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A criterion for a group to be nilpotent

Neil Flowers
Department of Mathematics, Youngstown State University, Youngstown Ohio, 44555 USA,
flowers@math.ysu.edu

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Abstract. Let G be a group with $|\pi(G)| \geq 3$. In this paper it is shown that G is nilpotent if and only if for every subgroup H of G with $|\pi(H)| \geq 2$ we have $P \cap H \in \mathrm{Syl}_p(H)$ for each $P \in \mathrm{Syl}_p(G)$ and for every $p \in \pi(G)$.

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In any intermediate text in finite group theory one is sure to find a theorem on the equivalence of the nilpotency of a group to a certain condition imposed on all of its subgroups. For instance, a finite group is nilpotent if and only if every subgroup is subnormal. Suppose G is a finite group and $H \leq G$. In [1], Peter Kleidman proved that $H \trianglelefteq\trianglelefteq G$ if and only if $P \cap H \in \mathrm{Syl}_p(H)$ for all primes p and $P \in \mathrm{Syl}_p(G)$. As an immediate consequence of [1], it follows that a finite group G is nilpotent if and only if every subgroup H of G, having order divisible by two or more primes, intersects every Sylow subgroup of G in a Sylow subgroup of H. The proof of the result in [1] uses the classification of finite simple groups.

Using only fundamental techniques, in this paper we show the latter equivalence of the nilpotency of a group to this strange condition imposed on all of its subgroups without the use of [1] or the classification of finite simple groups. In what follows all groups are assumed to be finite.

Theorem 1. *Let G be a group with $|\pi(G)| \geq 3$. Then G is nilpotent if and only if $P \cap H \in \mathrm{Syl}_p(H)$ for all $P \in \mathrm{Syl}_p(G)$ and all $H \leq G$ with $|\pi(H)| \geq 2$.*

Proof. First suppose G is nilpotent. Let $P \in \mathrm{Syl}_p(G)$ and $H \leq G$. Then $P \cap H$ is a p-group and so $P \cap H \leq P_1$ for some $P_1 \in \mathrm{Syl}_p(H)$. Since G is nilpotent, $P \trianglelefteq G$ and so $P_1 \leq P$. But then $P \cap H \leq P_1 \leq P \cap H$ and therefore $P \cap H = P_1$ is a Sylow p-subgroup of H.

Conversely, suppose all the subgroups of G, having orders divisible by at least two primes, satisfy the condition in the theorem. If G is not nilpotent, then there exists a nonnormal Sylow p-subgroup P of G, for some prime $p \in \pi(G)$. We claim that $P = N_G(P)$. If $N_G(P)$ is not a p-group, then there exists a p'-subgroup H of $N_G(P)$. Let $K = PH$. Then $K \leq G$ and, since $P \trianglelefteq K$, P is the unique Sylow p-subgroup of K. If $P_1 \in \mathrm{Syl}_p(G)$, then $P_1 \cap K \in \mathrm{Syl}_p(K)$

by assumption. Hence, $P_1 \cap K = P$ and so $P \leq P_1$. Now, since P_1 and P are both Sylow p-subgroups of G, we get $P = P_1$. Therefore, since P_1 was chosen arbitrarily, we have $P \trianglelefteq G$, a contradiction. Thus $N_G(P)$ is a p-group and, since $P \leq N_G(P)$, we have $P = N_G(P)$ as claimed.

Next we claim that $Q \cap R = O_p(G)$ for any distinct Sylow p-subgroups Q and R of G. To see this, let Q and R be Sylow p-subgroups of G with $|Q \cap R|$ maximal. If $N_G(Q \cap R)$ is not a p-group then, using the same argument in the second paragraph with P replaced by $Q \cap R$, we find that $Q \cap R$ lies in every Sylow p-subgroup of G and therefore $Q \cap R \leq O_p(G)$. Hence, $Q \cap R = O_p(G)$ and so, by the maximality of $|Q \cap R|$, the claim holds. On the other hand, if $N_G(Q \cap R)$ is a p-group, then there exists $S \in \mathrm{Syl}_p(G)$ such that $N_G(Q \cap R) \leq S$. Since $Q \cap R < Q$, we have $Q \cap R < N_Q(Q \cap R) \leq Q \cap S$ and so $Q \cap R < Q \cap S$. Similarly we get $Q \cap R < R \cap S$. Thus, by the maximality of $|Q \cap R|$ we get $Q = S = R$, a contradiction. Therefore the claim holds. Now the number of p-elements of G outside of $O_p(G)$ is

$$\begin{aligned} |G|/|N_G(P)|(|P| - |O_p(G)|) &= |G|/|P|(|P| - |O_p(G)|) \\ &= |G| - |G|(|O_p(G)|/|P|) \\ &\geq |G| - \frac{1}{2}|G| \\ &= \frac{1}{2}|G|. \end{aligned}$$

Hence G has at most one nonnormal Sylow subgroup and we conclude $G = PH$ where H is nilpotent. As $\pi(G) \geq 3$, let $Q \in \mathrm{Syl}_q(G)$ and $R \in \mathrm{Syl}_r(G)$ for distinct primes q and r different from p. Then $Q \trianglelefteq G$ and $R \trianglelefteq G$ and so PQ and PR are subgroups of G. Let $P_1 \in \mathrm{Syl}_p(PQ)$. Then $P_1 \in \mathrm{Syl}_p(G)$ and so $P_1 \cap PR \in \mathrm{Syl}_p(PR)$. But $P_1 \cap PR \leq PQ \cap PR = P$. Hence, since $P \in \mathrm{Syl}_p(PR)$, we get $P_1 \cap PR = P$ and consequently $P \leq P_1$. Now, since both P and P_1 are Sylow p-subgroups of PQ, we get $P = P_1$. Therefore P is the unique Sylow p-subgroup of PQ and so $P \trianglelefteq PQ$. But then $Q \leq N_G(P) = P$ and we get $Q = 1$, a contradiction. *QED*

There are groups, of order involving only two primes, all of whose subgroups satisfy the given property and yet fail to be nilpotent. For example, A_4 and D_{12}. However, if the subgroups of a nonnilpotent group G satisfy the given property and $|\pi(G)| = 2$ then, except for the last paragraph, all of the above arguments remain true. Thus for such a group G, the above argument shows $G = PQ$, where P is a p-group, Q is a q-group, $Q \trianglelefteq G$, $P = N_G(P)$, and $P \cap P^x = O_p(G)$ for all $x \in G \setminus P$.

References

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