} \) converges strongly to a common fixed point \( q \) of \( \{T_1, T_2, \ldots, T_N\} \), which solves the following variational inequality:

\[
\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \cap_{i=1}^{N} F(T_i).
\]

**Proof.** For each \( n \), let \( H_n = \sum_{i=1}^{N} \eta_i^{(n)} T_i \), then \( H_n \) is a \( k \)-strict pseudo-contraction by Lemma 6, \( k = \max\{k_i : 1 \leq i \leq N\} \). Moreover, \( S_n = \sigma I + (1 - \sigma) H_n \) is a non-expansive mapping and \( F(S_n) = F(H_n) = \cap_{i=1}^{N} F(T_i) \) by Lemma 8.

**Step 1.** Let \( p \in \cap_{i=1}^{N} F(T_i) \), from (17), we have

\[
||y_n - p||^2 = ||\beta_n(x_n - p) + (1 - \beta_n)(S_n x_n - p)||^2 \leq ||x_n - p||^2,
\]

and

\[
\begin{align*}
||x_{n+1} - p|| &= ||\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)(y_n - p)|| \\
&\leq \alpha_n \|\gamma f(x_n) - Bp\| + (1 - \alpha_n \tilde{\gamma}) \|x_n - p\| \\
&\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \tilde{\gamma}) \|x_n - p\| \\
&\leq (1 - \alpha_n(\tilde{\gamma} - \gamma \alpha)) \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| \\
&\leq \max\{\|x_0 - p\|, \|\gamma f(p) - Bp\| \},
\end{align*}
\]

Therefore, \( \{x_n\} \) is bounded, so is \( \{y_n\} \).

**Step 2.** Prove \( ||x_{n+1} - x_n|| \to 0 \). Let

\[
\gamma_n = 1 - \beta_n, \quad \eta_n = \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n} = \frac{\alpha_n(\gamma f(x_n) - B y_n)}{\gamma_n} + S_n x_n.
\]

Let \( M_3 \) be a constant such that

\[
\{||\gamma f(x_n) - B y_n||, \|T_1 x_n\|, \|T_2 x_n\|, \ldots, \|T_N x_n\|\} \leq M_3
\]
for $n \geq 0$. By $S_n = \sigma I + (1 - \sigma) \sum_{i=1}^{N} \eta_i^{(n)} T_i$, we have

$$S_{n+1}x_n - S_n x_n = (1 - \sigma) \sum_{i=1}^{N} (\eta_i^{(n+1)} - \eta_i^{(n)}) T_i x_n.$$  

Then

$$\|S_{n+1}x_{n+1} - S_n x_n\| \leq \|x_{n+1} - x_n\| + M_3 \sum_{i=1}^{N} |\eta_i^{(n+1)} - \eta_i^{(n)}|.$$  

Furthermore,

$$\|v_{n+1} - v_n\| \leq \frac{\alpha_n M_3}{\gamma_n} + \alpha_{n+1} M_3 + \|x_{n+1} - x_n\| + M_3 \sum_{i=1}^{N} |\eta_i^{(n+1)} - \eta_i^{(n)}|,$$

which implies that $\limsup_{n \to \infty} \{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\} = 0$. Using Lemma 2, we obtain $\lim_{n \to \infty} \|v_n - x_n\| = 0$, this shows that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \quad (18)$$  

Again from (17) and (18), we have

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0, \quad \lim_{n \to \infty} \|y_n - x_n\| = 0.$$  

Then

$$\lim_{n \to \infty} \|S_n x_n - x_n\| = \lim_{n \to \infty} \frac{1}{1 - \beta_n} \|y_n - x_n\| = 0. \quad (19)$$  

**Step 3.** Prove

$$\limsup_{n \to \infty} \langle \gamma f q - B q, x_n - q \rangle \leq 0, \quad q \in \cap_{i=1}^{N} F(T_i),$$

where $q$ is the unique solution of the following variational inequality

$$\langle \gamma f q - B q, p - q \rangle \leq 0, \quad \forall p \in \cap_{i=1}^{N} F(T_i). \quad (20)$$

Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{n \to \infty} \langle \gamma f q - B q, x_n - q \rangle = \lim_{n \to \infty} \langle \gamma f q - B q, x_{n_j} - q \rangle.$$  

For each $i$, since $\eta_i^{(n)}$ is bounded, there exists a subsequence of $\eta_i^{(n_j)}$ such that $\eta_i^{(n_j)} \to \eta_i \in (0, 1]$ as $j \to \infty$. At the
same time, since \( \{x_{n_j}\} \) is bounded, there exists a subsequence of \( \{x_{n_j}\} \) is still denoted by \( \{x_{n_j}\} \) such that \( x_{n_j} \rightarrow z \) as \( j \rightarrow \infty \). Define mapping \( S = \sigma I + (1 - \sigma) \sum_{i=1}^{N} \eta_i T_i \), then \( S \) is also non-expansive mapping and \( F(S) = F(S_n) = \cap_{i=1}^{N} F(T_i) \) by Lemma 7 and 8. Moreover, for all \( x \in C \),

\[
\|S_{n_j}x - Sx\| \leq \sum_{i=1}^{N} |\eta_i^{(n_j)} - \eta_i| \|T_i x\| \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.
\]

(21)

We claim \( z \in F(S) \). If not so, i.e. \( z \neq Sz \). Then using the Opial’s condition, from (19) and (21) we obtain

\[
\liminf_{j \to \infty} \|x_{n_j} - z\| < \liminf_{j \to \infty} \|x_{n_j} - Sz\|
\]

\[
\leq \liminf_{j \to \infty} (\|x_{n_j} - S_{n_j}x_{n_j}\| + \|S_{n_j}x_{n_j} - S_{n_j}z\| + \|S_{n_j}z - Sz\|)
\]

\[
\leq \liminf_{j \to \infty} \|x_{n_j} - z\|.
\]

(22)

This is a contradiction. So \( z \in F(S) = \cap_{i=1}^{N} F(T_i) \). It follows from (20) that

\[
\limsup_{n \to \infty} \langle \gamma f q - Bq, x_n - q \rangle = \lim_{j \to \infty} \langle \gamma f q - Bq, x_{n_j} - q \rangle = \langle \gamma f q - Bq, z - q \rangle \leq 0.
\]

(23)

Step 4. Prove \( \{x_n\} \) converge strongly to \( q \), where \( q \) is the unique solution of the following variational inequality

\[
\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \cap_{i=1}^{N} F(T_i).
\]

Reasoning as in Step 5 of Theorem 1, we obtain \( \{x_n\} \) converges strongly to \( q \). This completes the proof of Theorem 2.

**Remark 1.** Since a non-expansive mapping is a 0–strict pseudo-contraction mapping, our results are suitable to non-expansive mappings. In addition, our results remove the condition \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \) imposed on parameter \( \{\alpha_n\} \) in [1,2,7,8]. We remove also the condition \( \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty \) imposed on parameter \( \{\beta_n\} \) in [1].

**Remark 2.** The advantages of these results in the present paper are that fewer restrictions are imposed on the parameters \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\eta_i^{(n)}\} \). All of the results obtained in this paper can be viewed as a supplement to the results obtained in [1,2,7,8,14]
References


