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Strong convergence of the new modified composite iterative method for strict pseudo-contractions in Hilbert spaces

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Abstract. In this paper, we introduce and analyze a new modified Mann iterative scheme for strict pseudo-contraction mappings in Hilbert spaces. The results presented in this paper improve and extend the main results in [1] and many others.

Keywords: Strong convergence, Strict pseudo-contractions, Composite iterative scheme, Hilbert space

MSC 2000 classification: 47H07,47H09,47H10,47J25

1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let I be the identity mapping on H and C a closed convex subset of H.

A mapping T of H into itself is called a *k-strict pseudo-contraction* mapping, if $\forall\, x, y \in K$, $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\,\|(I-T)x - (I-T)y\|^2$, here $0 \leq k < 1$. We use $F(T)$ to denote the set of fixed points of T(i.e. $F(T) = \{x \in K : Tx = x\}$).

In Hilbert spaces, it is clear that a $k-$strict pseudo-contraction mappings is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1-k}{2}\,\|(I-T)x - (I-T)y\|^2\,, \tag{1}$$

i.e.

$$\frac{1-k}{2}\,\|(I-T)x - (I-T)y\|^2 \leq \langle (I-T)x - (I-T)y, x - y \rangle. \tag{2}$$

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Remark. Notice that a mapping $T : H \to H$ is called *non-expansive* mappings, if for all $x, y \in H$, $\|Tx - Ty\| \le \|x - y\|$. Therefore, a non-expansive mapping T is a $0-$strict pseudo-contractive mapping.

A linear bounded operator B is strongly positive if there exists a constant $\overline{\gamma} > 0$ with property $\langle Bx, x\rangle \ge \overline{\gamma}\, \|x\|^2\,,\ \forall\, x \in H$.

Marino and Xu [2] introduced a new iterative scheme by the viscosity approximation method:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad n \ge 0, \tag{3}$$

where,$S : H \to H$ is a non-expansive mapping. They proved that the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle \gamma f q - Bq, p - q\rangle \le 0,\ \ \forall\, p \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{p\in F(S)} \frac{1}{2}\langle Bp, p\rangle - h(p),\ \ \forall\, p \in F(S),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

The normal Mann's iterative process was introduced by Mann [3] in 1953 as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \ge 0, \tag{4}$$

where $\{\alpha_n\}$ is a real number sequence in (0,1).

If T is a non-expansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the normal Mann's iterative process (1.4) weakly converges to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [4],or more generally,in a uniformly convex Banach space such that its dual has the KK property as proved by Garcia Falset, Kaczor, Kuczumow and Reich in [5]). However,this scheme has only weak convergence even in a Hilbert space [6].Therefore, many authors try to modify normal Mann's iteration process to have strong convergence;see, e.g.,[7-12, 13, 14] and the references therein.

Yao et al. [14] considered the following iteration process.

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad n \ge 0, \end{cases} \tag{5}$$

where T is a non-expansive mapping of C into itself and f is an $\alpha-$contraction (i.e.$\|f(x)-f(y)\| \leq \alpha \|x-y\|, 0 \leq \alpha < 1$). They proved the sequence $\{x_n\}$ defined by (5) strongly converges to a fixed point of T provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Motivated by Marino and Xu[2,9] and Yao et al. [14], Marino et al. [1] introduced a composite iteration scheme as follows:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)\sum_{i=1}^{N} \eta_i T_i x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n B)y_n, \quad n \geq 0, \end{cases} \tag{6}$$

where f is an $\alpha-$contraction,γ is a suitable coefficient and B is a linear bounded strongly positive operator, T_i is a k_i-pseudo-contraction with $0 \leq k_i < 1$ and η_i is a positive constant such that $\eta_1+\eta_2+\cdots+\eta_N = 1$. They proved, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ defined by (6) converges to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$, which solves some variation inequality. To be more precisely, they obtained the next Theorems.

Theorem M1. [1]. *Let H be a Hilbert space and let for $i = 1, 2, \ldots, N$, T_i be a k_i-strict pseudo-contraction on H for some $0 \leq k_i < 1$ with $\Omega = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and f be an $\alpha-$contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in (0,1), satisfying the following conditions*

(M1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(M2) $\sum_{n=1}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1}-\beta_n| < \infty$.

(M3) $0 \leq \max_i\{k_i\} \leq \beta_n \leq \beta < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ defined by (6) converges strongly to some common fixed point q of $\{T_1, T_2, \ldots, T_N\}$, which solves the following variational inequality:

$$\langle \gamma f q - Bq, p-q\rangle \leq 0, \ \ \forall\, p \in \Omega.$$

Theorem M2. [1]. *Let H be a Hilbert space and let for $i = 1, 2, \ldots, N$, T_i be a k_i-strict pseudo-contraction on H for some $0 \leq k_i < 1$ with $\Omega = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and f be an $\alpha-$contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\eta_i^{(n)}\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in (0,1), satisfying the following conditions*

(M1') $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(M2') $\sum_{n=1}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1}-\beta_n| < \infty$;

(M3') *for every fixed* n, $\sum_{i=1}^{N}\eta_i^{(n)}=1$ *and* $\inf_n \eta_i^{(n)}>0$;

(M4') $0\le \max_i\{k_i\}\le\beta_n\le\beta<1$ *for all* $n\ge 0$;

(M5') $\sum_{n=0}^{\infty}|\eta_i^{(n+1)}-\eta_i^{(n)}|<\infty$ *(for* $i=1,2,\cdots,N$*).*

Let $\{x_n\}$ *be defined by*

$$\begin{cases} y_n=\beta_n x_n+(1-\beta_n)\sum_{i=1}^{N}\eta_i^{(n)}T_i x_n, \\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_n B)y_n, n\ge 0, \end{cases} \tag{7}$$

then $\{x_n\}$ *converges strongly to the common fixed point* q *of* $\{T_1,T_2,\cdots,T_N\}$, *which solves the following variational inequality:*

$$\langle \gamma f q - Bq, p-q\rangle \le 0,\ \ \forall\, p\in\Omega.$$

Inspired by Marino et al. [1], in this paper, our purpose is to introduce a modified composite iterative algorithm (given in next section 3) to approximate a common fixed point of finite family of strict pseudo-contraction mappings, which solves some variational inequality. Our results improve and extend the results of Marino et al. [1], Kim and Xu [8],Marino and Xu [2], Yao et al. [14].

2 Preliminaries

Lemma 1. *([15]). Let* $\{a_n\},\{b_n\},\{c_n\}$ *be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1}\le(1-\lambda_n)a_n+b_n+c_n,\ \forall\, n\ge n_0,$$

where n_0 *is some nonnegative integer and* $\{\lambda_n\}\subset(0,1)$ *with* $\Sigma_{n=0}^{\infty}\lambda_n=\infty$, $b_n=o(\lambda_n)$ *and* $\Sigma_{n=0}^{\infty}c_n<\infty$, *then* $\lim_{n\to\infty}a_n=0$.

Lemma 2. *([16]). Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in [0,1] with* $0<\liminf\beta_n\le\limsup\beta_n<1$. *Suppose* $x_{n+1}=\beta_n y_n+(1-\beta_n)x_n$ *for all integers* $n\ge 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0$, *then,* $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 3. *([17]). Let* E *be a real Banach space and* $J:E\to 2^{E^*}$ *be the normalized duality mapping, then for any* $x,y\in E$ *the following inequality holds:*

$$\|x+y\|^2\le\|x\|^2+2\langle y, j(x+y)\rangle,\quad \forall\, j(x+y)\in J(x+y).$$

Especially, when $E=H$, *then* $J=I$, *so from Lemma 3 we have that*

$$\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle,\quad \forall\, x,y\in H.$$

Lemma 4. *([2]). Assume that A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 5. *(Marino and Xu [2]). Let H be a real Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Let f be an $\alpha-$contraction. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $T : H \to H$ be a nonexpansive mapping. For $t \leq \|A\|^{-1}$, let x_t be the fixed point of the contraction $x_t \to \gamma f(x) + (I - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T, which solves the variational inequality*

$$\langle \gamma f\bar{x} - B\bar{x}, z - \bar{x}\rangle \leq 0, \ \ \forall\, z \in F(T).$$

Lemma 6. *(Acedo and Xu [7]). Let H be a real Hilbert space, K a closed convex subset of H. Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : K \to K$ is a k_i-strict pseudo-contraction for some $0 \leq k_i < 1$. Assume $\{\eta_i\}_{i=1}^{N}$ is a positive sequence such that $\sum_{i=1}^{n} \eta_i = 1$. Then $\sum_{i=1}^{n} \eta_i T_i$ is a $k-$strict pseudo-contraction, with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 7. *(Acedo and Xu [7]). Let $\{T_i\}$ and $\{\eta_i\}$ be given as in Lemma 6. Suppose that $\{T_i\}$ has a common fixed point. Then $F(\sum_{i=1}^{N} \eta_i T_i) = \bigcap_{i=1}^{N} F(T_i)$.*

Lemma 8. *(Zhou [18]). Let $T : H \to H$ be a $k-$strict pseudo-contraction. Define $S : H \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in H$. Then, as $\lambda \in [k, 1)$, S is non-expansive such that $F(S) = F(T)$.*

A Banach space E is said to satisfy Opial's condition [19] if, for any $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, the following inequality holds:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| ,$$

for all $y \in E$ with $y \neq x$. It is well-known that Hilbert spaces satisfies Opial's condition.

3 Main results

Theorem 1. *Let H be a real Hilbert space and let T be a $k-$strict pseudo-contraction on H for some $0 \leq k < 1$ with $F(T) \neq \emptyset$ and f be an $\alpha-$contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in (0,1), satisfying the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 0.$

Let $\{x_n\}$ *be defined by*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n, \quad n \ge 0, \end{cases} \tag{8}$$

where $S = \sigma I + (1-\sigma)T$, $k \le \sigma < 1$, *then* $\{x_n\}$ *defined by Theorem 1 converges strongly to a fixed point* q *of* T, *which solves the following variational inequality:*

$$\langle \gamma f q - Bq, p - q \rangle \le 0, \ \ \forall p \in F(T).$$

PROOF.

Step 1. Prove $\{x_n\}$ is bounded. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality,we may assume that $\alpha_n < \|B\|^{-1}$ for all $n \ge 0$. From Lemma 4, we know that $\|I - \alpha_n B\| \le 1 - \alpha_n \bar{\gamma}$. For all $p \in F(T)$, since S is a non-expansive mapping and $F(S) = F(T)$ by Lemma 8, we have

$$\|y_n - p\|^2 = \|\beta_n(x_n - p) + (1-\beta_n)(Sx_n - p)\|^2 \le \|x_n - p\|^2,$$

and

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)(y_n - p)\| \\ &\le \alpha_n \|\gamma f(x_n) - Bp\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\ &\le \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \bar{\gamma}) \|x_n - p\| \\ &\le (1 - \alpha_n(\bar{\gamma} - \gamma\alpha)) \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| \\ &\le \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \gamma\alpha}\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded, so is $\{y_n\}$.

Step 2. Prove $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Set $\gamma_n = 1 - \beta_n$ and $v_n = \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n} = \frac{\alpha_n(\gamma f(x_n) - By_n)}{1-\beta_n} + Sx_n$, then

$$\begin{aligned} \|v_{n+1} - v_n\| &\le \frac{\alpha_n}{1-\beta_n} M + \frac{\alpha_{n+1}}{1-\beta_{n+1}} M + \|Sx_{n+1} - Sx_n\| \\ &\le \frac{\alpha_n}{1-\beta_n} M + \frac{\alpha_{n+1}}{1-\beta_{n+1}} M + \|x_{n+1} - x_n\| \end{aligned} \tag{9}$$

where M is a constant satisfying $\|\gamma f(x_n) - By_n\| \le M$ for $n \ge 1$. It follows from (9), $\limsup_{n\to\infty}\{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\} = 0$, which implies that $\lim_{n\to\infty} \|v_n - x_n\| = 0$ by Lemma 2. From the definition of v_n we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{10}$$

Step 3. Prove $\|Tx_n - x_n\| \to 0$ as $n \to \infty$. In fact, from (8) and (10) we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \quad \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Hence

$$\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \frac{1}{1-\beta_n} \|x_n - y_n\| = 0,$$

which implies that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} (1-\sigma) \|Tx_n - x_n\| = 0. \tag{11}$$

Step 4. Let $q = \lim_{t\to 0^+} x_t$, where x_t is the fixed point (for $t \in (0, \|B\|^{-1})$ of contraction $x \mapsto tx + (I - tB)Sx$. From Lemma 5 and Lemma 8, $q \in F(S) = F(T)$, and

$$\langle \gamma f(q) - Bq, p - q\rangle \le 0, \quad \forall p \in F(T) \tag{12}$$

Claim $\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \le 0$. Take the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \lim_{j\to\infty} \langle \gamma f(q) - Bq, x_{n_j} - q\rangle. \tag{13}$$

Since $\{x_{n_j}\}$ is bounded, so there exists a subsequence of $\{x_{n_j}\}$ such that it converges weakly to a point $p \in K$. Without loss generality, let $\{x_{n_j}\}$ denote it and $x_{n_j} \rightharpoonup p$. By (11) and Opial's condition, $p \in F(S) = F(T)$. Thus from (12) and (13) we have that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \le 0. \tag{14}$$

Step 5. Prove that $\{x_n\}$ converges strongly to q. It follows from Lemma 3 and (8) that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(x_n) - Bq) + (I - \alpha_n B)(y_n - q)\|^2 \\
&\le (1 - \bar{\gamma}\alpha_n)^2 \|y_n - q\|^2 + 2\alpha_n\langle \gamma f(x_n) - Bp, x_{n+1} - q\rangle \\
&= (1 - \bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 \\
&\quad +2\alpha_n\langle \gamma f(x_n) - \gamma f(q) + \gamma f(q) - Bq, x_{n+1} - q\rangle \\
&\le (1 - \bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n\alpha\gamma \|x_n - q\| \|x_{n+1} - q\| \\
&\quad +2\alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\
&\le (1 - \bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 + \alpha_n\alpha\gamma(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \\
&\quad +2\alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle,
\end{aligned}$$

which yields that

$$\|x_{n+1} - q\|^2 \le (1 - \alpha_n \frac{2\bar{\gamma} - 2\alpha\gamma}{1 - \alpha_n\alpha\gamma}) \|x_n - q\|^2 + \frac{\alpha_n^2\bar{\gamma}^2}{1 - \alpha_n\alpha\gamma} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n\alpha\gamma}\langle \gamma f(q) - Bq, x_{n+1} - q\rangle. \quad (15)$$

By boundness of $\{x_n\}$ and the condition (i) and Lemma 1, $\{x_n\}$ converges strongly to q. This completes the proof of Theorem 1.

QED

Next, we give strong convergence theorems for a finite family of strict pseudo-contractions.

Theorem 2. *Let H be a real Hilbert space and for $i = 1, 2, \ldots, N$, let T_i be a k_i-strict pseudo-contraction on H for some $0 \le k_i < 1$ with $\bigcap_{i=1}^N F(T_i) \ne \emptyset$ and f be an $\alpha-$contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$, $\eta_i \in (0, 1)$ and $\sum_{i=1}^N \eta_i = 1$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in (0,1), satisfying the following conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ be defined by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Sx_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n B)y_n, \quad n \ge 0, \end{cases} \quad (16)$$

where $S = \sigma I + (1 - \sigma)\sum_{i=1}^N \eta_i T_i$, $k = \max\{k_i : 1 \le i \le N\} \le \sigma < 1$, then $\{x_n\}$ converges strongly to a common fixed point q of $\{T_1, T_2, \ldots, T_N\}$, which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q\rangle \le 0, \ \ \forall p \in \cap_{i=1}^N F(T_i).$$

Proof. Since $\sum_{i=1}^N \eta_i T_i$ is a $k-$strict pseudo-contraction mapping, by Theorem 1 we know Theorem 2 is true. This completes the proof of Theorem 2. QED

Theorem 3. *Let H be a real Hilbert space and for $i = 1, 2, \ldots, N$, let T_i be a k_i-strict pseudo-contraction on H for some $0 \le k_i < 1$ with $\bigcap_{i=1}^N F(T_i) \ne \emptyset$ and f be an $\alpha-$contraction. Let B be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Given the initial guess $x_0 \in H$ chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^\infty$, $\{\eta_i^{(n)}\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in (0,1), satisfying the following conditions* *(i)* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 0$;

(iii) $\sum_{i=1}^{N}\eta_i^{(n)} = 1$, $\inf_n \eta_i^{(n)} > 0$, $\lim_{n\to\infty}|\eta_i^{(n+1)} - \eta_i^{(n)}| = 0$ *for* $i = 1, 2, \ldots, N$.

Let $\{x_n\}$ *be defined by*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)S_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \quad n \ge 0, \end{cases} \tag{17}$$

where $S_n = \sigma I + (1-\sigma)\sum_{i=1}^{N}\eta_i^{(n)}T_i$, $k = \max\{k_i : 1 \le i \le N\} \le \sigma < 1$, *then* $\{x_n\}$ *converges strongly to a common fixed point* q *of* $\{T_1, T_2, \cdots, T_N\}$, *which solves the following variational inequality:*

$$\langle \gamma f q - Bq, p - q\rangle \le 0, \ \ \forall\, p \in \cap_{i=1}^{N}F(T_i).$$

Proof. For each n, let $H_n = \sum_{i=1}^{N}\eta_i^{(n)}T_i$, then H_n is a $k-$strict pseudo-contraction by Lemma 6, $k = \max\{k_i : 1 \le i \le N\}$. Moreover, $S_n = \sigma I + (1-\sigma)H_n$ is a non-expansive mapping and $F(S_n) = F(H_n) = \cap_{i=1}^{N}F(T_i)$ by Lemma 8.

Step 1. Let $p \in \cap_{i=1}^{N}F(T_i)$, from (17), we have

$$\|y_n - p\|^2 = \|\beta_n(x_n - p) + (1-\beta_n)(S_n x_n - p)\|^2 \le \|x_n - p\|^2,$$

and

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)(y_n - p)\| \\ &\le \alpha_n\|\gamma f(x_n) - Bp\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &\le (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| \\ &\le \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \gamma\alpha}\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded, so is $\{y_n\}$.

Step 2. Prove $\|x_{n+1} - x_n\| \to 0$. Let

$$\gamma_n = 1 - \beta_n, v_n = \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n} = \frac{\alpha_n(\gamma f(x_n) - By_n)}{\gamma_n} + S_n x_n.$$

Let M_3 be a constant such that

$$\{\|\gamma f(x_n) - By_n\|, \|T_1 x_n\|, \|T_2 x_n\|, \cdots, \|T_N x_n\|\} \le M_3$$

for $n \geq 0$. By $S_n = \sigma I + (1-\sigma)\sum_{i=1}^N \eta_i^{(n)} T_i$, we have

$$S_{n+1}x_n - S_n x_n = (1-\sigma)\sum_{i=1}^N (\eta_i^{(n+1)} - \eta_i^{(n)}) T_i x_n.$$

Then

$$\|S_{n+1}x_{n+1} - S_n x_n\| \leq \|x_{n+1} - x_n\| + M_3 \sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|.$$

Furthermore,

$$\|v_{n+1} - v_n\| \leq \frac{\alpha_n M_3}{\gamma_n} + \frac{\alpha_{n+1} M_3}{\gamma_{n+1}} + \|x_{n+1} - x_n\| + M_3 \sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|,$$

which implies that $\limsup_{n\to\infty}\{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\} = 0$. Using Lemma 2, we obtain $\lim_{n\to\infty}\|v_n - x_n\| = 0$, this shows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{18}$$

Again from (17) and (18), we have

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0, \quad \lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Then

$$\lim_{n\to\infty}\|S_n x_n - x_n\| = \lim_{n\to\infty}\frac{1}{1-\beta_n}\|y_n - x_n\| = 0. \tag{19}$$

Step 3. Prove

$$\limsup_{n\to\infty}\langle \gamma f q - Bq, x_n - q\rangle \leq 0, \;\; q \in \cap_{i=1}^N F(T_i),$$

where q is the unique solution of the following variational inequality

$$\langle \gamma f q - Bq, p - q\rangle \leq 0, \;\; \forall p \in \cap_{i=1}^N F(T_i). \tag{20}$$

Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f q - Bq, x_n - q\rangle = \lim_{n\to\infty}\langle \gamma f q - Bq, x_{n_j} - q\rangle.$$

For each i, since $\eta_i^{(n)}$ is bounded, there exists a subsequence of $\eta_i^{(n_j)}$ is still denoted by $\eta_i^{(n_j)}$ such that $\eta_i^{(n_j)} \to \eta_i \in (0,1]$ as $j \to \infty$. At the same time, since $\{x_{n_j}\}$ is bounded, there exists a subsequence of $\{x_{n_j}\}$ is still denoted by $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup z$ as $j \to \infty$. Define mapping $S = \sigma I + (1-\sigma)\sum_{i=1}^{N} \eta_i T_i$, then S is also non-expansive mapping and $F(S) = F(S_n) = \cap_{i=1}^{N} F(T_i)$ by Lemma 7 and 8. Moreover, for all $x \in C$,

$$\left\|S_{n_j}x - Sx\right\| \leq \sum_{i=1}^{N} |\eta_i^{(n_j)} - \eta_i| \, \|T_i x\| \to 0 \quad \text{as} \quad j \to \infty. \tag{21}$$

We claim $z \in F(S)$. If not so, i.e. $z \neq Sz$. Then using the Opial's condition, from (19) and (21) we obtain

$$\begin{aligned}\liminf_{j\to\infty} \left\|x_{n_j} - z\right\| &< \liminf_{j\to\infty} \left\|x_{n_j} - Sz\right\| \\ &\leq \liminf_{j\to\infty} (\left\|x_{n_j} - S_{n_j}x_{n_j}\right\| + \left\|S_{n_j}x_{n_j} - S_{n_j}z\right\| + \left\|S_{n_j}z - Sz\right\|) \\ &\leq \liminf_{j\to\infty} \left\|x_{n_j} - z\right\|. \end{aligned} \tag{22}$$

This is a contradiction. So $z \in F(S) = \cap_{i=1}^{N} F(T_i)$. It follows from (20) that

$$\begin{aligned}&\limsup_{n\to\infty} \langle \gamma f q - Bq, x_n - q\rangle \\ &\qquad = \lim_{j\to\infty} \langle \gamma f q - Bq, x_{n_j} - q\rangle = \langle \gamma f q - Bq, z - q\rangle \leq 0. \end{aligned} \tag{23}$$

Step 4. Prove $\{x_n\}$ converge strongly to q, where q is the unique solution of the following variational inequality

$$\langle \gamma f q - Bq, p - q\rangle \leq 0, \;\; \forall p \in \cap_{i=1}^{N} F(T_i).$$

Reasoning as in Step 5 of Theorem 1, we obtain $\{x_n\}$ converges strongly to q. This completes the proof of Theorem 2. QED

Remark 1. Since a non-expansive mapping is a 0−strict pseudo-contraction mapping, our results are suitable to non-expansive mappings. In addition, our results remove the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ imposed on parameter $\{\alpha_n\}$ in [1,2,7,8]. We remove also the condition $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ imposed on parameter $\{\beta_n\}$ in [1].

Remark 2. The advantages of these results in the present paper are that fewer restrictions are imposed on the parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\eta_i^{(n)}\}$. All of the results obtained in this paper can be viewed as a supplement to the results obtained in [1,2,7,8,14]

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