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The natural transformations $T^*T^{(r)} \to T^*T^{(r)}$

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Abstract. For natural numbers $r \geq 1$ and $n \geq 3$ a complete classification of natural transformations $A : T^*T^{(r)} \to T^*T^{(r)}$ over n-manifolds is given, where $T^{(r)}$ is the linear r-tangent bundle functor.

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In this paper let M be an arbitrary n-manifold.

In [3], Kolář and Radziszewski obtained a classification of all natural transformations $T^*TM \tilde{=} TT^*M \to T^*TM$. In [1], Gancarzewicz and Kolář obtained a classification of all natural affinors $T^*T^{(r)}M \to T^*T^{(r)}M$ on the linear r-tangent bundle $T^{(r)}M = (J^r(M, \mathbf{R})_0)^*$.

This note is a generalization of [1] and [3]. For natural numbers $n \geq 3$ and $r \geq 1$ we obtain a complete description of all natural transformations $A : T^*T^{(r)}M \to T^*T^{(r)}M$. It is following.

By [4], we have an (explicitly defined) isomorphism between the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$ and the algebra $C^\infty(\mathbf{R}^r)$ of smooth maps $\mathbf{R}^r \to \mathbf{R}$. In Section 1, we cite the result of [4].

Clearly, the set of all natural transformations $T^*T^{(r)}M \to T^{(r)}M$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$. In Section 3, we prove that if $n \geq 3$, then this module is free and r-dimensional, and we construct explicitly the basis of this module.

Let $\underline{B} : T^*T^{(r)}M \to T^{(r)}M$ be a natural transformation. A natural transformation $B : T^*T^{(r)}M \to T^*T^{(r)}M$ is called to be over $\underline{B}$ iff $q \circ B = \underline{B}$, where $q : T^*T^{(r)}M \to T^{(r)}M$ is the cotangent bundle projection. (Of course, any natural transformation $B : T^*T^{(r)}M \to T^*T^{(r)}M$ is over $\underline{B} = q \circ B$.) Clearly, the set of all natural transformations $T^*T^{(r)}M \to T^*T^{(r)}M$ over $\underline{B}$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$. In Section 5, we prove that this module is free and $(r+1)$-dimensional, and we construct explicitly the basis of this module.

In Sections 2 and 4, we cite or prove some technical facts. We use these facts in Sections 3 and 5.

Throughout this note the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \dots, x^n$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \dots, n$.

All natural operators, natural functions and natural transformations are over n-manifolds, i. e. the naturality is with respect to embeddings between n-manifolds.

All manifolds and maps are assumed to be of class C^∞.

1. The natural functions $T^*T^{(r)}M \to \mathbb{R}$

Example 1. ([4]) For any $s \in \{1, \dots, r\}$ we have a natural function $\lambda^{<s>} : T^*T^{(r)}M \to \mathbf{R}$ given by $\lambda^{<s>}(a) =< (A^{<s>} \circ \pi)(a), q(a) >$, where the map $q : T^*T^{(r)}M \to T^{(r)}M$ is the cotangent bundle projection, $A^{<s>} : (T^{(r)}M)^* \to (T^{(r)}M)^*$ is a fibre bundle morphism over id_M given by $A^{<s>}(j^r_x\gamma) = j^r_x(\gamma^s)$, $\gamma : M \to \mathbf{R}$, $\gamma(x) = 0$, γ^s is the s-th power of γ, $x \in M$, and $\pi : T^*T^{(r)}M \to (T^{(r)}M)^*$ is a fibre bundle morphism over id_M by $\pi(a) = a|V_{q(a)}T^{(r)}M \tilde{=} T^{(r)}_x M$, $a \in (T^*T^{(r)}M)_x M$, $x \in M$.

Proposition 1. *([4]) All natural functions $T^*T^{(r)}M \to \mathbf{R}$ are of the form $f \circ (\lambda^{<1>}, \dots, \lambda^{<r>})$, where $f \in C^\infty(\mathbf{R}^r)$.*

Hence (since the image of $(\lambda^{<1>}, \dots, \lambda^{<r>})$ is $\mathbf{R}^r$) we have the algebra isomorphism between natural functions $T^*T^{(r)}M \to \mathbf{R}$ and $C^\infty(\mathbf{R}^r)$.

2. The natural operators lifting functions from M to $T^*T^{(r)}M$

Example 2. ([5]) Denote $S(r) = \{(s_1, s_2) \in (\mathbf{N} \cup \{0\})^2 \ : \ 1 \le s_1 + s_2 \le r\}$. For $(s_1, s_2) \in S(r)$ and $L : M \to \mathbf{R}$ define $\lambda^{<s_1,s_2>}(L) : T^*T^{(r)}M \to \mathbf{R}$ by $\lambda^{<s_1,s_2>}(a) =< (A^{<s_1,s_2>}(L) \circ \pi)(a), q(a) >$, where $q : T^*T^{(r)}M \to T^{(r)}M$ and $\pi : T^*T^{(r)}M \to (T^{(r)}M)^*$ are as in Example 1 and $A^{<s_1,s_2>}(L) : (T^{(r)}M)^* \to (T^{(r)}M)^*$ is a fibre bundle morphism over id_M given by $A^{<s_1,s_2>}(L)(j^r_x\gamma) = j^r_x((L - L(x))^{s_2}\gamma^{s_1})$, $\gamma : M \to \mathbf{R}$, $\gamma(x) = 0$, $x \in M$. Clearly, given a pair $(s_1, s_2) \in S(r)$ the correspondence $\lambda^{<s_1,s_2>} : L \to \lambda^{<s_1,s_2>}(L)$ is a natural operator $T^{(0,0)} \rightsquigarrow T^{(0,0)}(T^*T^{(r)})$ in the sense of [2].

We see that $\lambda^{<0,s>} = \lambda^{<s>}$ for $s = 1, \dots, r$, where $\lambda^{<s>}$ is as in example 1, and the operators $\lambda^{<s,1>}$ for $s = 0, \dots, r-1$ are linear (in L) and $\lambda^{<s,1>}(1) = 0$.

Example 3. Given $L : M \to \mathbf{R}$ we have the vertical lifting $L^V : T^*T^{(r)}M \to \mathbf{R}$ of L defined to be the composition of L with the canonical projection $T^*T^{(r)}M$

$\to M$. The correspondence $L \to L^V$ is a natural operator

$$T^{(0,0)} \rightsquigarrow T^{(0,0)}(T^*T^{(r)}).$$

Proposition 2. *([5]) Let $C : T^{(0.0)} \rightsquigarrow T^{(0,0)}(T^*T^{(r)})$ be a natural operator. If $n \geq 3$, then there exists the uniquely determined (by C) smooth map $H : \mathbf{R}^{S(r)} \times \mathbf{R} \to \mathbf{R}$ such that $C(L) = H \circ ((\lambda^{<s_1,s_2>})_{(s_1,s_2)\in S(r)}, L^V)$ for any n-manifold M and any $L : M \to \mathbf{R}$.*

Corollary 1. *Let $C : T^{(0.0)} \rightsquigarrow T^{(0,0)}(T^*T^{(r)})$ be a linear natural operator with $C(1) = 0$. If $n \geq 3$, then there exists the uniquely determined (by C) smooth maps $H^0, \dots, H^{r-1} : \mathbf{R}^r \to \mathbf{R}$ such that $C(L) = \sum_{s=0}^{r-1} H^s \circ (\lambda^{<1>}, \dots, \lambda^{<r>}) \cdot \lambda^{<s,1>}(L)$ for any n-manifold M and any $L : M \to \mathbf{R}$.*

Proof. We have $\lambda^{<s_1,s_2>}(tL) = t^{s_2}\lambda^{<s_1,s_2>}(L)$ and $(tL)^V = tL^V$ for any $L : M \to \mathbf{R}$ and any $t \in \mathbf{R}$, and $1^V = 1$. Then the assertion is a consequence of Proposition 2 and the homogeneous function theorem, [2]. $\boxed{QED}$

3. The natural transformations $T^*T^{(r)}M \to T^{(r)}M$

3.1.

Every natural transformation $B : T^*T^{(r)}M \to T^{(r)}M$ induces a linear natural operator $\Phi(B) : T^{(0,0)}M \rightsquigarrow T^{(0,0)}(T^*T^{(r)}M)$ by $\Phi(B)(L)(a) =< B(a), j^r_x(L - L(x)) >$, $a \in (T^*T^{(r)}M)_x$, $x \in M$. Clearly, $\Phi(B)(1) = 0$. On the other hand every linear natural operator $C : T^{(0,0)}M \rightsquigarrow T^{(0,0)}(T^*T^{(r)}M)$ with $C(1) = 0$ induces a natural transformation $\Psi(C) : T^*T^{(r)}M \to T^{(r)}M$ by $< \Psi(C)(a), j^r_x\gamma > = C(\gamma)(a)$, $a \in (T^*T^{(r)}M)_x$, $\gamma : M \to \mathbf{R}$, $\gamma(x) = 0$, $x \in M$. ($\Psi(C)$ is well-defined as C is of order $\leq r$ because of Corollary 1.) It is easily seen that Ψ is inverse to Φ.

3.2.

The set of natural transformations $T^*T^{(r)}M \to T^{(r)}M$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$. Similarly, the set of natural operators $C : T^{(0,0)}M \rightsquigarrow T^{(0,0)}(T^*T^{(r)}M)$ with $C(1) = 0$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$. Clearly, the (described in 3.1.) bijection Ψ is an isomorphism of the modules. Hence from Corollary 1 we deduce.

Theorem 1. *If $n \geq 3$, then the $\Psi(\lambda^{<s,1>})$ for $s = 0, \dots, r-1$, where $\lambda^{<s,1>}$ are as in Example 2, form the basis (over the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$) of the module of natural transformations $T^*T^{(r)}M \to T^{(r)}M$.*

4. The natural functions $\underline{B}!(TT^{(r)}M) \to \mathbb{R}$

Let $\underline{B} : T^*T^{(r)}M \to T^{(r)}M$ be a natural transformation.

4.1.

Let $\underline{B}^!(TT^{(r)}M)$ be the pull-back of the tangent bundle $TT^{(r)}M$ of $T^{(r)}M$ with respect to $\underline{B}$. Any element from $\underline{B}^!(TT^{(r)}M)$ is of the form (a, y), where $a \in T^*T^{(r)}M$ and $y \in T_{\underline{B}(a)}T^{(r)}M$. Clearly, $\underline{B}^!(TT^{(r)}M)$ is a vector bundle over $T^*T^{(r)}M$ and $\underline{B}^!(TT^{(r)}M)$ is a natural bundle over n-manifolds M.

Example 4. The natural functions $\lambda^{<s>} : T^*T^{(r)}M \to \mathbf{R}$ for $s = 1, \dots, r$ (see Example 1) determine (by the pull-back with respect to the bundle projection $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M)$ the natural functions $\lambda^{<s>} : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$. Clearly, they are fibre constant with respect to the bundle projection $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M$.

Example 5. We have a natural function $\nu : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ such that $\nu(a, y) = v\gamma$, $a \in (T^*T^{(r)}M)_x$, $y \in T_{\underline{B}(a)}T^{(r)}M$, $x \in M$, $v = Tp(y) \in T_xM$, $p : T^{(r)}M \to M$ is the bundle projection, $\gamma : M \to \mathbf{R}$, $j^r_x\gamma = \pi(a)$, $\pi : T^*T^{(r)}M \to (T^{(r)}M)^*$ is as in Example 1. Clearly, ν is fibre linear with respect to the bundle projection $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M$.

Example 6. For $s = 2, \dots, r+1$ we have a natural function

$$\nu^{<s>} : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$$

such that $\nu^{<s>}(a, y) = d_{\underline{B}(a)}(C(\gamma^s))(y)$, $a \in (T^*T^{(r)}M)_x$, $y \in T_{\underline{B}(a)}T^{(r)}M$, $x \in M$, $\gamma : M \to \mathbf{R}$, $\gamma(x) = 0$, $j^r_x\gamma = \pi(a)$, $\pi : T^*T^{(r)}M \to (T^{(r)}M)^*$ is as in Example 1, γ^s is the s-th power of γ and $C : T^{(0,0)}M \rightsquigarrow T^{(0,0)}(T^{(r)}M)$ is a natural operator defined as follows. If $L : M \to \mathbf{R}$ then $C(L) : T^{(r)}M \to \mathbf{R}$, $C(L)(\omega) =< \omega, j^r_x(L - L(x)) >$, $\omega \in T^{(r)}_xM$, $x \in M$. (If $s = 2, \dots, r+1$ then $j^{r+1}_x(\gamma^s)$ is determined by a because of $j^r_x\gamma = \pi(a)$ is determined and $\gamma(x) = 0$. Hence $j^1_{\underline{B}(a)}(C(\gamma^s))$ is determined by a. Then the differential $d_{\underline{B}(a)}C(\gamma^s) : T_{\underline{B}(a)}T^{(r)}M \to \mathbf{R}$ is determined by a. Consequently $\nu^{<s>}$ is well-defined.) The $\nu^{<s>}$ are fibre linear with respect to $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M$.

The purpose of this section is to prove the following proposition.

Proposition 3. *Let $g : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ be a natural function. Then there exists the uniquely determined (by g) smooth map $f : \mathbf{R}^{2r+1} \to \mathbf{R}$ such that $g = f \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)$.*

We have the following corollary of Proposition 3.

Corollary 2. *Let $g : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ be a natural function such that g is fibre linear with respect to the bundle projection $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M$. Then there exists the uniquely determined (by g) smooth maps $f^2, \dots, f^{r+1}, f : \mathbf{R}^r \to \mathbf{R}$ such that $g = f \circ (\lambda^{<1>}, \dots, \lambda^{<r>}) \cdot \nu + \sum_{s=2}^{r+1} f^s \circ (\lambda^{<1>}, \dots, \lambda^{<r>}) \cdot \nu^{<s>}$.*

Proof. The assertion is a consequence of Proposition 3 and the homogeneous function theorem. QED

The proof of Proposition 3 will occupy the rest of this section.

Lemma 1. *Let $g, h : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ be natural functions. Suppose that $g(a, y) = h(a, y)$ for any $a \in (T^*T^{(r)}\mathbf{R}^n)_0$ and any $y \in T_{\underline{B}(a)}T^{(r)}\mathbf{R}^n$ with $\pi(a) = j^r_0(x^1)$ and $< a, T^{(r)}\partial_i(q(a)) >= 0$ for $i = 1, \dots, n$, where $T^{(r)}$ is also the complete lifting of vector fields to $T^{(r)}$ and where $q : T^*T^{(r)}\mathbf{R}^n \to T^{(r)}\mathbf{R}^n$ and $\pi : T^*T^{(r)}\mathbf{R}^n \to (T^{(r)}\mathbf{R}^n)^*$ are as in Example 1. Then $g = h$.*

Schema of the proof. The proof is quite similar to the proof of Lemma 5 in [4]. In [4], functions g and h depend only on a from $T^*T^{(r)}M$. Now, functions g and h depend on a (also from $T^*T^{(r)}M$) and y. Clearly, we can "trivialize" a in the same way as in the proofs of Lemmas 2–5 in [4]. Roughly speaking, in this way we obtain the proof of our lemma. QED

Proof of Proposition 3. Let $g : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ be a natural function. Define $f : \mathbf{R}^r \times \mathbf{R}^r \times \mathbf{R} \to \mathbf{R}$ by $f(\xi, \rho, \eta) = g(a_\xi, y_{\xi,\rho,\eta})$, $\xi = (\xi^1, \dots, \xi^r) \in \mathbf{R}^r$, $\rho = (\rho^2, \dots, \rho^{r+1}) \in \mathbf{R}^r$, $\eta \in \mathbf{R}$, where $a_\xi \in (T^*T^{(r)})_0$ is the unique form satisfying the conditions $\pi(a_\xi) = j^r_0(x^1)$, $< a_\xi, T^{(r)}\partial_i(q(a)) >= 0$ for $i = 1, \dots, n$, $< q(a_\xi), j^r_0(x^\alpha) >= 0$ for all $\alpha = (\alpha_1, \dots \alpha_n) \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\alpha| \le r$ and $\alpha_2 + \dots + \alpha_n \ge 1$ and $< q(a_\xi), j^r_0((x^1)^s) >= \xi^s$ for $s = 1, \dots, r$, and where $y_{\xi,\rho,\eta} = \eta T^{(r)}\partial_{1|\underline{B}(a_\xi)} + (\underline{B}(a_\xi), \sum_{p=1}^r \rho^{p+1}(j^r_0((x^1)^p))^*) \in T_{\underline{B}(a_\xi)}T^{(r)}\mathbf{R}^n$ (we use the standard identification $VT^{(r)}M = T^{(r)}M \times_M T^{(r)}M$). Here $(j^r_0(x^\alpha))^*$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\alpha| \le r$ is the basis dual to the $j^r_0(x^\alpha)$.

It is easy to compute that $\lambda^{<s>}(a_\xi, y_{\xi,\rho,\eta}) = \xi^s$ for $s = 1, \dots, r$, $\nu^{<s>}(a_\xi, y_{\xi,\rho,\eta}) = \rho^s + s\xi^{s-1}\eta$ for $s = 2, \dots, r+1$, and $\nu(a_\xi, y_{\xi,\rho,\eta}) = \eta$. Hence $g(a_\xi, y_{\xi,\rho,\eta}) = \overline{f} \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)(a_\xi, y_{\xi,\rho,\eta})$, where $\overline{f}(\xi, \rho, \eta) = f(\xi, (\rho^s - s\xi^{s-1}\eta)_{s=2}^{r+1}, \eta)$. We prove that $g = \overline{f} \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)$.

Let (a, y) be as in the assumption of Lemma 1. Let $c_t = (x^1, tx^2, \dots, tx^n) : \mathbf{R}^n \to \mathbf{R}^n$, $t \neq 0$. It is easy to see that $(T^*T^{(r)}c_t(a), TT^{(r)}c_t(y))$ tends (as t tends to 0) to some $(a_\xi, y_{\xi,\rho,\eta})$. By the invariance with respect to c_t we get $g(a, y) = g(T^*T^{(r)}c_t(a), TT^{(r)}c_t(y))$ and $\overline{f} \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)(a, y) = \overline{f} \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)(T^*T^{(r)}c_t(a), TT^{(r)}c_t(y))$ for any $t \neq 0$. Hence $g(a, y) = g(a_\xi, y_{\xi,\rho,\eta}) = \overline{f} \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)(a_\xi, y_{\xi,\rho,\eta}) = \overline{f} \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)(a, y)$. Therefore

$$g = \overline{f} \circ (\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)$$

because of Lemma 1.

Since the image of $(\lambda^{<1>}, \dots, \lambda^{<r>}, \nu^{<2>}, \dots, \nu^{<r+1>}, \nu)$ is $\mathbf{R}^{2r+1}$, the map f is uniquely determined by g. QED

5. The natural transformations $T^*T^{(r)}M \to T^*T^{(r)}M$ over $T^*T^{(r)}M \to T^{(r)}M$

Let $\underline{B} : T^*T^{(r)}M \to T^{(r)}M$ be a natural transformation. A natural transformation $B : T^*T^{(r)}M \to T^*T^{(r)}M$ is called to be over $\underline{B}$ iff $q \circ B = \underline{B}$, where $q : T^*T^{(r)}M \to T^{(r)}M$ is the cotangent bundle projection.

5.1.

Every natural transformation $B : T^*T^{(r)}M \to T^*T^{(r)}M$ over $\underline{B}$ induces a natural function $\Theta(B) : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ by $\Theta(B)(a, y) =< B(a), y >$, $(a, y) \in \underline{B}^!(TT^{(r)}M)$. ($B(a)$ and y are over $\underline{B}(a)$ and therefore we can take the contraction.) Clearly $\Theta(B)$ is fibre linear with respect to the bundle projection $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M$. On the other hand every natural function $g : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ such that g is fibre linear with respect to the bundle projection $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M$ induces a natural transformation $\Omega(g) : T^*T^{(r)}M \to T^*T^{(r)}M$ over $\underline{B}$ by $< \Omega(g)(a), y >= g(a, y)$, $a \in T^*T^{(r)}M$, $y \in T_{\underline{B}(a)}T^{(r)}M$. It is easily seen that Ω is inverse to Θ.

5.2.

The set of natural transformations $T^*T^{(r)}M \to T^*T^{(r)}M$ over $\underline{B}$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$. Similarly, the set of natural functions $g : \underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ such that g is fibre linear with respect to the bundle projection $\underline{B}^!(TT^{(r)}M) \to T^*T^{(r)}M$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to \mathbf{R}$. (We identify any natural function $T^*T^{(r)}M \to \mathbf{R}$ with the natural function $\underline{B}^!(TT^{(r)}M) \to \mathbf{R}$ by using the pull-back with respect to the obvious projection. Then the module operations are obvious.) Clearly, the (described in 5.1.) bijection Ω is an isomorphism of the modules. Hence from Corollary 2 we deduce.

Theorem 2. *The $\Omega(\nu^{<s>})$ for $s = 2, \dots, r+1$ and $\Omega(\nu)$, where ν and $\nu^{<s>}$ are as in Examples 5 and 6, form the basis (over the algebra of all natural functions $T^*T^{(r)}M \to \mathbf{R}$) of the module of natural transformations $T^*T^{(r)}M \to T^*T^{(r)}M$ over $\underline{B}$.*

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