Autocommutator subgroups with cyclic outer automorphism group

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Abstract. A criterion for the existence of groups admitting autocommutator subgroups with cyclic outer automorphism group is given. Also the classification of those finite groups $G$ such that $K(G) \cong H$ if $H$ is a centerless finite group with cyclic outer automorphism group and possible solutions $G$ if $|Z(H)| = 2$ and $H$ has a cyclic outer automorphism group is presented.

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1 Introduction

Let $G$ be a group and $\text{Aut}(G)$ denote its automorphism group. The autocommutator of element $g \in G$ and automorphism $\alpha \in \text{Aut}(G)$ is $[g, \alpha] = g^{-1}g^{\alpha}$ and the autocommutator subgroup of $G$ is $K(G) = [G, \text{Aut}(G)] = \langle [g, \alpha] : g \in G, \alpha \in \text{Aut}(G) \rangle$. In 1997, Hegarty [5] showed that for each finite group $H$ there are only finitely many finite groups $G$ satisfying $K(G) \cong H$.

Deaconescu and Wall [4] solved the equation $K(G) \cong H$, where $H \cong \mathbb{Z}$ is an infinite cyclic group or $H \cong \mathbb{Z}_p$ is a cyclic group of prime order $p$. They have shown that if $K(G) \cong \mathbb{Z}$, then $G \cong \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}_2$ or $D_{\infty}$ the infinite dihedral group, and if $G$ is a finite group such that $K(G) \cong \mathbb{Z}_p$, then $G \cong \mathbb{Z}_4$ if $p = 2$ and
\(G \cong \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_2, T\) or \(T \times \mathbb{Z}_2\) if \(p\) is odd, where \(T\) is a partial holomorph of \(\mathbb{Z}_p\) containing \(\mathbb{Z}_p\). Also they have noted that there exist finite groups \(H\) such that the equation \(K(G) \cong H\) has no solution and the symmetric group \(S_3\) as a complete group is an example of such a group. The fact that \(S_3\) is a complete group is very useful in determination of finite groups \(G\) with autocommutator subgroup isomorphic to \(S_3\) as we show the claim for each finite complete group in Corollary 1.

We intend to study finite groups, which have a structure rather similar to complete groups. First we give a criterion for the existence of a solution to the equation \(K(G) \cong H\), where \(H\) is an arbitrary finite group with cyclic outer automorphism group. Then we determine all solutions of the equation \(K(G) \cong H\) for each centreless group \(H\) with cyclic outer automorphism group and give all possible solutions, when \(|Z(H)| = 2\) and the outer automorphism group of \(H\) is cyclic. Finally we give some examples, illustrating our results.

2 Preliminaries

We begin with some useful results that will be used in the proof of our main theorems.

**Lemma 1.** If \(U\) and \(V\) are characteristic subgroups of \(G = U \times V\), then \(K(G) = K(U) \times K(V)\).

**Proof.** The proof is clear and also may be found in [2]. \(\Box\)

**Lemma 2.** If \(G = U \times V\), \(U \neq 1\) and \(U \cap K(G) = 1\), then \(U \cong \mathbb{Z}_2\).

**Proof.** See [4]. \(\Box\)

**Lemma 3.** Let \(G\) be a group. Then

1. \(C_G(K(G))' \subseteq Z(K(G))\) and \(\gamma_3(C_G(K(G))) = 1\);

2. if \(Z(K(G)) = 1\), then \(C_G(K(G))\) is abelian.

**Proof.** Clearly \(C_G(K(G))' \subseteq C_G(K(G)) \cap G' \subseteq C_G(K(G)) \cap K(G) = Z(K(G))\). In particular \([C_G(K(G))', C_G(K(G))] \subseteq [K(G), C_G(K(G))] = 1\) and clearly if \(Z(K(G)) = 1\), then \(C_G(K(G))\) is abelian. \(\Box\)

**Lemma 4.** Let \(G\) be a group such that \(G = K(G)C_G(K(G))\). Then

1. \(K(G) = K(K(G))Z(K(G))\);

2. if \(Z(K(G)) = 1\), then \(K(G) = K(K(G))\) and \(C_G(K(G)) \cong 1\) or \(\mathbb{Z}_2\).
Proof. (1) Since \( K(G) \) and \( C_G(K(G)) \) are characteristic subgroups of \( G \), we have

\[
K(G) = [G, \text{Aut}(G)] \\
= [K(G), \text{Aut}(G)][C_G(K(G)), \text{Aut}(G)] \\
\subseteq [K(G), \text{Aut}(G)](C_G(K(G)) \cap K(G)) \\
= K(K(G)) Z(K(G)) \subseteq K(G).
\]

Hence \( K(G) = K(K(G)) Z(K(G)) \).

(2) It follows using part (1) and Lemma 2. \( \square \)

**Lemma 5.** If \( U \) and \( V \) are finite groups with no common direct factor, then

\[
K(U \times V) = K(U) \text{Im} (\text{Hom}(V, Z(U))) \times K(V) \text{Im} (\text{Hom}(U, Z(V))),(\text{Im}(\text{Hom}(V, Z(U)))) \text{ and } \text{Im} (\text{Hom}(U, Z(V))) \text { are the union of the images of all corresponding homomorphisms, respectively. In particular } K(U \times V) = K(U) \times 1 \text{ if and only if } V \cong 1, \text{ or } V \cong \mathbb{Z}_2, U \text{ has no subgroups of index } 2 \text{ and } \Omega_1(Syl_2(Z(U))) \subseteq K(U), \text{ where } Syl_2(Z(U)) \text{ is the Sylow 2-subgroup of } Z(U). \]

Proof. The result is a direct consequence of [1, Theorem 3.2]. \( \square \)

The following lemma is crucial in determination of the structure of groups under considerations.

**Lemma 6.** Let \( G \) be a group, \( K(G) = H \), \( A = C_G(H) \) and \( \{x_1, \ldots, x_n\} \) a right transversal to \( HA \) in \( G \). If \( \alpha \in \text{Aut}(A) \) fixes \( Z(H) \) elementwise, then the map \( \bar{\alpha} : G \to G \), which is defined by \( (hx_i)^{\bar{\alpha}} = ha^\alpha x_i a_i \) (\( a_i \in A \)) is an automorphism of \( G \) if and only if \( (x_i x_j^{-1})^{\bar{\alpha}} = x_i^{\alpha} x_j^{a_i} x_j^{-1} \), for each \( i, j, k \) such that \( HA x_i x_j = HA x_k \).

Proof. Let \( a_1, \ldots, a_n \) be in \( A \) and let \( h a x_i, h a' x_j \in G \) be arbitrary elements such that \( HA x_i x_j = HA x_k \). Then \( x_i x_j = h'' a'' x_k \) for some \( h'' \in H \) and \( a'' \in A \) and hence \( h a x_i, h a' x_j = h h'' a'' a x_k \). Now the map \( \bar{\alpha} \) is a homomorphism if and only if

\[
(h h'' a'' a x_k a_k) = (h h'' a'' a a x_k a_k) = (h h'' a'' a a x_k a_k) \bar{\alpha}. \]

\[
= (h a x_i, h a' x_j) \bar{\alpha} \\
= (h a x_i) \bar{\alpha} (h a' x_j) \bar{\alpha} \\
= h a x_i a_i, h a a x_j a_j \bar{\alpha} \\
= h h'' a'' a a x_k a_k a_i a_i x_j x_j^{-1} x_k .
\]
Therefore $a^{\mu a} = a'' a_i^{-1} a_j^{-1} a_k^{-1} x_k$. Since $x_i x_j = h'' a'' x_k$, we get $x_j = h'' a'' x_i x_j^{-1} x_k$ so that

$$(x_i x_j x_k^{-1})^\alpha = h'' a''$$

$$= h'' a'' x_i a_i x_j a_j a_k a_k^{-1} x_k$$

$$= x_i a_i h'' a_j a_k a_k^{-1} x_k$$

$$= x_i a_i a_j a_k a_k^{-1} x_k$$

$$= x_i x_j x_k^{-1}$$

as required. The other conditions are easy to verify and the proof is complete.

Lemma 7. Let $H$ be a centreless group and $G$ be a group such that $\text{Inn}(H)$ is the automorphism group of $G$. Then $\text{Aut}(G) \cong \mathcal{N}_{\text{Aut}(H)}(G)$, where the isomorphism comes from the conjugation of elements of $\mathcal{N}_{\text{Aut}(H)}(G)$ on $G$.

Proof. See [8, Lemma 1.1].

3 Main results

We first obtain a criterion for the existence of groups admitting an automommutator subgroup with cyclic outer automorphism group.

Theorem 1. Let $H$ be a group with cyclic outer automorphism group. If $H$ is the automommutator subgroup of a group, then $H = K(H)Z(H)$.

Proof. Let $G$ be an arbitrary group such that $K(G) = H$ and put $A = C_G(H)$. As $HA/A \cong H/Z(H)$ and $G/H$ is isomorphic to a subgroup of $\text{Aut}(H)$, there exist elements $x$ and $y$ such that $\langle x \rangle \leq \langle y \rangle$ and $G = HA \langle x \rangle \leq M = HA \langle y \rangle$, where $M/A \cong \text{Aut}(H)$.

If $\alpha \in \text{Aut}(G)$, then $\alpha|_H \in \text{Aut}(H)$ and so there exists an element $g \in M$ such that $\alpha|_H = \theta_g|_H$, where $\theta_g$ is the automorphism of $G$ defined by conjugation by $g$. Put $\beta = \alpha g^{-1}$, then $\beta|_H$ is the identity map and so $h^\beta = (h^x)^\beta = h^{x\beta}$ for each $h \in H$. Hence $[x, \beta] \in A \cap H = Z(H)$. Let $g = hay^i$, where $h \in H$ and $a \in A$. Then

$$[x, \alpha] = [x, \beta g] = [x, \theta_g] [x, \beta]^{g} = [x, g][x, \beta]^g$$

$$= [x, hay^i][x, \beta]^g = [x, a]^{g^i} [x, h]^{ag^i} [x, \beta]^g \in K(H)Z(H).$$

Now since $H$ and $A$ are characteristic subgroups of $G$, we have

$$H = [G, \text{Aut}(G)] = [H, \text{Aut}(G)][A, \text{Aut}(G)][\langle x \rangle, \text{Aut}(G)] \subseteq K(H)Z(H) \subseteq H.$$ 

Therefore $H = K(H)Z(H)$. 

QED
**Theorem 2.** Let $G$ be a finite group, $K(G) = H$ such that $\text{Out}(H)$ is cyclic and let $A = C_G(H)$. Then

1. $Z(H) = 1$ if and only if $H = K(H)$ and either $G \cong K$, or $G \cong K \times \mathbb{Z}_2$ such that $K$ has no subgroups of index 2, for some $\text{Inn}(H)$.

2. If $Z(H) \cong \mathbb{Z}_2$, then $G/Z(H) \cong K$ or $K \times \mathbb{Z}_2$ for some $\text{Inn}(H) \leq K \leq \text{Aut}(G)$.

**Proof.** Since $\text{Out}(H)$ is cyclic and $G/HA$ is isomorphic to a subgroup of $\text{Out}(H)$ and $HA/A \cong H/Z(H) \cong \text{Inn}(H)$ there exists an element $x \in G$ such that $G = HA\langle x \rangle$, where $x^n = ha \in HA$ for some $n$. Utilizing Lemma 4(1), we may write $A = B \times C$, where $B$ is the Sylow 2-subgroup of $A$ and $C$ is a group of odd order. Moreover $[A, \text{Aut}(G)] \triangleleft \Omega = Z(H)$, and hence $C \subseteq Z(G)$.

If $\gcd(n, |a|) = 1$, then $a = a^n$ for some $a' \in A$. Since $|Z(H)| \leq 2$, we have $[x, a^{-1}] \in A \cap H = Z(H) \subseteq Z(G)$ and so

$$(a^{-1}x)^n = a^{-n}x^n[x, a^{-1}]^{(n)} = h[x, a^{-1}]^{(n)} \in H$$

and we may assume without loss of generality that $a = 1$. Now if $\gcd(n, |a|) > 1$, then let $p$ to be a prime divisor of $\gcd(n, |a|)$ and $a'$ be an element of order $p$ in $A$. If $\alpha \in \text{Aut}(A)$ fixes $a$, then by Lemma 6, $\alpha$ can be extended to an automorphism $\alpha'$ of $G$ such that $x^\alpha = xa'$, so that $a' = [x, \alpha] \in A \cap H = Z(H)$. Hence $p = 2$ and $\Omega_1(B) = Z(H) \neq 1$. Now if $a = bc$ for some $b \in B$ and $c \in C$, then $c = c'^n$ for some $c' \in C$ and so by replacing $x$ by $c^{-1}x$ we may assume without loss of generality that $c = 1$. Hence in both cases $G \cong HB\langle x \rangle \times C$ and by Lemma 2, we have $C = 1$ and $A$ is a 2-group. We have two cases:

Case 1. $A$ is abelian with $|Z(H)| = 2$ and $x^n \in H$.

Let $A = \langle a' \rangle \times D$, where $Z(H) \subseteq \langle a' \rangle$. If $\alpha \in \text{Aut}(A)$ such that $a'^\alpha = a'^{-1}b$ for some $b \in \Omega_1(D)$ with $|b| < |a'|$ and $\alpha|_D$ is an arbitrary automorphism of $D$, then by Lemma 6, $\alpha$ can be extended to an automorphism of $G$, by fixing $x$, and it follows that $[A, \alpha] \subseteq H$. Hence $a^4 = 1$, $K(D) = 1$ that is $D \cong 1$ or $\mathbb{Z}_2$ and if $D \neq 1$, then $a^2 = 1$. Therefore $A \cong \mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. $Z(H) = 1$, or $|Z(H)| = 2$ and either $A' = Z(H)$ or $\Omega_1(A) = Z(H)$.

Since an arbitrary automorphism $\alpha$ of $A$ fixes $Z(H)$ elementwise, $\alpha$ can be extended, by Lemma 6, to an automorphism $\bar{\alpha}$ of $G$ such that $x^{\bar{\alpha}} = x[a, \alpha]$. Therefore

$$K(A) = [A, \text{Aut}(A)] = [A, \text{Aut}(G)] \subseteq A \cap H = Z(H).$$

Hence $A \cong 1$ or $\mathbb{Z}_2$ if $Z(H) = 1$ and by [4, Theorem 2], $A \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$ if $Z(H) \cong \mathbb{Z}_2$. 

Now if $Z(H) = 1$, then $x^n \in H$ and so $G = H \langle x \rangle A \cong H \langle x \rangle \times A \cong K \times A$, where $\text{Inn}(H)$ Char $K \leq \text{Aut}(H)$. Hence either $G \cong K$, or $G \cong K \times \mathbb{Z}_2$ and by Lemma 5, $K$ has no subgroups of index 2. Also by Theorem 1, $H = K(H)Z(H) = K(H)$, as required. Conversely assume that $G \cong K$, or $K \times \mathbb{Z}_2$, where $K$ has no subgroups of index 2, for some $\text{Inn}(H)$ Char $\text{Aut}(H)$. By Lemma 7, $\text{Aut}(K) \cong N_{\text{Aut}(H)}(K) = \text{Aut}(H)$, which follows in conjunction with Lemma 5 that

$$K(G) = K(K) = [K, \text{Aut}(K)] = [K, \text{Aut}(H)] = K(H) = H,$$

as required.

Finally if $Z(H) \cong \mathbb{Z}_2$, then $G = H \langle x \rangle A$ with $H \langle x \rangle \cap A = Z(H)$. It follows that

$$G/Z(H) \cong H\langle x \rangle/Z(H) \times A/Z(H) \cong K$$

or $K \times \mathbb{Z}_2$ for some $\text{Inn}(H) \leq K \leq \text{Aut}(H)$. The proof is complete. \qedsymbol

**Corollary 1.** If $G$ is a finite group and $K(G) = H$ is a complete group, then $H$ is perfect and $G \cong H$ or $H \times \mathbb{Z}_2$. Conversely if $H$ is a centerless perfect group and $G \cong H$ or $H \times \mathbb{Z}_2$, then $K(G) = H$.

**Proof.** The result follows from Theorem 2 or from Lemma 4. \qedsymbol

**Example 1.** Let $G$ be a finite group such that $K(G) = H$ is a simple group with cyclic outer automorphism group. Then atlas of finite simple groups [3] gives that $H$ is isomorphic to a sporadic simple group or one of the groups $A_n$ ($n \neq 6$), $PSL_n(p^m)$ with $p^m > 3$ ($n = 2, p$ odd, $m$ even, or $n = p = 2$, or $\text{gcd}(n, p^m - 1) = m = 1$ and $n > 2$), $O_{2n+1}(p^m)$ with $n > 1$ ($n = p = m + 1 = 2$, or $n = m = 2$, or $p, m$ odd), $PSO_{2n}(p^m)$ with $n > 2$ ($p = 2$ or $p, m$ odd), $O_2^-(p^m)$ with $n > 3$ ($p = m + 1 = 2$), $E_6(p^m)$ ($m = 1$ and $3 \nmid p - 1$), $E_7(p^m)$ ($p = 2$ or $p, m$ odd), $E_8(p^m)$, $F_4(p^m)$ ($p$ odd or $p = m + 1 = 2$), $G_2(p^m)$ ($p \neq 3$ or $p = m + 2 = 3$), $PSU_n(p^{2m})$ with $n > 2$ ($\text{gcd}(n, p^{2m} + 1) = 1$, or $\text{gcd}(n, p^{2m} + 1) = 2$ and $m$ odd), $O_2^-(p^{2m})$ with $n > 3$ ($p = 2$, or $\text{gcd}(4, p^{2m} + 1) = 2$ and $m$ odd, or $m = 1$), $2E_6(p^{2m})$ ($3 \nmid p^{2m} + 1$ or $m = 1$), $3D_4(p^{2m})$, $Sz(2^{2m+1})$, $2F_4(2^{2n+1})$ or $ Ree(3^{2n+1})$.

In this case the structure of the group $G$ is provided by Theorem 2(1).

**Example 2.** According to Corollary 1, [7, Theorem 13.5.9] and the atlas of finite simple groups [3], there is a finite group $G$ with $K(G) \cong \text{Aut}(H)$, where $H$ is a non-abelian simple group if and only if $\text{Aut}(H)$ is perfect and $\text{Out}(H) = 1$, or equivalently $H$ is a complete simple group. Hence $H$ is isomorphic to $M_{11}$, $M_{23}$, $M_{24}$, $Co_1$, $Co_2$, $Co_3$, $Fi_{23}$, $Th$, $B$, $M$, $J_1$, $J_4$, $Ly$, $Ru$, $PSO_{2n}(2)$ ($n > 2$), $E_7(2)$, $E_8(p)$, $F_4(p)$ ($p > 2$) and $G_2(p)$ ($p \neq 3$), where $p$ is a prime and $n$ is a natural number. In this case $G \cong H$ or $H \times \mathbb{Z}_2.$
If \( K(G) \cong D_2 \cong \mathbb{Z}_2 \), then by [4, Theorem 2], \( G \cong \mathbb{Z}_4 \) and if \( G \) is abelian and \( K(G) \cong D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), then by [2], \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( \mathbb{Z}_4 \times \mathbb{Z}_2 \). Note that we don’t know all finite solutions to the equation \( K(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

It can be easily verified that for \( n > 2 \)

\[
\text{Out}(D_{2n}) \cong \begin{cases} 
\mathbb{Z}_{\varphi(n)/2}, & n \text{ odd,} \\
\mathbb{Z}_{\varphi(n)}, & n = 2p^m \text{ and } p \equiv 3 \pmod{4}, \\
\mathbb{Z}_{\varphi(n)/2} \times \mathbb{Z}_2, & \text{otherwise.}
\end{cases}
\]

Hence \( \text{Out}(D_{2n}) \) \((n > 2)\) is cyclic if and only if \( n \) is odd, \( n = 4p^m \) and \( p \equiv 3 \pmod{4} \), or \( n = 4 \).

**Example 3.** There is no finite group \( G \) such that \( K(G) \cong D_8, D_{4p^m} \) with \( p \equiv 3 \pmod{4}, D_{2n} \) with odd \( n \), or even \( D_{\infty} \). For otherwise if \( K(G) = H \), then by Theorem 1, \( H = K(H)Z(H) = K(H) \subset H \), which is impossible.

According to the above example we may pose the following conjecture.

**Conjecture 1.** There is no finite group \( G \) such that \( K(G) \cong D_{2n} \) \((n > 2)\).

As another application of Theorem 1 we have:

**Example 4.** There is no finite group \( G \) such that \( K(G) \cong QD_{2n} \) \((n > 3)\), the quasi-dihedral group of order \( 2^n \). To see this, let

\[
H = QD_{2n} = \langle a, b : a^{2n-1} = b^2 = 1, a^b = a^{2n-2-1}\rangle,
\]

where \( n > 3 \). Let \( \alpha \) be an endomorphism of \( H \). A simple computation shows that \( \varphi \) is an automorphism if and only if \( a^\varphi = a^i \) and \( b^\varphi = a^{2j}b \) for some odd integer \( i \) and integer \( j \). In particular \( \text{Aut}(H) = \langle \beta \rangle \times \langle \alpha \rangle \), where \( \alpha \) and \( \beta \) are defined by \( a^u = a^u, b^u = b, a^\beta = a \) and \( b^\beta = a^2b \), in which \( u \) is a primitive root modulo \( 2^{n-1} \). It can be easily verified that \( \beta \in \text{Inn}(H) \) and \( \alpha^i \in \text{Inn}(H) \) if and only if \( |\alpha^i| \leq 2 \). It follows that \( \text{Out}(H) = \langle \alpha \text{ Inn}(G) \rangle \cong \mathbb{Z}_{2^{n-3}} \) is cyclic. Hence by Theorem 1, we should have \( H = K(H)Z(H) \), which is a contradiction for \( K(H)Z(H) = K(H) = \langle a^2 \rangle \subset H \).

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**References**


