The natural transformations $T^*T^{(r)} \to T^*T^{(r)}$

Włodzimierz M. Mikulski

Institute of Mathematics, Jagiellonian University, Kraków, Reymonta 4, Poland,
wlodzimierz.mikulski@im.uj.edu.pl

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Abstract. For natural numbers $r \geq 1$ and $n \geq 3$ a complete classification of natural transformations $A : T^*T^{(r)} \to T^*T^{(r)}$ over $n$-manifolds is given, where $T^{(r)}$ is the linear $r$-tangent bundle functor.

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In this paper let $M$ be an arbitrary $n$-manifold.

In [3], Kolar and Radziszewski obtained a classification of all natural transformations $T^*T^r = TT^*M \to T^*TM$. In [1], Gancerzewicz and Kolar obtained a classification of all natural affinors $T^*T^{(r)}M \to T^*T^{(r)}M$ on the linear $r$-tangent bundle $T^{(r)}M = (J^r(M, R)_0)^*$.  

This note is a generalization of [1] and [3]. For natural numbers $n \geq 3$ and $r \geq 1$ we obtain a complete description of all natural transformations $A : T^*T^{(r)}M \to T^*T^{(r)}M$. It is following.

By [4], we have an (explicitly defined) isomorphism between the algebra of natural functions $T^*T^{(r)}M \to R$ and the algebra $C^\infty(R^r)$ of smooth maps $R^r \to R$. In Section 1, we cite the result of [4].

Clearly, the set of all natural transformations $T^*T^{(r)}M \to T^{(r)}M$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to R$. In Section 3, we prove that if $n \geq 3$, then this module is free and $r$-dimensional, and we construct explicitly the basis of this module.

Let $\overline{B} : T^*T^{(r)}M \to T^{(r)}M$ be a natural transformation. A natural transformation $B : T^*T^{(r)}M \to T^*T^{(r)}M$ is called to be over $\overline{B}$ iff $q \circ B = \overline{B}$, where $q : T^*T^{(r)}M \to T^{(r)}M$ is the cotangent bundle projection. (Of course, any natural transformation $B : T^*T^{(r)}M \to T^*T^{(r)}M$ is over $\overline{B} = q \circ B$.) Clearly, the set of all natural transformations $T^*T^{(r)}M \to T^*T^{(r)}M$ over $\overline{B}$ is (in obvious way) a module over the algebra of natural functions $T^*T^{(r)}M \to R$. In Section 5, we prove that this module is free and $(r + 1)$-dimensional, and we construct explicitly the basis of this module.
In Sections 2 and 4, we cite or prove some technical facts. We use these facts in Sections 3 and 5.

Throughout this note the usual coordinates on \( \mathbb{R}^n \) are denoted by \( x^1, \ldots, x^n \) and \( \partial_i = \frac{\partial}{\partial x^i}, \; i = 1, \ldots, n. \)

All natural operators, natural functions and natural transformations are over \( n \)-manifolds, i.e. the naturality is with respect to embeddings between \( n \)-manifolds.

All manifolds and maps are assumed to be of class \( C^\infty. \)

1. The natural functions \( T^*T^{(r)}M \to \mathbb{R} \)

Example 1. ([4]) For any \( s \in \{1, \ldots, r\} \) we have a natural function \( \lambda^{<s>} : T^*T^{(r)}M \to \mathbb{R} \) given by \( \lambda^{<s>}(a) = \langle (A^{<s>} \circ \pi)(a), q(a) \rangle, \) where the map \( q : T^*T^{(r)}M \to T^{(r)}M \) is the cotangent bundle projection, \( A^{<s>} : (T^{(r)}M)^* \to (T^{(r)}M)^* \) is a fibre bundle morphism over \( id_M \) given by \( A^{<s>} (j^r_x \gamma) = j^s_x (\gamma^s), \) \( \gamma: M \to \mathbb{R}, \; \gamma(x) = 0, \; \gamma^s \) is the \( s \)-th power of \( \gamma, \; x \in M, \) and \( \pi : T^*T^{(r)}M \to (T^{(r)}M)^* \) is a fibre bundle morphism over \( id_M \) by \( \pi(a) = a|V_{q(a)}T^{(r)}M = T^{(r)}M, \) \( a \in (T^*T^{(r)}M)_x, \; x \in M. \)

Proposition 1. ([4]) All natural functions \( T^*T^{(r)}M \to \mathbb{R} \) are of the form \( f \circ (\lambda^{<1>}, \ldots, \lambda^{<r>}), \) where \( f \in C^\infty(\mathbb{R}^r). \)

Hence (since the image of \( (\lambda^{<1>}, \ldots, \lambda^{<r>}) \) is \( \mathbb{R}^r \)) we have the algebra isomorphism between natural functions \( T^*T^{(r)}M \to \mathbb{R} \) and \( C^\infty(\mathbb{R}^r). \)

2. The natural operators lifting functions from \( M \) to \( T^*T^{(r)}M \)

Example 2. ([5]) Denote \( S(r) = \{(s_1, s_2) \in (\mathbb{N} \cup \{0\})^2 : 1 \leq s_1 + s_2 \leq r\}. \)

For \( (s_1, s_2) \in S(r) \) and \( L : M \to \mathbb{R} \) define \( \lambda^{<s_1,s_2>} : T^*T^{(r)}M \to \mathbb{R} \) by \( \lambda^{<s_1,s_2>} (a) = \langle (A^{<s_1,s_2>} \circ \pi)(a), q(a) \rangle, \) where \( q : T^*T^{(r)}M \to T^{(r)}M \) and \( \pi : T^*T^{(r)}M \to (T^{(r)}M)^* \) are as in Example 1 and \( A^{<s_1,s_2>} : (T^{(r)}M)^* \to (T^{(r)}M)^* \) is a fibre bundle morphism over \( id_M \) given by \( A^{<s_1,s_2>} (j^r_x \gamma) = j^s_x ((L - L(x))^s \gamma^s), \) \( \gamma : M \to \mathbb{R}, \; \gamma(x) = 0, \; x \in M. \)

Clearly, given a pair \( (s_1, s_2) \in S(r) \) the correspondence \( \lambda^{<s_1,s_2>} : L \to \lambda^{<s_1,s_2>} (L) \) is a natural operator \( T^{(0,0)} \sim T^{(0,0)}(T^*T^{(r)}) \) in the sense of [2].

We see that \( \lambda^{<0,s>} = \lambda^{<s>} \) for \( s = 1, \ldots, r \), where \( \lambda^{<s>} \) is as in example 1, and the operators \( \lambda^{<s,1>} \) for \( s = 0, \ldots, r - 1 \) are linear (in \( L \)) and \( \lambda^{<s,1>} (1) = 0. \)

Example 3. Given \( L : M \to \mathbb{R} \) we have the vertical lifting \( L^V : T^*T^{(r)}M \to \mathbb{R} \) of \( L \) defined to be the composition of \( L \) with the canonical projection \( T^*T^{(r)}M \)
→ M. The correspondence \( L \to L^V \) is a natural operator

\[
T^{(0,0)} \sim T^{(0,0)}(T^*(T^r)).
\]

**Proposition 2.** ([5]) Let \( C : T^{(0,0)} \sim T^{(0,0)}(T^*(T^r)) \) be a natural operator. If \( n \geq 3 \), then there exists the uniquely determined (by \( C \)) smooth map \( H : \mathbb{R}^S(r) \times \mathbb{R} \to \mathbb{R} \) such that \( C(L) = H \circ ((\lambda^{<s_1,s_2>})_{(s_1,s_2) \in S(r)}), L^V) \) for any \( n \)-manifold \( M \) and any \( L : M \to \mathbb{R} \).

**Corollary 1.** Let \( C : T^{(0,0)} \sim T^{(0,0)}(T^*(T^r)) \) be a linear natural operator with \( C(1) = 0 \). If \( n \geq 3 \), then there exists the uniquely determined (by \( C \)) smooth maps \( H^0, \ldots, H^{r-1} : \mathbb{R}^r \to \mathbb{R} \) such that \( C(L) = \sum_{s=0}^{r-1} H^s \circ (\lambda^{<1>, \ldots, \lambda^{<r>}}). \lambda^{<s,1>} (L) \) for any \( n \)-manifold \( M \) and any \( L : M \to \mathbb{R} \).

**Proof.** We have \( \lambda^{<s_1,s_2>} (L) = t^{s_2} \lambda^{<s_1,s_2>} (L) \) and \((tL)^V = tL^V) \) for any \( L : M \to \mathbb{R} \) and any \( t \in \mathbb{R} \), and \( 1^V = 1 \). Then the assertion is a consequence of Proposition 2 and the homogeneous function theorem, [2].

### 3. The natural transformations \( T^*T^r M \to T^r M \)

#### 3.1.

Every natural transformation \( B : T^*T^r M \to T^r M \) induces a linear natural operator \( \Phi(B) : T^{(0,0)}M \sim T^{(0,0)}(T^*T^r M) \) by \( \Phi(B)(L)(a) = \langle B(a), j^x_L(L(x)) \rangle, a \in (T^*T^r M)_x, x \in M \). Clearly, \( \Phi(B)(1) = 0 \). On the other hand every linear natural operator \( C : T^{(0,0)}M \sim T^{(0,0)}(T^*T^r M) \) with \( C(1) = 0 \) induces a natural transformation \( \Psi(C) : T^*T^r M \to T^r M \) by \( \langle \Psi(C)(a), j^x_C(\gamma) \rangle = C(\gamma)(a), a \in (T^*T^r M)_x, \gamma : M \to \mathbb{R}, \gamma(x) = 0, x \in M \). (\( \Psi(C) \) is well-defined as \( C \) is of order \( \leq r \) because of Corollary 1.) It is easily seen that \( \Psi \) is inverse to \( \Phi \).

#### 3.2.

The set of natural transformations \( T^*T^r M \to T^r M \) is (in obvious way) a module over the algebra of natural functions \( T^*T^r M \to \mathbb{R} \). Similarly, the set of natural operators \( C : T^{(0,0)}M \sim T^{(0,0)}(T^*T^r M) \) with \( C(1) = 0 \) is (in obvious way) a module over the algebra of natural functions \( T^*T^r M \to \mathbb{R} \). Clearly, the (described in 3.1.) bijection \( \Psi \) is an isomorphism of the modules. Hence from Corollary 1 we deduce.

**Theorem 1.** If \( n \geq 3 \), then the \( \Psi(\lambda^{<s,1>}) \) for \( s = 0, \ldots, r - 1 \), where \( \lambda^{<s,1>} \) are as in Example 2, form the basis (over the algebra of natural functions \( T^*T^r M \to \mathbb{R} \)) of the module of natural transformations \( T^*T^r M \to T^r M \).
4. The natural functions $B!(TT^{(r)}M) \to \mathbb{R}$

Let $B : T^*T^{(r)}M \to T^{(r)}M$ be a natural transformation.

4.1.

Let $B^!(TT^{(r)}M)$ be the pull-back of the tangent bundle $TT^{(r)}M$ of $T^{(r)}M$ with respect to $B$. Any element from $B^!(TT^{(r)}M)$ is of the form $(a, y)$, where $a \in T^*T^{(r)}M$ and $y \in T_{B(a)}T^{(r)}M$. Clearly, $B^!(TT^{(r)}M)$ is a vector bundle over $T^*T^{(r)}M$ and $B^!(TT^{(r)}M)$ is a natural bundle over $n$-manifolds $M$.

Example 4. The natural functions $\lambda^{<s>} : T^*T^{(r)}M \to \mathbb{R}$ for $s = 1, \ldots, r$ (see Example 1) determine (by the pull-back with respect to the bundle projection $B^!(TT^{(r)}M) \to T^*T^{(r)}M$) the natural functions $\lambda^{<s>} : B^!(TT^{(r)}M) \to \mathbb{R}$. Clearly, they are fibre constant with respect to the bundle projection $B^!(TT^{(r)}M) \to T^*T^{(r)}M$.

Example 5. We have a natural function $\nu : B^!(TT^{(r)}M) \to \mathbb{R}$ such that $\nu(a, y) = \nu(x), a \in (T^*T^{(r)}M)_x, y \in T_{B(a)}T^{(r)}M, x \in M, (0 < s) \nu(x, y) = 0$. $j^1_\gamma = \pi(a), \pi : T^*T^{(r)}M \to (T^{(r)}M)^\ast$ as in Example 1. Clearly, $\nu$ is fibre linear with respect to the bundle projection $B^!(TT^{(r)}M) \to T^*T^{(r)}M$.

Example 6. For $s = 2, \ldots, r + 1$ we have a natural function

$$\nu^{<s>} : B^!(TT^{(r)}M) \to \mathbb{R}$$

such that $\nu^{<s>}(a, y) = d_{B(a)}(C(\gamma^s))(y), a \in (T^*T^{(r)}M)_x, y \in T_{B(a)}T^{(r)}M, x \in M, \gamma : M \to \mathbb{R}, \gamma(x) = 0$. $j^1_\gamma = \pi(a), \pi : T^*T^{(r)}M \to (T^{(r)}M)^\ast$ as in Example 1, $\gamma^s$ is the $s$-th power of $\gamma$ and $C : T^{(0,0)}M \to T^{(0,0)}(T^{(r)}M)$ is a natural operator defined as follows. If $L : M \to \mathbb{R}$ then $C(L) : T^{(r)}M \to \mathbb{R}$, $C(L)(\omega) = \omega, j^1_\gamma(L - L(x))$, $\omega \in T_x^rM, x \in M$. (If $s = 2, \ldots, r + 1$ then $j^1_\gamma(\gamma^s)$ is determined by $a$ because of $j^1_\gamma = \pi(a)$ is determined and $\gamma(x) = 0$. Hence $j^1_{B(a)}(C(\gamma^s))$ is determined by $a$. Then the differential $d_{B(a)}C(\gamma^s) : T_{B(a)}T^{(r)}M \to \mathbb{R}$ is determined by $a$. Consequently $\nu^{<s>}$ is well-defined.) The $\nu^{<s>}$ are fibre linear with respect to $B^!(TT^{(r)}M) \to T^*T^{(r)}M$.

The purpose of this section is to prove the following proposition.

**Proposition 3.** Let $g : B^!(TT^{(r)}M) \to \mathbb{R}$ be a natural function. Then there exists the uniquely determined (by $g$) smooth map $f : \mathbb{R}^{2r+1} \to \mathbb{R}$ such that $g = f \circ (\lambda^{<1>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \ldots, \nu^{<r+1>}, \nu)$.

We have the following corollary of Proposition 3.
Corollary 2. Let \( g : \mathcal{B}^1(T^r T^r M) \to \mathbb{R} \) be a natural function such that \( g \) is fibre linear with respect to the bundle projection \( \mathcal{B}^1(T^r T^r M) \to T^* T^r T^r M \). Then there exists the uniquely determined (by complete lifting of vector fields to \( \mathcal{B}^1(T^r T^r M) \)) smooth maps \( f^2, \ldots, f^{r+1}, f : \mathbb{R}^r \to \mathbb{R} \) such that \( g = f \circ (\lambda^{<11>}, \ldots, \lambda^{<r>}) \cdot \nu + \sum_{s=2}^{r+1} f^s \circ (\lambda^{<11>}, \ldots, \lambda^{<r>}) \cdot \nu^{<s>} \).

**Proof.** The assertion is a consequence of Proposition 3 and the homogeneity function theorem.

The proof of Proposition 3 will occupy the rest of this section.

**Lemma 1.** Let \( g, h : \mathcal{B}^1(T^r T^r M) \to \mathbb{R} \) be natural functions. Suppose that \( g(a, y) = h(a, y) \) for any \( a \in (T^* T^r M)_0 \) and any \( y \in T_{\mathcal{B}(a)} T^r M \) with \( \pi(a) = j_0^T(x^1) \) and \( q < a, T^r \partial_i(q(a)) >= 0 \) for \( i = 1, \ldots, n \), where \( T^r \) is also the complete lifting of vector fields to \( T^r \) and where \( q : T^* T^r M \to T^r M \) and \( \pi : T^* T^r M \to (T^r M)^* \) are as in Example 1. Then \( g = h \).

**Schema of the proof.** The proof is quite similar to the proof of Lemma 5 in [4]. In [4], functions \( g \) and \( h \) depend only on \( a \) from \( T^* T^r M \). Now, functions \( g \) and \( h \) depend on \( a \) (also from \( T^* T^r M \)) and \( y \). Clearly, we can “trivialize” \( a \) in the same way as in the proofs of Lemmas 2–5 in [4]. Roughly speaking, in this way we obtain the proof of our lemma.

**Proof of Proposition 3.** Let \( g : \mathcal{B}^1(T^r T^r M) \to \mathbb{R} \) be a natural function. Define \( f : \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R} \) by \( f(\xi, \rho, \eta) = g(a \xi, y \rho, \eta), \xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^r, \rho = (\rho^1, \ldots, \rho^{r+1}) \in \mathbb{R}^r, \eta \in \mathbb{R} \), where \( a \xi \in (T^* T^r M)_0 \) is the unique form satisfying the conditions \( \pi(a \xi) = j_0^T(x^1) \), \( q < a \xi, T^r \partial_i(q(a)) >= 0 \) for \( i = 1, \ldots, n \), \( q < a \xi, j_0^T(x^a) >= 0 \) for all \( a = (a_1, \ldots, a_n) \in (\mathbb{N} \cup \{0\})^n \) with \( 1 \leq \lvert a \rvert \leq r \) and \( a_2 + \cdots + a_n \geq 1 \) and \( q(a \xi), j_0^T((x^1)^a) >= \xi^s \) for \( s = 1, \ldots, r \), and where \( y \rho, \eta = \eta T^r \partial_1(a \xi) + (B(a \xi), \sum_{p=1}^{r+1} \rho^{p+1}(j_0^T((x^a)^a))^p) \in T\mathcal{B}(a \xi) T^r M \) (we use the standard identification \( VT^r M = T^r M \times_M T^r M \)). Here \( (j_0^T((x^a)^a)^s \) for \( a = (a_1, \ldots, a_n) \in (\mathbb{N} \cup \{0\})^n \) with \( 1 \leq \lvert a \rvert \leq r \) is the basis dual to the \( j_0^T(x^a) \).

It is easy to compute that \( \lambda^{<s>} (a \xi, y \rho, \eta) = \xi^s \) for \( s = 1, \ldots, r \), \( \nu^{<s>} (a \xi, y \rho, \eta) = \rho^s + s \xi^{s-1} \eta \) for \( s = 2, \ldots, r+1 \), and \( \nu (a \xi, y \rho, \eta) = \eta \). Hence \( g(a \xi, y \rho, \eta) = f_0 (\lambda^{<11>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \nu^{<r+1>}) (a \xi, y \rho, \eta) \), where \( f_0 (\xi, \rho, \eta) = f(\xi, \rho^s) \). We prove that \( g = f_0 (\lambda^{<11>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \nu^{<r+1>}, \nu) \).

Let \( (a, y) \) be as in the assumption of Lemma 1. Let \( c_t = (x^1, tx^2, \ldots, tx^n) : \mathbb{R}^n \to \mathbb{R}^n, t \neq 0 \). It is easy to see that \( (T^* T^r c_t(a), T\mathcal{B}(c_t(a))) \) (tends as \( t \) tends to 0) to some \( (a \xi, y \rho, \eta) \). By the invariance with respect to \( c_t \) we get \( g(a, y) = g(T^* T^r c_t(a), T\mathcal{B}(c_t(a))) \) and \( f_0 (\lambda^{<11>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \nu^{<r+1>}) (a, y) = f_0 (\lambda^{<11>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \nu^{<r+1>}, \nu) (T^* T^r c_t(a), T\mathcal{B}(c_t(a))) \) for any \( t \neq 0 \). Hence \( g(a, y) = g(a \xi, y \rho, \eta) = f_0 (\lambda^{<11>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \nu^{<r+1>}, \nu) (a \xi, y \rho, \eta) = f_0 (\lambda^{<11>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \nu^{<r+1>}, \nu) (a, y) \). Therefore \( g = f_0 (\lambda^{<11>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \nu^{<r+1>}, \nu) \).
because of Lemma 1.

Since the image of \((\lambda^{<1>}, \ldots, \lambda^{<r>}, \nu^{<2>}, \ldots, \nu^{<r+1>}, \nu)\) is \(\mathbb{R}^{2r+1}\), the map \(f\) is uniquely determined by \(g\).

5. The natural transformations \(T^*T^{(r)}M \to T^*T^{(r)}M\) over \(T^*T^{(r)}M \to T^{(r)}M\)

Let \(B : T^*T^{(r)}M \to T^{(r)}M\) be a natural transformation. A natural transformation \(B : T^*T^{(r)}M \to T^*T^{(r)}M\) is called to be over \(B\) if \(q \circ B = B\), where \(q : T^*T^{(r)}M \to T^{(r)}M\) is the cotangent bundle projection.

5.1.

Every natural transformation \(B : T^*T^{(r)}M \to T^*T^{(r)}M\) over \(B\) induces a natural function \(\Theta(B) : B^! (TT^{(r)}M) \to \mathbb{R}\) by \(\Theta(B)(a, y) = (B(a), y)\), \((a, y) \in B^! (TT^{(r)}M)\). \((B(a)\) and \(y\) are over \(B(a)\) and therefore we can take the contraction.) Clearly \(\Theta(B)\) is fibre linear with respect to the bundle projection \(B^! (TT^{(r)}M) \to T^*T^{(r)}M\). On the other hand every natural function \(g : B^! (TT^{(r)}M) \to \mathbb{R}\) such that \(g\) is fibre linear with respect to the bundle projection \(B^! (TT^{(r)}M) \to T^*T^{(r)}M\) induces a natural transformation \(\Omega(g) : T^*T^{(r)}M \to T^*T^{(r)}M\) over \(B\) by \(\Omega(g)(a, y) = g(a, y)\), \(a \in T^*T^{(r)}M\), \(y \in T_{B(a)}T^{(r)}M\). It is easily seen that \(\Omega\) is inverse to \(\Theta\).

5.2.

The set of natural transformations \(T^*T^{(r)}M \to T^*T^{(r)}M\) over \(B\) is (in obvious way) a module over the algebra of natural functions \(T^*T^{(r)}M \to \mathbb{R}\). Similarly, the set of natural functions \(g : B^! (TT^{(r)}M) \to \mathbb{R}\) such that \(g\) is fibre linear with respect to the bundle projection \(B^! (TT^{(r)}M) \to T^*T^{(r)}M\) is (in obvious way) a module over the algebra of natural functions \(T^*T^{(r)}M \to \mathbb{R}\). (We identify any natural function \(T^*T^{(r)}M \to \mathbb{R}\) with the natural function \(B^! (TT^{(r)}M) \to \mathbb{R}\) by using the pull-back with respect to the obvious projection. Then the module operations are obvious.) Clearly, the (described in 5.1.) bijection \(\Omega\) is an isomorphism of the modules. Hence from Corollary 2 we deduce.

**Theorem 2.** The \(\Omega(\nu^{<s>})\) for \(s = 2, \ldots, r + 1\) and \(\Omega(\nu)\), where \(\nu\) and \(\nu^{<s>}\) are as in Examples 5 and 6, form the basis (over the algebra of all natural functions \(T^*T^{(r)}M \to \mathbb{R}\)) of the module of natural transformations \(T^*T^{(r)}M \to T^*T^{(r)}M\) over \(B\).
The natural transformations $T^*T^{(r)} \rightarrow T^*T^{(r)}$

References

[1] J. Gane


