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Heegaard splittings of the Brieskorn homology spheres that are equivalent after one stabilization

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Abstract. We construct a class of 3-manifolds M_q which are homeomorphic to the Brieskorn homology spheres $\sum(2,3,q)$, where $(2,3,q)$ are relatively prime. Also, we show that M_q is a 2-fold cyclic branched covering of S^3 over a knot K_q which is inequivalent with torus knot $T(3,q)$ for $q \geq 7$. Moreover, we show that two inequivalent Heegaard splittings of $\sum(2,3,q)$ of genus 2 associated with $T(3,q)$ and K_q are equivalent after single stabilization.

Keywords: Crystallization, Heegaard splitting,Crystallization move, Brieskorn homology sphere

MSC 2000 classification: primary 57M12, secondary 57M15, 57M50

1. Introduction

In [1] and [12], they independently showed that the Brieskorn homology sphere $\sum(2,3,7)$ admits two inequivalent Heegaard splittings of genus 2 associated with the torus knot $T(3,7)$ and K_7 (Figure 3.2, $q=7$), which are inequivalent knot types but yield the 2-fold cyclic branched covering spaces homeomorphic to $\sum(2,3,7)$.

Then we have the following natural question via Reidemeister and Singer's stable equivalence theorem of Heegaard splittings ([9] and [11]). What is the minimum number of stabilizations necessary to yield a stable equivalence of two inequivalent Heegaard splittings of a 3-manifold M which is a 2-fold cyclic branched covering associated with two different knots?

In this paper, we construct a class of 3-manifolds M_q which are homeomorphic to the Brieskorn homology sphere $\sum(2,3,q)$, where $(2,3,q)$ are relatively prime ([7]). Also, we show that M_q is a 2-fold cyclic branched covering of S^3 over a knot K_q which is inequivalent with torus knot $T(3,q)$ for $q \geq 7$. Moreover, we show that two inequivalent Heegaard splittings of $\sum(2,3,q)$ of genus 2

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associated with $T(3,q)$ and K_q are equivalent after single stabilization by using crystallizations of M_q.

Among various representations of 3-manifolds, M. Pezzana and his group introduced a method utilizing a 4-colored regular graph known as a crystallization. Since crystallizations are totally combinatorial, they have many advantages. In particular, they are quite convenient means to see interplay between links L and 2-fold cyclic branched coverings of S^3 over L if links L have bridge presentations ([4]).

In [3], Ferry and Gagliardi introduced polyhedral cut and glue moves or moves of type A of crystallizations representing 3-manifolds so that we can transform one crystallization to the other if both represent homeomorphic 3-manifolds, like Singer moves in Heegaard diagrams and Kirby moves in framed links.

Recently in [12], they introduced LCG moves ($\equiv$ linear cut and glue moves) of crystallizations, 4-colored 4-regular graphs representing 3-manifolds, from a point of view of them as extended Heegaard diagrams. In fact, LCG moves are particular kinds of moves of type A which transform crystallizations via replacements of 2-residues, i. e., meridians in extended Heegaard diagrams.

Also, they showed that LCG moves are considered as crystallization theoretical versions of geometric T-transformations in Heegaard diagrams of 3-manifolds.

We will approach the problem with Ferri's algorithm to construct 2-fold cyclic branched coverings of S^3 over links and LCG moves as tools.

2. Preliminaries

Throughout this paper, all spaces and maps are piecewise linear (PL) in the sense of [10]. Manifolds are always assumed to be closed, connected and orientable. For basic graph theory, we refer to [5].

An edge-coloration on graph $\Gamma = (V(\Gamma), E(\Gamma))$ is map $r : E(\Gamma) \to \Delta = \{0,1,2,3\}$ such that $r(e) \neq r(f)$ for each pair e, f of adjacent edges. The pair (Γ, r) is a 4-colored graph if r is regular of degree 4.

For each $\Delta' \subset \Delta$, we set $\Gamma_{\Delta'} = (V(\Gamma), r^{-1}(\Delta'))$; each connected component of $\Gamma_{\Delta'}$ is called a Δ'-residue. An k-residue is a Δ'-residue such that the cardinality of Δ' is k. For each $i \in \Delta$, we set $\hat{\imath} = \Delta - \{i\}$. (Γ, r) is said to be contracted if $\Gamma_{\hat{\imath}}$ is connected for each $i \in \Delta$. A contracted graph representing a closed 3-manifold M^3 is said to be a crystallization of M; every closed connected 3-manifold admits a crystallization ([6] and [8]).

Given a 4-colored graph (Γ, r), a subgraph θ of Γ formed by two vertices X, Y joined by h edges $(1 \leq h \leq 3)$ with colors $c_1, \ldots, c_h \in \Delta$ will be called a

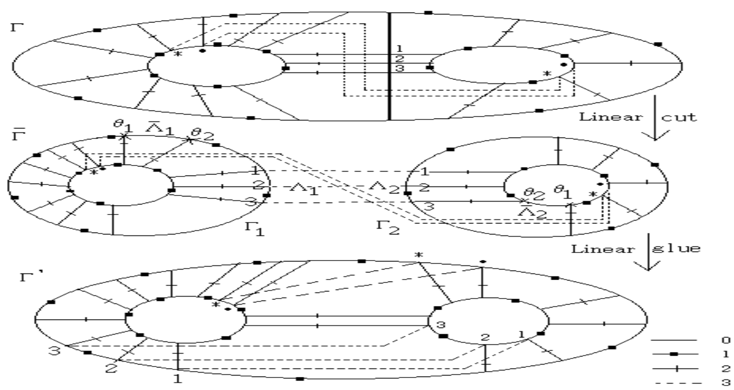

Figure 1. Linear (Polyhedral) cut and glue move

dipole of type h iff X and Y belong to distinct components of $\Gamma_{\Delta-\{c_1,\ldots,c_h\}}$. If $h=1$ or $h=3$, the dipole is said to be degenerate.

Cancelling a dipole θ of type h means (a) in $\Gamma_{\Delta-\{c_1,\ldots,c_h\}}$ replacing the components containing X and Y by their connected sum with respect to these vertices, (b) leaving the edges of colors $c_1,\ldots,c_h$ not adjacent with X and Y. Adding θ means the converse process.

Move I is defined as the addition or cancellation of a non-degenerate dipole.

Move II is defined as the addition or cancellation of a dipole of type 1.

Two crystallizations are said to be (I,II)-equivalent iff they can be joined by a finite sequence of moves I and/or II.

In [3], a generalization of move II, called move A or polyhedral cut and glue, is defined in which the degenerate dipole is substituted by a more complicated subgraph. See Fig. 1 for an example.

A handlebody H_g is a tubular neighbourhood of a graph, i. e., the topological product of a (small) disk with a graph. A Heegaard splitting of a closed connected 3-manifold M is a pair $(H_g,\bar{H}_g)$ of handlebodies of genus g if $M^3 = H_g \cup \bar{H}_g$ and $H_g \cap \bar{H}_g = \partial H_g = \partial \bar{H}_g$ is a closed orientable surface of genus g. The genus of the splitting is the genus of the surface ∂H_g.

Two Heegaard splittings $(H_g,\bar{H}_g)$ and $(H'_g,\bar{H}'_g)$ of a 3-manifold M are said to be equivalent if there is a (orientation preserving) homeomorphism $\phi : H_g \cup \bar{H}_g \to H'_g \cup \bar{H}'_g$ with $\phi(H_g) = H'_g$, $\phi(\bar{H}_g) = \bar{H}'_g$.

A stabilization of the Heegaard splitting $(H_g,\bar{H}_g)$ of M is the Heegaard splitting of M obtained by taking the connected sum $(H_g,\bar{H}_g)\,\#_n(T_1,\bar{T}_1)$, where $(T_1,\bar{T}_1)$ is the Heegaard splitting of S^3 of genus 1. It is well-known that every closed 3-manifold M has a Heegaard splitting and any two Heegaard splittings

$(H_g, \bar{H}_g)$ and $(H'_g, \bar{H}'_g)$ of a 3-manifold M are stably equivalent.

Now we introduce LCG moves and their relations with geometric T-transformations. An intermediary 3-gem $\bar{\Gamma}$ gets involved in a move from one crystallization Γ to the other Γ'. A passage from Γ (resp. $\bar{\Gamma}$) to $\bar{\Gamma}$ (resp. Γ') is said to be a cut (resp. glue). And we employ the term "linear" in our version of crystallization moves to point out that two subgraphs of the $\bar{\delta}$ 3-residues involved in gluing are linear trees, where δ is some fixed color in $\Delta = \{0, 1, 2, 3\}$.

In LCG moves, the intermediary 3-gem $\bar{\Gamma}$ is characterized by its 3-residues as follows. The 3-gem $\bar{\Gamma}$ has two $\bar{\delta}$ 3-residues Γ^1 and Γ^2 whereas for the other colors c, $\bar{\Gamma}$ has single $\bar{c}$ 3-residue. In order to describe a LCG move from Γ to Γ' through $\bar{\Gamma}$ we introduce one more concept of a 'linearly extended 1-dipole' determined by δ edges lying between two linear trees, which may be thought of as a kind of generalization of the standard 1-dipole of type δ. Thus Γ (resp. Γ') is obtained from $\bar{\Gamma}$ by eliminating a linearly extended 1-dipole between a pair (m^1, m^2) (resp. (n^1, n^2)) of $\alpha\beta$-2residues such that m^i (resp. n^i) belongs to Γ^1 (resp. Γ^2). See Figure 1 and 2.

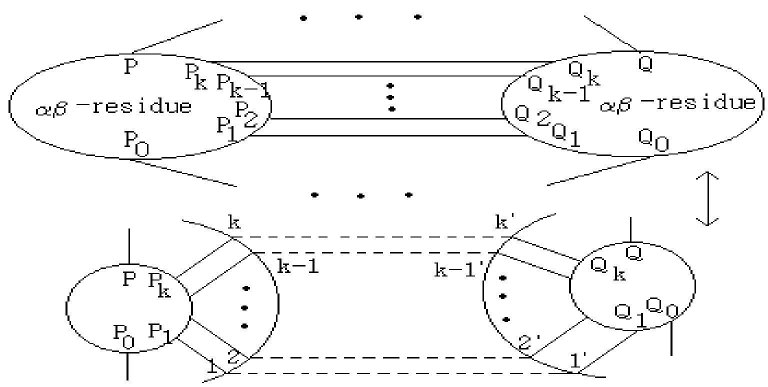

Figure 2. Inserting or eliminating linear extended 1-dipole

As in Figure 1, it is very convenient to understand LCG move using auxiliary lines l in Γ. A line l in 3-gem is said to be a $(i, j; k)$ if a line l cutting $\{i, j\}$-residue meets at least one k-colored edges, and the line in Figure 1 is a $(0, 1; 2)$. In this paper all auxiliary lines l in 4-colored graphs Γ are presented by thick lines.

Now we explain how LCG moves are related to geometric T transformations. It is known that Γ yields the natural Heegaard splitting $M^3(\Gamma) = H_{\alpha\beta} \cup_{h(\Gamma)} H_{\gamma\delta}$. Similarly Γ' yields a natural Heegaard splitting $M^3(\Gamma') = H'_{\alpha\beta} \cup_{h(\Gamma')} H'_{\gamma\delta}$. From the 3-residues of $\bar{\Gamma}$, we also have a Heegaard splitting $M^3(\bar{\Gamma}) = \bar{H}_{\alpha\beta} \cup_{h(\bar{\Gamma})} \bar{H}_{\gamma\delta}$

naturally associated with $\bar{\Gamma}$. The handlebody $\bar{H}_{\gamma\delta}$ is analogously constructed from the $\bar{\beta}$ and $\bar{\alpha}$ 3-residues of $\bar{\Gamma}$ and hence has a system of extended meridian disks corresponding to a set $\mathcal{M}_{\gamma\delta}$ of all $\gamma\delta$ 3-residues of $\bar{\Gamma}$ but for the handlebody $\bar{H}_{\alpha\beta}$, there are two choices of systems of extended meridian disks such that each choice yields an equivalence of two Heegaard splittings $M^3(\Gamma)$ with $M^3(\bar{\Gamma})$ and that of $M^3(\Gamma')$with $M^3(\bar{\Gamma})$ respectively. Hence we may view a LCG move from Γ to Γ' through $\bar{\Gamma}$ as an equivalence of two Heegaard splittings $M^3(\Gamma)$ and $M^3(\Gamma')$ which is induced by a geometric T-transformation replacing one system of meridian disks of $\bar{H}_{\alpha\beta}$ arising from $\bar{M}_{\alpha\beta}\backslash\{m^1, m^2\}$ by the other arising from $\bar{M}_{\alpha\beta}\backslash\{n^1, n^2\}$, where $\bar{M}_{\alpha\beta}$ is a set of all $\alpha\beta$ 2-residues of $\bar{\Gamma}$.

According to relations between the pairs (m^1, m^2) and (n^1, n^2), we may classify LCG moves and identify those inducing geometric T-transformations of elementary Nielsen types. For more details, see [12].

3. The construction of 3-manifolds M_q

In this section, we will construct a class of 3-manifolds M_q which are homeomorphic to the Brieskorn homology spheres $\sum(2,3,q)$, where $(2,3,q)$ are relatively prime. First, we show that LCG moves between crystallizations are equivalent to well-known crystallization moves.

Theorem 1. *Let M, M' be closed 3-manifolds and (Γ, r), (Γ', r') two crystallizations of them. Then the following statements are equivalent:*
(1) M is homeomorphic with M',
(2) (Γ, r) and (Γ', r') are equivalence by Move I and II,
(3) (Γ, r) and (Γ', r') are equivalence by Move A, and
(4) (Γ, r) and (Γ', r') are equivalence by LCG moves.

Proof. It is already known that the statements (1), (2), and (3) are equivalent ([3]). So we will show that (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) hold.
(4) $\Rightarrow$ (3) is clear since a LCG move is a special case of a Move A.
(2) $\Rightarrow$ (4): Move II can be regarded as a LCG move since a dipole of type 1 is a linearly extended 1-dipole.
Cancelling of a 2-dipole corresponds to a LCG move accomplished by (1) isolating a vertex of a 2-dipole by means of a linear cut and (2) doing a linear gluing. Figure 3.1 illustrates, as an example, how the move of Figure 3.1 can be obtained by such an operation. In Figure 3.1, small and capital letters are assumed to be joined by the remaining color 3 among 4 colors. A line l in Figure 3.1 is used to obtain a 3-gem $\bar{\Gamma}$ by a linear cut, where l is $(0, 1; 2)$. Then by linear glue along edges $\{Bx\}$ and $\{bX\}$ with the same color, we obtain the crystallization in Figure 3.1. Therefore, a Move I is a LCG move. $\boxed{QED}$

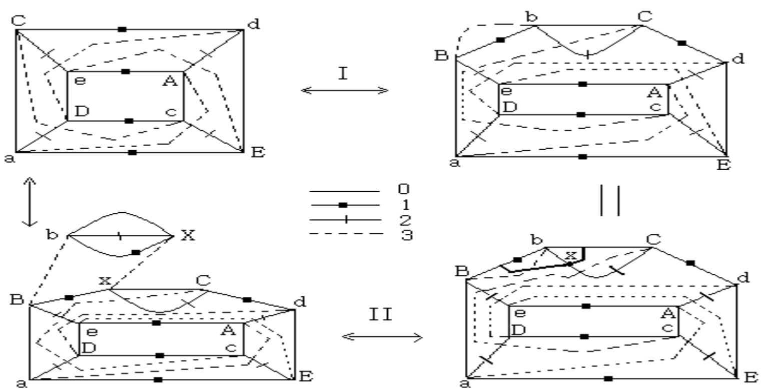

Figure 3. Move I, II and LCG Move

Throughout this paper, we assume that a crystallization move is one of the statements in Theorem 1.

Now we construct a class of 3-manifolds M_q which are consequently the same as the Brieskorn homology spheres $\sum(2,3,q)$ where $(2,3,q)$ are relatively prime, and we describe the method to obtain crystallizations of 3-manifolds M_q which are 2-fold cyclic branched coverings of S^3 over knots K_q.

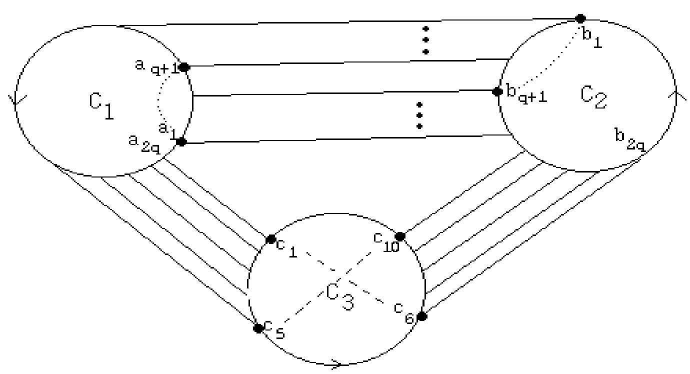

Figure 4. Crystallization of M_q

Let C_1, C_2, C_3 be circles on the plane having $2q$, $2q$, 10 vertices, respectively. Draw 5 parallel arcs joining each pair of vertices on C_1 and C_3, 5 parallel arcs joining each pair of vertices on C_2 and C_3, and $2q-5$ parallel arcs joining each pair of vertices on C_1 and C_2.

As in Figure 4, label the vertices of C_3 by $c_1, \ldots, c_{10}$, the vertices of C_1 (resp. C_2) by $a_1, \ldots, a_{2q}$ (resp. $b_1, \ldots, b_{2q}$) counter-clockwise so that a_{2q} (resp. b_{2q}) is joined with c_1 (resp. c_6).

Let $\mathcal{V}$ be the set of vertices and $\mathcal{C}$ be the set of edges in the interiors of C_i $(i = 1, 2, 3)$ joining vertices by the reflections with respect to the axes $\overline{a_1 a_{q+1}}$ and $\overline{b_1 b_{q+1}}$ on C_i $(i = 1, 2)$, and by the reflection with respect to the axis $\overline{c_5 c_{10}}$ (resp. $\overline{c_1 c_6}$) on C_3 if $q \equiv 1$(resp. $\equiv 2$)(mod 3).

Let $\mathcal{C}$ be the set of edges in the exteriors of C_i $(i = 1, 2, 3)$. Call α the involution on $\mathcal{V}$ which interchanges the end points of the edges of $\mathcal{C}$, leaving fixed the points of the axes; call β the involution on $\mathcal{V}$ which interchanges the end points of the edges of $\mathcal{C}$.

Label all edges on C_i $(i = 1, 2, 3)$ alternatively with colors 0, 1, starting arbitrary; label all edges of $\mathcal{D}$ with color 2; draw a further set $\mathcal{D}'$ of edges, each connecting a pair of points of $\mathcal{V}$ which correspond under the involution $\alpha\beta\alpha$ and label the elements of $\mathcal{D}'$ with color 3.

The graph Γ obtained by the above construction is a 4 colored regular graph. It follows from [2] that such a graph is the crystallization of a 3-manifold, denoted by M_q, which is a 2-fold cyclic branched covering space of S^3 over a knot K_q, where K_q is a knot with 3-bridge presentation so that $\overline{a_1 a_{q+1}}$, $\overline{b_1 b_{q+1}}$ and $\overline{c_5 c_{10}}$ (or $\overline{c_1 c_6}$) are overbridges, and the edges of $\mathcal{C} \cup \mathcal{D}$ are underbridges.

Theorem 2. *The manifold M_q is the 2-fold cyclic branched covering of S^3 over the knot K_q.*

Remark 1. For $q = 5$, the knot K_5 is exactly the torus knot $T(3, 5)$.

Now we show that the knot K_q and the torus knot $T(3, q)$ are inequivalent knot types so that their 2-fold cyclic branched covering of S^3 have inequivalent Heegaard splittings of genus 2.

Theorem 3. *For $q \geq 7$, the knot K_q and the torus knot $T(3, q)$ are inequivalent knot types.*

PROOF. It is well-known that the genus of the torus knot $T(3, q)$ is $q - 1$. Let F be a Seifert surface with boundary K_q. Then F has the genus $g(F) = (1 - d + b)/2$, where d is the number of Seifert circles and b is the number of crossings. From the construction of K_q, we obtain $b = 2(q - 1) + 4$. Consider the knot diagram of K_q on S^2 and the underbridge L with end points a_{q+1} and b_{q+1}. Then it is not hard to see that edges of L belong to 7 different Seifert circles. This implies $d = 7$. Therefore, $g(F) = q - 2$ and the genus of the knot K_q is at most $q - 2$. As a result, the knot K_q and the torus knot $T(3, q)$ are inequivalent knot types. $\boxed{QED}$

4. The stabilization of Heegaard splittings

In this section, we show that two inequivalent Heegaard splittings of genus 2 of M_q and the Brieskorn homology sphere $\sum(2,3,q)$ associated with $T(3,q)$ and K_q are equivalent after single stabilization.

Consider crystallizations of M_q and the Brieskorn homology sphere $\sum(2,3,q)$ by Ferri's algorithm to construct 2-fold cyclic branched coverings of S^3. Note that these crystallizations correspond to Heegaard splittings of genus 2 ([2]).

By the fact that all Heegaard splittings of a 3-manifold M are stably equivalent, we claim:

Theorem 4. *Two inequivalent Heegaard splittings of M_q and the Brieskorn homology sphere $\sum(2,3,q)$ are equivalent after single stabilization for $q \geq 7$.*

Proof. By Ferri's algorithm to construct crystallizations of 2-fold cyclic branched covering of S^3, we obtain crystallizations corresponding to Heegaard splittings of genus 2 of $\sum(2,3,q)$ and M_q (first figures in Figure 5 and 8). The second figure in Figure 5 (resp. Figure 8) is obtained by adding a dipole of type 2, that corresponds to a single stabilization of Heegaard splitting. Figure 7 (resp. Figure 11) is obtained from Figure 6 (resp. Figure 9 and Figure 10) by a finite sequence of polyhedral cut and glue moves to decrease the number of vertices of crystallization, so that we have isomorphic two crystallizations (second figures in Figure 7 and 11). As a consequence, we have equivalent two Heegaard splittings of genus 3. In the figures, the same numbers and letters are joined by 3-colored edges, and we execute polyhedral cut and glue moves along the thick lines.

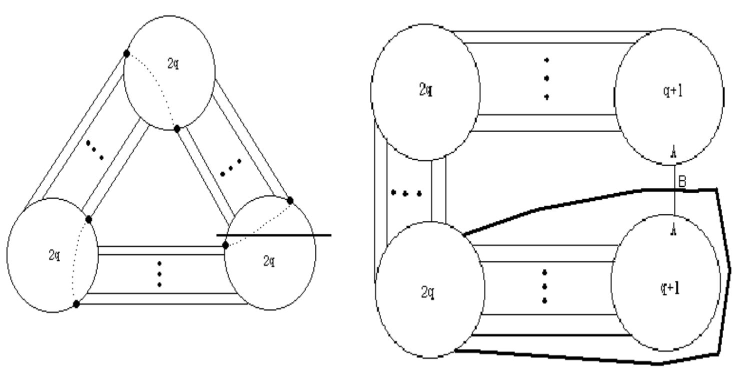

Figure 5.

$\boxed{QED}$

Corollary 1. *M_q and the Brieskorn homology sphere $\sum(2,3,q)$ are homeomorphic.*

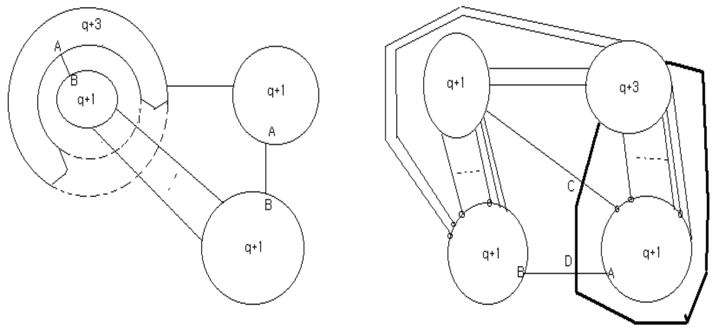

Figure 6.

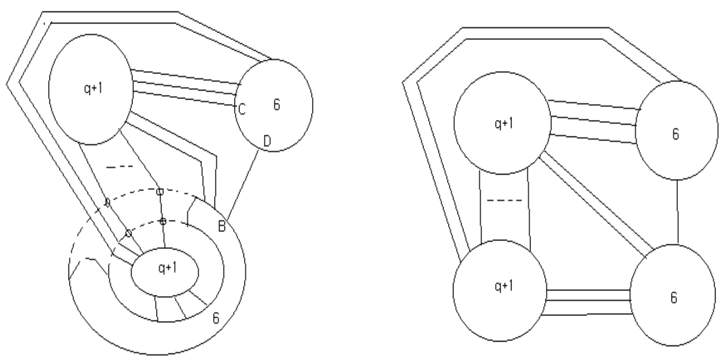

Figure 7.

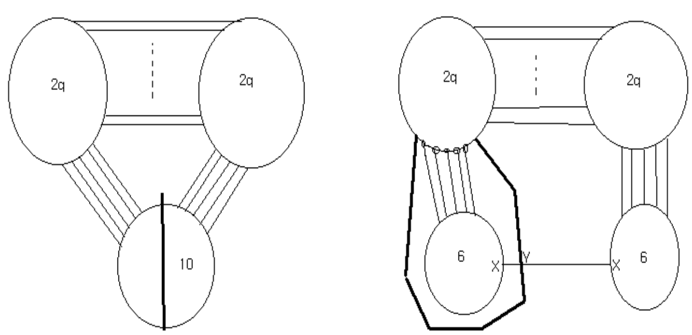

Figure 8.

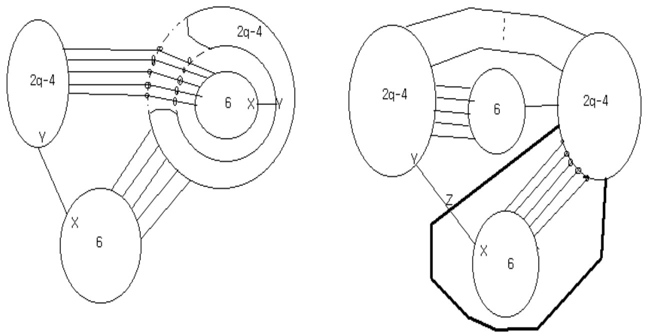

Figure 9.

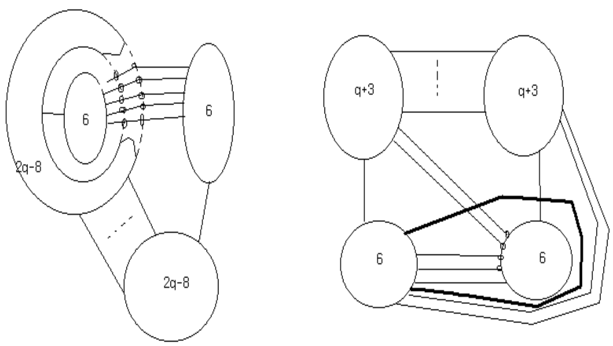

Figure 10.

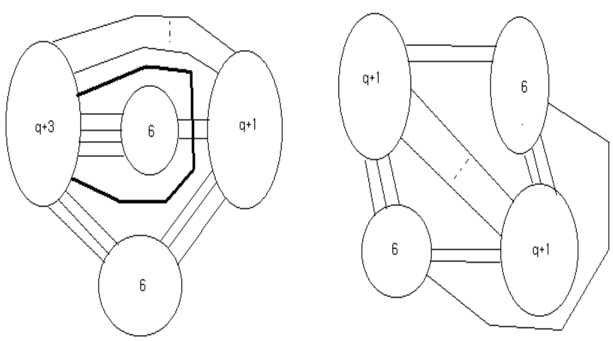

Figure 11.

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