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Autocommutator subgroups with cyclic outer automorphism group

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Abstract. A criterion for the existence of groups admitting autocommutator subgroups with cyclic outer automorphism group is given. Also the classification of those finite groups G such that $K(G) \cong H$ if H is a centerless finite group with cyclic outer automorphism group and possible solutions G if $|Z(H)| = 2$ and H has a cyclic outer automorphism group is presented.

Keywords: Autocommutator subgroup, outer automorphism

MSC 2000 classification: primary 20D45, secondary 20E34,20F99

1 Introduction

Let G be a group and $\mathrm{Aut}(G)$ denote its automorphism group. The *autocommutator* of element $g \in G$ and automorphism $\alpha \in \mathrm{Aut}(G)$ is $[g, \alpha] = g^{-1}g^{\alpha}$ and the *autocommutator subgroup* of G is $K(G) = [G, \mathrm{Aut}(G)] = \langle [g, \alpha] : g \in G, \alpha \in \mathrm{Aut}(G)\rangle$. In 1997, Hegarty [5] showed that for each finite group H there are only finitely many finite groups G satisfying $K(G) \cong H$.

Deaconescu and Wall [4] solved the equation $K(G) \cong H$, where $H \cong \mathbb{Z}$ is an infinite cyclic group or $H \cong \mathbb{Z}_p$ is a cyclic group of prime order p. They have shown that if $K(G) \cong \mathbb{Z}$, then $G \cong \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}_2$ or D_∞ the infinite dihedral group, and if G is a finite group such that $K(G) \cong \mathbb{Z}_p$, then $G \cong \mathbb{Z}_4$ if $p = 2$ and

$G \cong \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_2$, T or $T \times \mathbb{Z}_2$ if p is odd, where T is a partial holomorph of $\mathbb{Z}_p$ containing $\mathbb{Z}_p$. Also they have noted that there exist finite groups H such that the equation $K(G) \cong H$ has no solution and the symmetric group S_3 as a complete group is an example of such a group. The fact that S_3 is a complete group is very useful in determination of finite groups G with autocommutator subgroup isomorphic to S_3 as we show the claim for each finite complete group in Corollary 1.

We intend to study finite groups, which have a structure rather similar to complete groups. First we give a criterion for the existence of a solution to the equation $K(G) \cong H$, where H is an arbitrary finite group with cyclic outer autmorphism group. Then we determine all solutions of the equation $K(G) \cong H$ for each centreless group H with cyclic outer automorphism group and give all possible solutions, when $|Z(H)| = 2$ and the outer automorphism group of H is cyclic. Finally we give some examples, illustrating our results.

2 Preliminaries

We begin with some useful results that will be used in the proof of our main theorems.

Lemma 1. *If U and V are characteristic subgroups of $G = U \times V$, then $K(G) = K(U) \times K(V)$.*

Proof. The proof is clear and also may be found in [2]. QED

Lemma 2. *If $G = U \times V$, $U \neq 1$ and $U \cap K(G) = 1$, then $U \cong \mathbb{Z}_2$.*

Proof. See [4]. QED

Lemma 3. *Let G be a group. Then*

(1) $C_G(K(G))' \subseteq Z(K(G))$ and $\gamma_3(C_G(K(G))) = 1$;

(2) if $Z(K(G)) = 1$, then $C_G(K(G))$ is abelian.

Proof. Clearly $C_G(K(G))' \subseteq C_G(K(G)) \cap G' \subseteq C_G(K(G)) \cap K(G) = Z(K(G))$. In particular $[C_G(K(G))', C_G(K(G))] \subseteq [K(G), C_G(K(G))] = 1$ and clearly if $Z(K(G)) = 1$, then $C_G(K(G))$ is abelian. QED

Lemma 4. *Let G be a group such that $G = K(G)C_G(K(G))$. Then*

(1) $K(G) = K(K(G))Z(K(G))$;

(2) If $Z(K(G)) = 1$, then $K(G) = K(K(G))$ and $C_G(K(G)) \cong 1$ or $\mathbb{Z}_2$.

PROOF. (1) Since $K(G)$ and $C_G(K(G))$ are characteristic subgroups of G, we have

$$\begin{aligned} K(G) &= [G, \mathrm{Aut}(G)] \\ &= [K(G), \mathrm{Aut}(G)][C_G(K(G)), \mathrm{Aut}(G)] \\ &\subseteq [K(G), \mathrm{Aut}(K(G))](C_G(K(G)) \cap K(G)) \\ &= K(K(G))Z(K(G)) \subseteq K(G). \end{aligned}$$

Hence $K(G) = K(K(G))Z(K(G))$.

(2) It follows using part (1) and Lemma 2. QED

Lemma 5. *If U and V are finite groups with no common direct factor, then*

$$K(U \times V) = K(U)\,\mathrm{Im}(\mathrm{Hom}(V, Z(U))) \times K(V)\,\mathrm{Im}(\mathrm{Hom}(U, Z(V))),$$

where $\mathrm{Im}(\mathrm{Hom}(V, Z(U)))$ *and* $\mathrm{Im}(\mathrm{Hom}(U, Z(V)))$ *are the union of the images of all corresponding homomorphisms, respectively. In particular $K(U \times V) = K(U) \times 1$ if and only if $V \cong 1$, or $V \cong \mathbb{Z}_2$, U has no subgroups of index 2 and $\Omega_1(Syl_2(Z(U))) \subseteq K(U)$, where $Syl_2(Z(U))$ is the Sylow 2-subgroup of $Z(U)$.*

PROOF. The result is a direct consequence of [1, Theorem 3.2]. QED

The following lemma is crucial in determination of the structure of groups under considerations.

Lemma 6. *Let G be a group, $K(G) = H$, $A = C_G(H)$ and $\{x_1, \dots, x_n\}$ a right transversal to HA in G. If $\alpha \in \mathrm{Aut}(A)$ fixes $Z(H)$ elementwise, then the map $\bar{\alpha} : G \to G$, which is defined by $(hax_i)^{\bar{\alpha}} = ha^{\alpha}x_ia_i$ ($a_i \in A$) is an automorphism of G if and only if $(x_ix_jx_k^{-1})^{\bar{\alpha}} = x_i^{\bar{\alpha}}x_j^{\bar{\alpha}}x_k^{\bar{\alpha}-1}$, for each i, j, k such that $HAx_ix_j = HAx_k$.*

PROOF. Let $a_1, \dots, a_n$ be in A and let $hax_i, h'a'x_j \in G$ be arbitrary elements such that $HAx_ix_j = HAx_k$. Then $x_ix_j = h''a''x_k$ for some $h'' \in H$ and $a'' \in A$ and hence $hax_ih'a'x_j = hh'^{x_i^{-1}}h''aa'^{x_i^{-1}}a''x_k$. Now the map $\bar{\alpha}$ is a homomorphism if and only if

$$\begin{aligned} hh'^{x_i^{-1}}h''a^{\alpha}a'^{x_i^{-1}\alpha}a''^{\alpha}x_ka_k &= (hh'^{x_i^{-1}}h''aa'^{x_i^{-1}}a''x_k)^{\bar{\alpha}} \\ &= (hax_ih'a'x_j)^{\bar{\alpha}} \\ &= (hax_i)^{\bar{\alpha}}(h'a'x_j)^{\bar{\alpha}} \\ &= ha^{\alpha}x_ia_ih'a'^{\alpha}x_ja_j \\ &= hh'^{x_i^{-1}}h''a^{\alpha}a'^{\alpha x_i^{-1}}a''a_i^{x_i^{-1}}a_j^{x_k^{-1}}x_k. \end{aligned}$$

Hence $a''^{\alpha} = a'' a_i^{x_i^{-1}} a_j^{x_k^{-1}} a_k^{-1 x_k}$. Since $x_i x_j = h'' a'' x_k$, we get $x_j = h''^{x_i} a''^{x_i} x_i^{-1} x_k$ so that

$$\begin{aligned}(x_i x_j x_k^{-1})^{\bar{\alpha}} &= h'' a''^{\alpha} \\ &= h'' a'' x_i a_i x_i^{-1} x_k a_j a_k^{-1} x_k^{-1} \\ &= x_i a_i h''^{x_i} a''^{x_i} x_i^{-1} x_k a_j a_k^{-1} x_k^{-1} \\ &= x_i a_i x_j a_j a_k^{-1} x_k^{-1} \\ &= x_i^{\bar{\alpha}} x_j^{\bar{\alpha}} x_k^{\bar{\alpha}-1},\end{aligned}$$

as required. The other conditions are easy to verify and the proof is complete. QED

Lemma 7. *Let H be a centreless group and G be a group such that* $\mathrm{Inn}(H)$ *char* $G \leq \mathrm{Aut}(H)$. *Then* $\mathrm{Aut}(G) \cong N_{\mathrm{Aut}(H)}(G)$*, where the isomorphism comes from the conjugation of elements of $N_{\mathrm{Aut}(H)}(G)$ on G.*

Proof. See [8, Lemma 1.1]. QED

3 Main results

We first obtain a criterion for the existence of groups admitting an autocommutator subgroup with cyclic outer automorphism group.

Theorem 1. *Let H be a group with cyclic outer automorphism group. If H is the autocommutator subgroup of a group, then $H = K(H)Z(H)$.*

Proof. Let G be an arbitrary group such that $K(G) = H$ and put $A = C_G(H)$. As $HA/A \cong H/Z(H)$ and G/H is isomorphic to a subgroup of $\mathrm{Aut}(H)$, there exist elements x and y such that $\langle x\rangle \leq \langle y\rangle$ and $G = HA\langle x\rangle \trianglelefteq M = HA\langle y\rangle$, where $M/A \cong \mathrm{Aut}(H)$.

If $\alpha \in \mathrm{Aut}(G)$, then $\alpha|_H \in \mathrm{Aut}(H)$ and so there exists an element $g \in M$ such that $\alpha|_H = \theta_g|_H$, where θ_g is the automorphism of G defined by conjugation by g. Put $\beta = \alpha\theta_g^{-1}$, then $\beta|_H$ is the identity map and so $h^x = (h^x)^\beta = h^{x^\beta}$ for each $h \in H$. Hence $[x, \beta] \in A \cap H = Z(H)$. Let $g = hay^i$, where $h \in H$ and $a \in A$. Then

$$\begin{aligned}[x, \alpha] &= [x, \beta\theta_g] = [x, \theta_g][x, \beta]^{\theta_g} = [x, g][x, \beta]^g \\ &= [x, hay^i][x, \beta]^g = [x, a]^{y^i}[x, h]^{ay^i}[x, \beta]^g \in K(H)Z(H).\end{aligned}$$

Now since H and A are characteristic subgroups of G, we have

$$H = [G, \mathrm{Aut}(G)] = [H, \mathrm{Aut}(G)][A, \mathrm{Aut}(G)][\langle x\rangle, \mathrm{Aut}(G)] \subseteq K(H)Z(H) \subseteq H.$$

Therefore $H = K(H)Z(H)$. QED

Theorem 2. *Let G be a finite group, $K(G) = H$ such that* $\mathrm{Out}(H)$ *is cyclic and let $A = C_G(H)$. Then*

(1) $Z(H) = 1$ *if and only if* $H = K(H)$ *and either* $G \cong K$, *or* $G \cong K \times \mathbb{Z}_2$ *such that K has no subgroups of index 2, for some* $\mathrm{Inn}(H)$ *Char* $K \leq \mathrm{Aut}(H)$.

(2) if $Z(H) \cong \mathbb{Z}_2$, *then* $G/Z(H) \cong K$ *or* $K \times \mathbb{Z}_2$ *for some* $\mathrm{Inn}(H) \leq K \leq \mathrm{Aut}(G)$.

Proof. Since $\mathrm{Out}(H)$ is cyclic and G/HA is isomorphic to a subgroup of $\mathrm{Out}(H)$ and $HA/A \cong H/Z(H) \cong \mathrm{Inn}(H)$ there exists an element $x \in G$ such that $G = HA\langle x\rangle$, where $x^n = ha \in HA$ for some n. Utilizing Lemma 4(1), we may write $A = B \times C$, where B is the Sylow 2-subgroup of A and C is a group of odd order. Moreover $[A, \mathrm{Aut}(G)] \subseteq A \cap H = Z(H)$, and hence $C \subseteq Z(G)$.

If $\gcd(n, |a|) = 1$, then $a = a'^n$ for some $a' \in A$. Since $|Z(H)| \leq 2$, we have $[x, a'^{-1}] \in A \cap H = Z(H) \subseteq Z(G)$ and so

$$(a'^{-1}x)^n = a'^{-n}x^n[x, a'^{-1}]^{\binom{n}{2}} = h[x, a'^{-1}]^{\binom{n}{2}} \in H$$

and we may assume without loss of generality that $a = 1$. Now if $\gcd(n, |a|) > 1$, then we let p to be a prime divisor of $\gcd(n, |a|)$ and a' be an element of order p in A. If $\alpha \in \mathrm{Aut}(A)$ fixes a, then by Lemma 6, α can be extended to an automorphism $\bar{\alpha}$ of G such that $x^{\bar{\alpha}} = xa'$, so that $a' = [x, \bar{\alpha}] \in A \cap H = Z(H)$. Hence $p = 2$ and $\Omega_1(B) = Z(H) \neq 1$. Now if $a = bc$ for some $b \in B$ and $c \in C$, then $c = c'^n$ for some $c' \in C$ and so by replacing x by $c'^{-1}x$ we may assume without loss of generality that $c = 1$. Hence in both cases $G \cong HB\langle x\rangle \times C$ and by Lemma 2, we have $C = 1$ and A is a 2-group. We have two cases:

Case 1. A is abelian with $|Z(H)| = 2$ and $x^n \in H$.

Let $A = \langle a'\rangle \times D$, where $Z(H) \subseteq \langle a'\rangle$. If $\alpha \in \mathrm{Aut}(A)$ such that $a'^{\alpha} = a'^{-1}b$ for some $b \in \Omega_1(D)$ with $|b| < |a'|$ and $\alpha|_D$ is an arbitrary automorphism of D, then by Lemma 6, α can be extednted to an automorphism of G, by fixing x, and it follows that $[A, \alpha] \subseteq H$. Hence $a^4 = 1$, $K(D) = 1$ that is $D \cong 1$ or $\mathbb{Z}_2$ and if $D \neq 1$, then $a^2 = 1$. Therefore $A \cong \mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. $Z(H) = 1$, or $|Z(H)| = 2$ and either $A' = Z(H)$ or $\Omega_1(A) = Z(H)$.

Since an arbitrary automorphism α of A fixes $Z(H)$ elementwise, α can be extended, by Lemma 6, to an automorphism $\bar{\alpha}$ of G such that $x^{\bar{\alpha}} = x[a, \alpha]$. Therefore

$$K(A) = [A, \mathrm{Aut}(A)] = [A, \mathrm{Aut}(G)] \subseteq A \cap H = Z(H).$$

Hence $A \cong 1$ or $\mathbb{Z}_2$ if $Z(H) = 1$ and by [4, Theorem 2], $A \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$ if $Z(H) \cong \mathbb{Z}_2$.

Now if $Z(H) = 1$, then $x^n \in H$ and so $G = H\langle x\rangle A \cong H\langle x\rangle \times A \cong K \times A$, where $\mathrm{Inn}(H)$ Char $K \leq \mathrm{Aut}(H)$. Hence either $G \cong K$, or $G \cong K \times \mathbb{Z}_2$ and by Lemma 5, K has no subgroups of index 2. Also by Theorem 1, $H = K(H)Z(H) = K(H)$, as required. Conversely assume that $G \cong K$, or $K \times \mathbb{Z}_2$, where K has no subgroups of index 2, for some $\mathrm{Inn}(H)$ Char $\mathrm{Aut}(H)$. By Lemma 7, $\mathrm{Aut}(K) \cong N_{\mathrm{Aut}(H)}(K) = \mathrm{Aut}(H)$, which follows in conjunction with Lemma 5 that

$$K(G) = K(K) = [K, \mathrm{Aut}(K)] = [K, \mathrm{Aut}(H)] = K(H) = H,$$

as required.

Finally if $Z(H) \cong \mathbb{Z}_2$, then $G = H\langle x\rangle A$ with $H\langle x\rangle \cap A = Z(H)$. It follows that

$$G/Z(H) \cong H\langle x\rangle/Z(H) \times A/Z(H) \cong K$$

or $K \times \mathbb{Z}_2$ for some $\mathrm{Inn}(H) \leq K \leq \mathrm{Aut}(H)$. The proof is complete. QED

Corollary 1. *If G is a finite group and $K(G) = H$ is a complete group, then H is perfect and $G \cong H$ or $H \times \mathbb{Z}_2$. Conversely if H is a centerless perfect group and $G \cong H$ or $H \times \mathbb{Z}_2$, then $K(G) = H$.*

Proof. The result follows from Theorem 2 or from Lemma 4. QED

Example 1. Let G be a finite group such that $K(G) = H$ is a simple group with cyclic outer automorphism group. Then atlas of finite simple groups [3] gives that H is isomorphic to a sporadic simple group or one of the groups A_n $(n \neq 6)$, $PSL_n(p^m)$ with $p^m > 3$ ($n = 2$, p odd, m even, or $n = p = 2$, or $\gcd(n, p^m - 1) = m = 1$ and $n > 2$), $O_{2n+1}(p^m)$ with $n > 1$ ($n = p = m+1 = 2$, or $n = m = 2$, or p, m odd), $PSp_{2n}(p^m)$ with $n > 2$ ($p = 2$ or p, m odd), $O^+_{2n}(p^m)$ with $n > 3$ ($p = m+1 = 2$), $E_6(p^m)$ ($m = 1$ and $3 \nmid p-1$), $E_7(p^m)$ ($p = 2$ or p, m odd), $E_8(p^m)$, $F_4(p^m)$ (p odd or $p = m+1 = 2$), $G_2(p^m)$ ($p \neq 3$ or $p = m+2 = 3$), $PSU_n(p^{2m})$ with $n > 2$ ($\gcd(n, p^{2m}+1) = 1$, or $\gcd(n, p^{2m}+1) = 2$ and m odd), $O^-_{2n}(p^{2m})$ with $n > 3$ ($p = 2$, or $\gcd(4, p^{mn} + 1) = 2$ and m odd, or $m = 1$), ${}^2E_6(p^{2m})$ ($3 \nmid p^m+1$ or $m = 1$), ${}^3D_4(p^{3m})$, $Sz(2^{2m+1})$, ${}^2F_4(2^{2n+1})$ or $Ree(3^{2n+1})$. In this case the structure of the group G is provided by Theorem 2(1).

Example 2. According to Corollary 1, [7, Theorem 13.5.9] and the atlas of finite simple groups [3], there is a finite group G with $K(G) \cong \mathrm{Aut}(H)$, where H is a non-abelian simple group if and only if $\mathrm{Aut}(H)$ is perfect and $\mathrm{Out}(H) = 1$, or equivalently H is a complete simple group. Hence H is isomorphic to M_{11}, M_{23}, M_{24}, Co_1, Co_2, Co_3, Fi_{23}, Th, B, M, J_1, J_4, Ly, Ru, $PSp_{2n}(2)$ $(n > 2)$, $E_7(2)$, $E_8(p)$, $F_4(p)$ $(p > 2)$ and $G_2(p)$ $(p \neq 3)$, where p is a prime and n is a natural number. In this case $G \cong H$ or $H \times \mathbb{Z}_2$.

If $K(G) \cong D_2 \cong \mathbb{Z}_2$, then by [4, Theorem 2], $G \cong \mathbb{Z}_4$ and if G is abelian and $K(G) \cong D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then by [2], $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$. Note that we don't know all finite solutions to the equation $K(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

It can be easily verified that for $n > 2$

$$\mathrm{Out}(D_{2n}) \cong \begin{cases} \mathbb{Z}_{\frac{\varphi(n)}{2}}, & n \text{ odd}, \\ \mathbb{Z}_{\varphi(n)}, & n = 2p^m \text{ and } p \equiv 3 \pmod 4, \\ \mathbb{Z}_{\frac{\varphi(n)}{2}} \times \mathbb{Z}_2, & \text{otherwise}. \end{cases}$$

Hence $\mathrm{Out}(D_{2n})$ $(n > 2)$ is cyclic if and only if n is odd, $n = 4p^m$ and $p \equiv 3 \pmod 4$, or $n = 4$.

Example 3. There is no finite group G such that $K(G) \cong D_8$, D_{4p^m} with $p \equiv 3 \pmod 4$, D_{2n} with odd n, or even D_∞. For otherwise if $K(G) = H$, then by Theorem 1, $H = K(H)Z(H) = K(H) \subset H$, which is impossible.

According to the above example we may pose the following conjecture.

Conjecture 1. *There is no finite group G such that $K(G) \cong D_{2n}$ $(n > 2)$.*

As another application of Theorem 1 we have:

Example 4. There is no finite group G such that $K(G) \cong QD_{2^n}$ $(n > 3)$, the quasi-dihedral group of order 2^n. To see this, let

$$H = QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, a^b = a^{2^{n-2}-1} \rangle,$$

where $n > 3$. Let α be an endomorphism of H. A simple computation shows that φ is an automorphism if and only if $a^\varphi = a^i$ and $b^\varphi = a^{2j}b$ for some odd integer i and integer j. In particular $\mathrm{Aut}(H) = \langle \beta \rangle \rtimes \langle \alpha \rangle$, where α and β are defined by $a^\alpha = a^u$, $b^\alpha = b$, $a^\beta = a$ and $b^\beta = a^2 b$, in which u is a primitive root modulo 2^{n-1}. It can be easily verified that $\beta \in \mathrm{Inn}(H)$ and $\alpha^t \in \mathrm{Inn}(H)$ if and only if $|\alpha^t| \leq 2$. It follows that $\mathrm{Out}(H) = \langle \alpha\, \mathrm{Inn}(G) \rangle \cong \mathbb{Z}_{2^{n-3}}$ is cyclic. Hence by Theorem 1, we should have $H = K(H)Z(H)$, which is a contradiction for $K(H)Z(H) = K(H) = \langle a^2 \rangle \subset H$.

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