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Saturated classes of bases

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Abstract. In this paper we consider classes consisting of pairs (B, X), where B is a base of cardinality $\leq \tau$ for the open subsets of a space X. Such classes are called classes of bases. For such a class $\mathbb{P}$ we define the notion of a universal element: an element (B^T, T) of $\mathbb{P}$ is said to be universal in $\mathbb{P}$ if for every $(B^X, X) \in \mathbb{P}$ there exists an embedding i_T^X of X into T such that $B^X = \{(i_T^X)^{-1}(U) : U \in B^T\}$. We define also the notion of a (weakly) saturated class of bases similar to that of a saturated class of spaces in [2] and a saturated class of subsets in [3]. For the (weakly) saturated classes of bases we prove the universality property (that is, in any such class there exist universal elements) and the intersection property (that is, the intersection of not more than τ many saturated classes of bases is also saturated). We give some relations between these classes and the classes of spaces and classes of subsets. Furthermore, we give a method of construction of saturated classes of bases by saturated classes of subsets.

Also, we consider classes consisting of triads (Q, B, X), where Q is a subset of a space X and B is a set of open subsets of X such that the set $\{Q \cap U : U \in B\}$ is a base for the open subsets of the subspace Q. Such classes are called classes of p-bases (positional bases). For such classes we also define the notion of a universal element and the notion of a saturated class of p-bases and prove the universality and the intersection properties. Some examples are given.

Keywords: Universal space, Containing space, Saturated class of spaces, Saturated class of subsets, Saturated class of bases, Saturated class of p-bases.

MSC 2000 classification: 54C25, 54E99.

Introduction

Agreement. All spaces considered in the paper are assumed to be T_0-spaces of weight less than or equal to a given infinite cardinal denoted by τ. The notions and notations introduced in the papers [2] and [3] are assumed to be known.

The universality problem is to determine whether there are universal elements in a given class of spaces. Problems concerning universal and containing spaces appeared in topology in its early development, when special classes of separable metrizable spaces were considered. At first, the construction of universal elements had more or less a heuristic nature. With the consideration of more general classes of spaces, some methods of construction of universal elements appeared. The methods using factorization theorems seem to be the most important.

The containing spaces constructed in [2] provide us with a new point of view on the universality problem. Suppose that we have a class $I\!P$ of T_0-spaces of weight $\leq \tau$. The basic idea is to find a universal element in $I\!P$ among the corresponding containing spaces, that is, for a suitable (indexed) collection $\mathbf{S}$ of elements of $I\!P$ to find a pair $(\mathbf{M}, \mathrm{R})$, where $\mathbf{M}$ is a co-mark of $\mathbf{S}$ and R is an $\mathbf{M}$-admissible family of equivalence relation on $\mathbf{S}$, such that the containing space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ to be an element of $I\!P$ (and, therefore, $\mathrm{T}(\mathbf{M}, \mathrm{R})$ will be a universal element in $I\!P$). In such a manner we can prove in a unified way the existence of a universal element for many (well-known) classes of spaces. (A generalization of the construction of containing spaces given in [2] is considered in [4]).

The above idea becomes more fruitful if from the class $I\!P$ of spaces we require the existence of "sufficiently many" pairs $(\mathbf{M}, \mathrm{R})$ for which $\mathrm{T}(\mathbf{M}, \mathrm{R}) \in I\!P$. This requirement can be formulated more precisely as follows: for every (indexed) collection $\mathbf{S}$ of elements of $I\!P$ there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ with the property that for every co-mark $\mathbf{M}$, which is a co-extension of $\mathbf{M}^+$, there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R, which is a final refinement of R^+, the containing space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ belongs to $I\!P$. Classes satisfying this condition are called saturated. Thus, in our method of construction of universal spaces, the notion of a saturated class of spaces is a very natural object of research. We note that many of the well-known classes of spaces in which there are universal elements are saturated. (However, in [2] this is proved only for the class of all T_0-spaces, for the class of all regular T_0-spaces, and for the class of all completely regular T_0-spaces).

Saturated classes of spaces have of course the universality property, that is, in any such class there exist universal elements. They have also the intersection property, that is, the intersection of not more than τ many saturated classes is also a saturated class. (We note that in general the classes of spaces in which there exist universal elements do not have this property). So, any "new" saturated class "multiply" the number of known classes of spaces in which there are universal elements. Saturated classes are also convenient to use for (inductive) construction of other saturated classes of spaces. (In [5] saturated classes of spaces are used for the construction of "saturated classes" of mappings and, therefore, for the construction of classes of mappings in which there are universal elements).

In [3] the further investigation of the method considered in [2] is given. Suppose that for every element X of an indexed collection $\mathbf{S}$ of spaces a subset Q^X of X is given. Then, the indexed collection $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ of spaces can be considered. For any co-mark $\mathbf{M}$ of $\mathbf{S}$ the trace of $\mathbf{M}$ on $\mathbf{Q}$, which is a co-mark of $\mathbf{Q}$ denoted by $\mathbf{M}|_{\mathbf{Q}}$, is defined by a natural way. Also, for any family R of equivalence relation on $\mathbf{S}$ the trace of R on $\mathbf{Q}$, which is a family of equivalence relations on $\mathbf{Q}$ denoted by $\mathrm{R}|_{\mathbf{Q}}$, is defined. In the case, where $\mathrm{R}|_{\mathbf{Q}}$ is an $\mathbf{M}|_{\mathbf{Q}}$-admissible, that is, R is $(\mathbf{M}, \mathbf{Q})$-admissible, we can consider the containing space $\mathrm{T}|_{\mathbf{Q}} \equiv \mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$. In [3] it is proved that this space can be considered by a natural way as a (specific) subset of the space $\mathrm{T} \equiv \mathrm{T}(\mathbf{M}, \mathrm{R})$.

The specific subsets of the containing spaces can be used in order to verify that a given class of spaces is saturated (and, therefore, in order to verify that in this class there is a universal space). This can be done as follows. Suppose that the elements of a class $I\!P$ of spaces is defined by a property P, that is, a space X belongs to $I\!P$ if and only if X satisfies property P. For the formulation of this property we use some subsets of a space, whose "composition" can be considered as a "structure" on this space. If for "sufficiently many" pairs $(\mathbf{M}, \mathrm{R})$, where $\mathbf{M}$ is a co-mark of an indexed collection $\mathbf{S}$ of elements of $I\!P$ and R is an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$, the "composition" of the corresponding specific subsets defines the same "structure" on the containing spaces $\mathrm{T}(\mathbf{M}, \mathrm{R})$, then these containing spaces belong to $I\!P$, which means that the class $I\!P$ is saturated.

The universal spaces in $I\!P$ constructed in such a manner (that is, by transfered the given "structure" on elements of $I\!P$ to the corresponding containing spaces) have some additional property: the embeddings of elements of $I\!P$ into these universal spaces "preserve" the "structure". Thus, we obtain new type of universal elements. The consideration of such universal elements suggest an idea to us to consider classes of spaces with a given "structure". For example, in parallels to the class of all separable metrizable countable-dimensional spaces (in which there exist universal elements) we can consider the class $I\!P$ of all pairs $(\{Q_i : i \in \omega\}, X)$, where $\{Q_i : i \in \omega\}$ is a countable set of zero-dimensional subsets of a separable metrizable space X (the "structure" on X), whose union is X. We can define the notion of a universal element in this class preserving the "structure" as follows: an element $(\{Q_i^T : i \in \omega\}, T)$ of $I\!P$ is said to be universal (respectively, properly universal) in $I\!P$ if for every $(\{Q_i^X : i \in \omega\}, X) \in I\!P$ there exists an embedding i_T^X of X into T such that $i_T^X(Q_i^X) \subset Q_i^T$ (respectively, $(i_T^X)^{-1}(Q_i^T) = Q_i^X$), $i \in \omega$. Of course, such universal elements are "more strong" than the usual universal spaces in the class of all countable-dimensional spaces. (Although in the present paper we do not consider such kind of classes we note that the class $I\!P$ is "saturated" and, therefore, in this class there exist universal elements preserving the "structure").

The simplest "structure" is that, which is defined by a subset of a space. In [3] classes consisting of pairs (Q, X), where Q is a subset of a space X, is considered. Such a class is called a class of subsets. The investigation of such classes actually means the investigation of relative properties of subsets of spaces. For a class of subsets a new notion of a universal element and a notion of a saturated class of subsets are defined and the universality and intersection properties are proved. It is also proved that the classes consisting of all pairs (Q, X) such that: (a) Q is a closed subset of X, (b) Q is an open subset of X, and (c) Q is a nowhere dense subset of X, are saturated classes of spaces. (In [3] it is announced also the following result: the class of all pairs (Q, X), where Q is a Borel set of the multiplicative (additive) class α in a separable metrizable space X, is a saturated class of subsets. It is interesting to compare the existence of universal elements in this class with the result of [6] (see also [1]).

An other simple "structure" on a space is the given base for the open subsets of the space. In Section 1 of the present paper we consider classes consisting of pairs (B, X), where B is a base for the open subsets of a space X. Such classes are called classes of bases. For the classes of bases, as for the classes of spaces in [2] and for the classes of subsets in [3], we define a new notion of a universal element: an element (B^T, T) of a class $I\!\!P$ of bases is called universal if for every $(B^X, X) \in I\!\!P$ there exists an embedding i_T^X of X into T such that $B^X = \{(i_T^X)^{-1}(U) : U \in B^T\}$. Also we define the notion of a (weakly) saturated class of bases and prove the universality and intersection properties for such classes. We give some relations between the (weakly) saturated classes of bases and the saturated classes of spaces and subsets. In particular, we prove that the class of bases consisting of all pairs (B, X) such that the pair (U, X) belongs to a given saturated class of subsets for any element U of B, is a saturated class of bases.

In Section 2 we consider the so-called classes of p-bases consisting of triads (Q, B, X), where Q is a subset of a space X and B is a set of open subsets of X such that the set $\{Q \cap U : U \in B\}$ is a base for the open subsets of the subspace Q. (Such a set B is called p-base (positional base) for Q in X). For such classes we define the notion of a universal element and the notion of a saturated class of p-base and give some results, which are similar to that of the classes of spaces. Some examples of the saturated classes of p-bases are given.

Although, the dimensional-like functions are not investigated in the present paper we note that the saturated classes of bases and saturated classes of p-bases are convenient to use for the construction of different such functions df having the "saturation property", that is, for any ordinal $\alpha \in \tau^+$ or integer $n \in \omega$ the class of all spaces X such that $df(X) \leq \alpha$ or $df(X) \leq n$ is a saturated class of spaces. Moreover, it is possible to construct dimensional-like functions having the "saturation property" with the domain not only the class of spaces but, for example, the class of all subsets, the class of all bases or the class of all p-bases.

Below, we define some specific notions and notations, which are used in the paper. This definition is slightly more general than the corresponding definition of [2].

1 Definition. Let $\mathbf{S}$ be an indexed collection of spaces. An $\mathbf{S}$-*indexed set of families* is an indexed set $\mathbf{F} \equiv \{F^X : X \in \mathbf{S}\}$, where F^X is a family of subsets of X. The family F^X is denoted also by $\mathbf{F}(X)$.

An $\mathbf{S}$-indexed set $\mathbf{F} \equiv \{F^X : X \in \mathbf{S}\}$ of families is said to be a *co-base for* $\mathbf{S}$ if F^X is a base for X of cardinality less than or equal to τ containing the empty set. (We underline the fact that the cardinality of F^X is $\leq \tau$).

An $\mathbf{S}$-*indexed set of* τ-*indexed families* is an indexed set

$$\mathbf{N} \equiv \{N^X : X \in \mathbf{S}\},$$

where N^X is a τ-indexed family $\{N^X_\delta : \delta \in \tau\}$ of subsets of X. The indexed family N^X is denoted also by $\mathbf{N}(X)$.

An $\mathbf{S}$-indexed set $\mathbf{N} \equiv \{N^X : X \in \mathbf{S}\}$ of τ-indexed families is said to be a *co-indication* of an $\mathbf{S}$-indexed set $\mathbf{F} \equiv \{F^X : X \in \mathbf{S}\}$ of families if N^X is an indication of F^X for every $X \in \mathbf{S}$. (We underline the fact that the indexing set of N^X is the set τ). The above notion is frequently used in the case, where $\mathbf{F}$ is a co-base for $\mathbf{S}$. In this case $\mathbf{N}$ is said to be a *co-mark* of $\mathbf{S}$.

Let $\mathbf{N}^0 \equiv \{\{N^{0,X}_\delta : \delta \in \tau\} : X \in \mathbf{S}\}$ and $\mathbf{N}^1 \equiv \{\{N^{1,X}_\delta : \delta \in \tau\} : X \in \mathbf{S}\}$ be two $\mathbf{S}$-indexed sets of τ-indexed families. It is said that $\mathbf{N}^1$ is a *co-extension* of $\mathbf{N}^0$ if there exists an one-to-one mapping θ of τ into itself such that $N^{0,X}_\delta = N^{1,X}_{\theta(\delta)}$ for every $X \in \mathbf{S}$. The mapping θ is called an *indicial mapping* of this co-extension or an *indicial mapping from* $\mathbf{N}^0$ *to* $\mathbf{N}^1$. (We note that in general θ is not unique).

Suppose that for every $X \in \mathbf{S}$ a subset Q^X of X is given. Then, the $\mathbf{S}$-indexed set $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ is called a *restriction* of $\mathbf{S}$. The set Q^X is denoted also by $\mathbf{Q}(X)$.

2 Remark. According to our agreement all notions and notations of the papers [2] and [3] are assumed to be known. Below, we indicate some of these specific notion and notations, which are used in the present paper:

An *admissible* family $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ of equivalence relations on $\mathbf{S}$ (see [2]).

The *set* $\mathrm{C}^\diamondsuit(\mathrm{R})$ and its *elements* $\mathbf{L}$ (see [2]).

An $\mathbf{M}$-*admissible* family R of equivalence relations on $\mathbf{S}$ (see [2]).

A family R is a *final refinement* of a family R^+ (see [2]).

The *space* T and an *element* or *point* $\mathbf{a}$ of T (see [2]).

The *natural embedding* of $X \in \mathbf{S}$ into T (see [2]).

The *subset* $\mathrm{T}(\mathbf{L})$ of T (see [2]).

The *subset* $\mathrm{T}(\mathbf{L}|_{\mathbf{Q}})$ of T (see [3]).

The $\theta(\tau)$-*standard* base $\mathrm{B}^{\mathbf{L}}_{\theta(\tau)}$ for $\mathrm{T}(\mathbf{L})$ and its *elements* $U^{\mathbf{L}}_{\delta}(\mathbf{L})$ (see [2]).
A $I\!\!K$-*restriction* $\mathbf{Q}$ of $\mathbf{S}$ (see [3]).
An $(\mathbf{M}, \mathbf{Q})$-*admissible* family R of equivalence relations on $\mathbf{S}$ (see [3]).
A *complete* restriction $\mathbf{Q}$ of $\mathbf{S}$ (see [3]).
An $(\mathbf{M}, \mathrm{R})$-*complete* restriction $\mathbf{Q}$ of $\mathbf{S}$ (see [3]).
The *operators* **Cl**, **Bd**, and **Int** (see [3]).
A *saturated* class of spaces and a *saturated class of subsets* (see [2] and [3]).
A *complete* saturated class of subsets (see [3]).
A *marked* space and a *mark* of a marked space (see [2]).
The *minimal algebra* of X containing the given set of subset of X (see [2]).
The *s-algebra* A^X_s of X, $s \in \mathcal{F}$ (see [2]).
The *natural isomorphism* i of A^X_s onto A^Y_s (see [2]).
The *mapping* d^X_s (see [2]).

1 Saturated classes of bases

3 Definition. The class consisting of all pairs (B, X), where B is a base of cardinality $\leq \tau$ for the open subsets of a space X containing the empty set, is called the *class of all bases* and is denoted by $\mathbb{P}(\text{base})$. Any subclass of $\mathbb{P}(\text{base})$ is called a *class of bases*. Such a class $I\!\!P$ is said to be *topological* if for every homeomorphism h of a space X onto a space Y the condition $(B, X) \in I\!\!P$ implies that $(\{h(V) : V \in B\}, Y) \in I\!\!P$. In what follows, an arbitrary considered class of bases is assumed to be topological.

4 Definition. Let $I\!\!P$ be a class of bases. We say that a base B for a space X is a $I\!\!P$-*base* if $(B, X) \in I\!\!P$.

A co-base $\mathbf{B}$ for an indexed collection $\mathbf{S}$ of spaces is said to be a $I\!\!P$-*co-base* if for every $X \in \mathbf{S}$, $\mathbf{B}(X)$ is a $I\!\!P$-base for X.

5 Definition. A non-empty class $I\!\!P$ of bases is said to be *saturated* (respectively, *weakly saturated*) if for every indexed collection $\mathbf{S}$ of spaces, a $I\!\!P$-co-base $\mathbf{B}$ for $\mathbf{S}$, and for every (respectively, for some) co-indication $\mathbf{N}$ of $\mathbf{B}$ there exists a co-extension $\mathbf{M}^+$ of $\mathbf{N}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every element $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$, we have $(\mathrm{B}^{\mathbf{L}}_{\theta(\tau)}, \mathrm{T}(\mathbf{L})) \in I\!\!P$, where θ is an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

The considered co-mark $\mathbf{M}^+$ is called an *initial co-mark* (*corresponding to the co-indication* $\mathbf{N}$ *of* $\mathbf{B}$ *and the class* $I\!\!P$), and the family R^+ is called an *initial* *family* (*corresponding to the co-mark* $\mathbf{M}$*, the co-indication* $\mathbf{N}$ *of* $\mathbf{B}$*, and the class* $I\!P$).

Obviously, the class of all bases is saturated and any saturated class of bases is weakly saturated.

The following lemma shows that the base $\mathrm{B}^{\mathbf{L}}_{\theta(\tau)}$ of the definition of a (weakly) saturated class of bases is independent of the choice of the indicial mapping θ. This lemma is easily proved.

6 Lemma. *Let* $\mathbf{S}$ *be an indexed collection of spaces,* $\mathbf{M}$ *a co-mark of* $\mathbf{S}$*, and* R *an* $\mathbf{M}$*-admissible family of equivalence relations on* $\mathbf{S}$*. Suppose that* $\mathbf{M}$ *is a co-extension of a co-indexed* $\mathbf{S}$*-family* $\mathbf{N}$*. If* θ_0 *and* θ_1 *are two indicial mappings from* $\mathbf{N}$ *to* $\mathbf{M}$*, then for every* $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$*,* $\mathrm{B}^{\mathbf{L}}_{\theta_0(\tau)}\mathrm{B}^{\mathbf{L}}_{\theta_1(\tau)}$.

The following proposition is the intersection property of the saturated classes of basses.

7 Proposition. *The non-empty intersection of not more than* τ *many saturated classes of bases is also a saturated class of bases.*

Proof. Suppose that for every $\delta \in \tau$, $I\!P_\delta$ is a saturated class of bases and let

$$I\!P = \cap\{I\!P_\delta : \delta \in \tau\}.$$

Let $\mathbf{S}$ be an indexed collection of spaces and $\mathbf{B}$ a $I\!P$-co-base for $\mathbf{S}$. It is clear that $\mathbf{B}$ is a $I\!P_\delta$-co-base for every $\delta \in \tau$. Consider an arbitrary co-indication $\mathbf{N}$ of $\mathbf{B}$.

For every $\delta \in \tau$ denote by $\mathbf{M}^+_\delta$ an initial co-mark of $\mathbf{S}$ corresponding to the co-indication $\mathbf{N}$ of $\mathbf{B}$ and the class $I\!P_\delta$, and let $\mathbf{M}^+$ be a co-extension of all $\mathbf{M}_\delta$ (see Lemma 3.2 of [2]). Consider an arbitrary co-extension

$$\mathbf{M} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\}$$

of $\mathbf{M}^+$. Then, $\mathbf{M}$ is also a co-extension of $\mathbf{N}$. We denote by θ an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

Let R^+_δ be an initial family of equivalence relations on $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $I\!P_\delta$. Let also R^+ be a family of equivalence relations on $\mathbf{S}$, which is a final refinement of all R^+_δ (see Lemma 3.1 of [2]). Consider an arbitrary admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and let $\mathbf{L}$ be an element of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$.

Since $I\!P_\delta$, $\delta \in \tau$, is saturated, the base $\mathrm{B}^{\mathbf{L}}_{\theta(\tau)}$ for the space $\mathrm{T}(\mathbf{L})$ is a $I\!P_\delta$-base. By the definition of $I\!P$, $\mathrm{B}^{\mathbf{L}}_{\theta(\tau)}$ is also a $I\!P$-base. Hence, $\mathbf{M}^+$ is an initial co-mark corresponding to the co-indication $\mathbf{N}$ of $\mathbf{B}$ and the class $I\!P$, and R^+ is an initial family corresponding to the co-mark $\mathbf{M}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $I\!P$. Thus, $I\!P$ is a saturated class of bases. $\boxed{QED}$

Similarly we can prove the following proposition.

8 Proposition. *The non-empty intersection of a saturated class of bases and a weakly saturated class of bases is a weakly saturated class of bases.*

9 Definition. An element (B^T, T) of a class $I\!P$ of bases is said to be *universal in* $I\!P$ if for every element (B^X, X) of $I\!P$ there exists an embedding h of X into T such that $B^X = \{h^{-1}(V) : V \in B^T\}$.

The following proposition is proved similarly to the Proposition 3.4 of [2].

10 Proposition. *In any weakly saturated class of bases there exist universal elements.*

11 Definition. Let $I\!P$ be a class of bases. The class of all spaces X such that $(B, X) \in I\!P$ for some base B for X is said to be the *space-component* of $I\!P$.

The following two propositions are easily proved. They give some relations between the (weakly) saturated classes and saturated classes of spaces.

12 Proposition. *The space-component of any weakly saturated class of bases is a saturated class of spaces.*

13 Proposition. *Let $I\!P$ be a (weakly) saturated class of bases and $I\!E$ a saturated class of spaces. Then, the class*

$$\{(B, X) \in I\!P : X \in I\!E\}$$

is a (weakly) saturated class of bases.

In the next proposition we give a method of construction of saturated classes of bases by a given saturated class of subsets.

14 Proposition. *Let $I\!K$ be a saturated class of subsets. Then, the class $I\!P$ of bases consisting of all pairs $(B, X) \in \mathbb{P}(\text{base})$ such that $(U, X) \in I\!K$ for every element U of B, is a saturated class.*

Proof. Let $\mathbf{S}$ be an indexed collection of spaces, $\mathbf{B}$ a $I\!P$-co-base for $\mathbf{S}$, and let

$$\mathbf{N} \equiv \{\{V_\eta^X : \eta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-indication of $\mathbf{B}$.

For every $\eta \in \tau$ we consider the restriction

$$\mathbf{V}_\eta \equiv \{V_\eta^X : X \in \mathbf{S}\}$$

of $\mathbf{S}$. Since $\mathbf{B}$ is a $I\!P$-co-base, this restriction is a $I\!K$-restriction of $\mathbf{S}$. Since $I\!K$ is a saturated class there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ such that for every $\eta \in \tau$,

$\mathbf{M}^+$ is an initial co-mark corresponding to the $I\!\!K$-restriction $\mathbf{V}_\eta$. We can also suppose that $\mathbf{M}^+$ is a co-extension of $\mathbf{N}$. Let

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-extension of $\mathbf{M}^+$. Then, $\mathbf{M}$ is also a co-extension of $\mathbf{N}$. Denote by θ an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

Let R^+ be a family of equivalence relations on $\mathbf{S}$ such that for every $\eta \in \tau$, R^+ is an initial family corresponding to the co-mark $\mathbf{M}$ and the $I\!\!K$-restriction $\mathbf{V}_\eta$. Let R be an arbitrary admissible family of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+. We prove that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the co-indication $\mathbf{N}$ of $\mathbf{B}$ and the class $I\!\!P$, and R^+ is an initial family corresponding to the co-mark $\mathbf{M}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $I\!\!P$. For this purpose, we denote by $\mathbf{L}$ an arbitrary element of $\mathrm{C}^\diamondsuit(\mathrm{R})$ and consider the base $\mathrm{B}^{\mathbf{L}}_{\theta(\tau)}$ for $\mathrm{T}(\mathbf{L})$. Any element $U_\delta^{\mathrm{T}}(\mathbf{H})$ of this base coincides with the subset $\mathrm{T}(\mathbf{H}|_{\mathbf{V}_\eta})$ of $\mathrm{T}(\mathbf{L})$, where $\eta = \theta^{-1}(\delta)$. By construction, $(\mathrm{T}(\mathbf{H}|_{\mathbf{V}_\eta}), \mathrm{T}(\mathbf{L})) \in I\!\!K$. Therefore, $(\mathrm{B}^{\mathbf{L}}_{\theta(\tau)}, \mathrm{T}(\mathbf{L})) \in I\!\!P$. Hence, $\mathbf{M}^+$ is an initial co-mark and R^+ is an initial family. Thus, $I\!\!P$ is a saturated class of bases. $\boxed{QED}$

The given below Proposition based on Corollaries 4.5–4.7 of [3] and the following unpublished result of the author:

The class $\mathrm{I\!P}(\mathrm{F}_\sigma)$ (respectively, $\mathrm{I\!P}(\mathrm{G}_\delta)$) of subsets consisting of all pairs (Q, X) such that Q is an F_σ-subset of X (respectively, a G_δ-subset of X) is a complete saturated class of subsets. (This result is a consequence of a more general result concerning the Borel type sets).

15 Proposition. *The non-empty class consisting of all pairs (B, X) of* $\mathrm{I\!P}$ (base) *satisfying one of the following conditions:*

(1) $(U, X) \in \mathrm{I\!P}(\mathrm{F}_\sigma)$ *for every* $U \in B$,

(2) $(U, X) \in \mathbf{Cl}^{-1}(\mathrm{I\!P}(\mathrm{G}_\delta))$ *for every* $U \in B$,

(3) $(U, X) \in \mathbf{Bd}^{-1}(\mathrm{I\!P}(\mathrm{G}_\delta))$ *for every* $U \in B$,

(4) $(U, X) \in \mathbf{Int}(\mathbf{Cl}(\mathrm{I\!P}(\mathrm{Op})))$ *for every* $U \in B$,

is a saturated class of bases. Therefore, in such a class there exist universal elements.

16 Corollary. *The class of all spaces having a base B satisfying one of the given below conditions is saturated and, therefore, in this class there exists a universal space:*

(1) *The elements of B are* F_σ*-sets,*

(2) *The boundaries of elements of B are* G_δ*-sets,*

(3) *The closures of elements of B are* G_δ*-sets,*

(4) *The elements of B are regular open. (An open subset U of a space X is said to be* regular open *if* $U = \mathrm{Int}_X(\mathrm{Cl}(U))$*).*

Proof. We prove only the second case. The other cases are proved similarly. Let $\mathbb{E}$ be the class of all spaces having a base B satisfying condition (2) of the corollary. Denote by $\mathbb{P}$ the class of subsets consisting of all pairs $(B, X) \in \mathbb{P}(\text{base})$ such that the closure of each element of B is a G_δ-set. By Proposition 15, $\mathbb{P}$ is a saturated class. Obviously, the space-component of $\mathbb{P}$ coincides with $\mathbb{E}$. By Proposition 12 the class $\mathbb{E}$ is saturated. $\boxed{QED}$

2 Saturated classes of p-bases

17 Definition. Let Q be a subset of a space X. A set B of open subsets of a space X containing the empty set is said to be *p-base* if the set $\{Q \cap U : U \in B\}$ is a base for the open subsets of the subspace Q.

A p-base B for Q in X is said to be *pos-base* if for every $x \in Q$ and an open neighbourhood U of x in X there exists an element V of B such that $x \in V \subset U$.

A p-base B for Q in X is said to be *ps-base* if B is base for the open subsets of X.

18 Definition. The class consisting of all triads (Q, B, X), where Q is a subset of a space X and B is a p-base for Q in X of cardinality $\leq \tau$, is called the *class of all p-bases* and is denoted by $\mathbb{P}(\text{p-base})$. Any subclass $\mathbb{P}$ of $\mathbb{P}(\text{p-base})$ is called a *class of p-bases*. Such a class $\mathbb{P}$ is said to be *topological* if for every homeomorphism h of a space X onto a space Y the condition $(Q, B, X) \in \mathbb{P}$ implies that $(h(Q), \{h(V) : V \in B\}, Y) \in \mathbb{P}$.

In what follows an arbitrary considered class of p-bases is assumed to be topological.

19 Definition. The class of all elements (Q, B, X) of $\mathbb{P}(\text{p-base})$, where B is a pos-base for Q in X (respectively, a base for X), is called the *class of all pos-bases* (respectively, the *class of all ps-bases*) and is denoted by $\mathbb{P}(\text{pos-base})$ (respectively, by $\mathbb{P}(\text{ps-base})$).

20 Definition. Let $\mathbf{S}$ is an indexed collection of spaces and $\mathbf{Q}$ a restriction of $\mathbf{S}$. The $\mathbf{S}$-indexed family $\mathbf{B} \equiv \{\mathbf{B}(X) : X \in \mathbf{S}\}$ is said to be a *co-p-base* for $\mathbf{Q}$ in $\mathbf{S}$ if $\mathbf{B}(X)$ is a p-base for $\mathbf{Q}(X)$ in X.

21 Definition. Let $\mathbb{P}$ be a class of p-bases. We say that a p-base B for a subset Q in a space X is a *$\mathbb{P}$-p-base* if $(Q, B, X) \in \mathbb{P}$.

A co-p-base $\mathbf{B}$ for a restriction $\mathbf{Q}$ of an indexed collection $\mathbf{S}$ of spaces is called a *$\mathbb{P}$-co-p-base* if for every $X \in \mathbf{S}$, $\mathbf{B}(X)$ is a $\mathbb{P}$-p-base for $\mathbf{Q}(X)$ in X.

22 Definition. A non-empty class $\mathbb{P}$ of p-bases is said to be *saturated* (respectively, *weakly saturated*) if for every indexed collection $\mathbf{S}$ of spaces, a restriction $\mathbf{Q}$ of $\mathbf{S}$, a $\mathbb{P}$-co-p-base $\mathbf{B}$ for $\mathbf{Q}$ in $\mathbf{S}$, and for every (respectively, for some) co-indication $\mathbf{N}$ of $\mathbf{B}$ there exists a co-mark $\mathbf{M}^+$, which is a co-extension of $\mathbf{N}$, satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $(\mathbf{M},\mathbf{Q})$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every elements $\mathbf{L}$, $\mathbf{H}$, and $\mathbf{E}$ of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$ for which $\mathbf{E} \subset \mathbf{H} \subset \mathbf{L}$, we have $(\mathrm{T}(\mathbf{E}|_{\mathbf{Q}}), \mathrm{B}^{\mathbf{H}}_{\theta(\tau)}, \mathrm{T}(\mathbf{L})) \in \mathbb{P}$, where θ is an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

The considered co-mark $\mathbf{M}^+$ is called an *initial co-mark* (*corresponding to the restriction* $\mathbf{Q}$*, the co-indication* $\mathbf{N}$ *of* $\mathbf{B}$*, and the class* $\mathbb{P}$) and the family R^+ is called an *initial family* (*corresponding to the co-mark* $\mathbf{M}$*, the restriction* $\mathbf{Q}$*, the co-indication* $\mathbf{N}$ *of* $\mathbf{B}$*, and the class* $\mathbb{P}$).

It is easy to see that the classes $\mathbb{P}$(p-base), $\mathbb{P}$(pos-base), and $\mathbb{P}$(ps-base) are saturated and that any saturated class of p-bases is also weakly saturated.

The following proposition, which is the intersection property of the (weakly) saturated classes of p-bases, is proved similarly to Proposition 7.

23 Proposition. *The following statements are true:*

(1) *The non-empty intersection of not more than* τ *many saturated classes of p-bases is also a saturated class of p-bases.*

(2) *The non-empty intersection of a saturated class of p-bases and a weakly saturated class of p-bases is weakly saturated.*

24 Definition. Let $\mathbb{P}$ be a class of p-bases. The class consisting of all spaces X such that $(Q,B,X) \in \mathbb{P}$ for some subset Q of X and some p-base B for Q in X is said to be the *space-component* of $\mathbb{P}$.

The class consisting of all spaces Q such that $(Q,B,X) \in \mathbb{P}$ for some space X (therefore, $Q \subset X$) and some p-base B for Q in X is said to be the *subset-component* of $\mathbb{P}$.

The class of subsets consisting of all pairs (Q,X) such that $(Q,B,X) \in \mathbb{P}$ for some p-base B for Q in X is called the *adjacent to* $\mathbb{P}$ *class of subsets.*

25 Definition. A class $\mathbb{P}$ of p-bases is said to be *complete* if the adjacent to $\mathbb{P}$ class of subsets is complete.

26 Definition. An element (Q^T,B^T,T) of a class $\mathbb{P}$ of p-bases is said to be *universal* in $\mathbb{P}$ (respectively, *properly universal* in $\mathbb{P}$) if for every element $(Q^Z,B^Z,Z) \in \mathbb{P}$ there exists an embedding h of Z into T such that $Q^Z \subset h^{-1}(Q^T)$ (respectively, $Q^Z = h^{-1}(Q^T)$) and $B^Z = \{h^{-1}(V) : V \in B^T\}$.

27 Proposition. *In any (complete) weakly saturated class of p-bases there exist (properly) universal elements.*

Proof. Let $\mathbb{P}$ be a complete weakly saturated class of p-bases. Since $\mathbb{P}$ is a topological class there exists a set $\mathbb{P}_0$ of elements of $\mathbb{P}$ such that for every element $(Q^Y,B^Y,Y) \in \mathbb{P}$ there exists an element $(Q^X,B^X,X) \in \mathbb{P}_0$ and a homeomorphism h of Y onto X such that $h(Q^Y) = Q^X$ and

$$B^Y = \{h^{-1}(V) : V \in B^X\}.$$

Denote by $\mathbf{S}$ the indexed collection of spaces, which is defined as follows. The indexing set of $\mathbf{S}$ is the set $I\!P_0$ and the element of $\mathbf{S}$ having as index an element (Q^X, B^X, X) of $I\!P_0$ is the space X. Therefore, for every $X \in \mathbf{S}$ there exist a unique determined subset Q^X of X and a unique determined p-base B^X for Q^X in X such that $(Q^X, B^X, X) \in I\!P_0$.

Consider the restriction

$$\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\} \text{ of} \mathbf{S}$$

and the $I\!P$-co-p-base

$$\mathbf{B} \equiv \{B^X : X \in \mathbf{S}\}$$

for $\mathbf{Q}$ in $\mathbf{S}$. Since $I\!P$ is a weakly saturated class there exists a co-indication $\mathbf{N}$ of $\mathbf{B}$ and an initial co-mark $\mathbf{M}^+$ of $\mathbf{S}$ corresponding to the restriction $\mathbf{Q}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $I\!P$. (Therefore, $\mathbf{M}^+$ is a co-extension of $\mathbf{N}$). Let $\mathbf{M}$ be an arbitrary co-extension of $\mathbf{M}^+$. Then, $\mathbf{M}$ is also a co-extension of $\mathbf{N}$. Denote by θ an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

Let R^+ be an initial family of equivalence relations on $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$, the restriction $\mathbf{Q}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $I\!P$. Let also R be an arbitrary family of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+.

Since $I\!P$ is a complete class by Lemma 3.4 of [3] we can suppose that $\mathbf{M}$ and R are chosen in such a manner that $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction.

By construction, $(\mathrm{T}|_{\mathbf{Q}}, \mathrm{B}^{\mathrm{T}}_{\theta(\tau)}, \mathrm{T}) \in I\!P$. We prove that this element is properly universal in $I\!P$. Indeed, let $(Q^Y, B^Y, Y) \in I\!P$. There exists an element (Q^X, B^X, X) of $I\!P_0$ and a homeomorphism h of Y onto X such that $h(Q^Y) = Q^X$ and $B^Y = \{h^{-1}(V) : V \in B^X\}$. Denote by e^X_{T} the natural embedding of X into T and set $f = e^X_{\mathrm{T}} \circ h$.

We prove that $f^{-1}(Q^{\mathrm{T}}) = Q^Y$ and $B^Y = \{f^{-1}(V) : V \in \mathrm{B}^{\mathrm{T}}_{\theta(\tau)}\}$. For this it suffices to prove that $(e^X_{\mathrm{T}})^{-1}(Q^{\mathrm{T}}) = Q^X$ and $B^X = \{(e^X_{\mathrm{T}})^{-1}(V) : V \in \mathrm{B}^{\mathrm{T}}_{\theta(\tau)}\}$. However, the first relation follows immediately by the fact that $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction and the second by the fact that $I\!P$ is a weakly saturated class. Thus, in $I\!P$ there exists a properly universal element. The case, where the considered class is weakly saturated is proved similarly. $\boxed{QED}$

The next proposition is proved similarly to Proposition 14.

28 Proposition. *Let $I\!P$ be a saturated class of p-bases and $I\!K$ a saturated class of subsets. Then, the class of p-bases consisting of all elements $(Q, B, X) \in$ IP(p-base) such that $(U, X) \in I\!K$ for every $U \in B$ is a saturated class.*

The following two propositions are easily proved.

29 Proposition. *Let $I\!P$ be a (complete) (weakly) saturated class of p-bases, $I\!F$ a saturated class of subsets, and $I\!E$ a saturated class of spaces. Then, the classes*

$$\begin{aligned}&\{(Q,B,X)\in I\!P : (Q,X)\in I\!F\},\\ &\{(Q,B,X)\in I\!P : Q\in I\!E\},\\ &\{(Q,B,X)\in I\!P : X\in I\!E\},\\ &\{(Q,B,X)\in I\!P : B \textit{ is a pos-base for } Q \textit{ in } X\},\\ &\{(Q,B,X)\in I\!P : B \textit{ is a base for } X\}\end{aligned}$$

are also (complete) (weakly) saturated classes of p-bases.

30 Proposition. *Let $I\!P$ be a weakly saturated class of p-bases. Then, the space-component of $I\!P$, the subset-component of $I\!P$, and the adjacent to $I\!P$ class of subsets are saturated classes.*

Now, we define different kinds of "regularity" of p-bases.

31 Definition. Let B be a p-base for a subset Q in a space X. Then, B is said to be: (a) a T_3-*p-base*, (b) a T_3^s-*p-base*, (c) a T_3^c-*p-base*, and (d) a T_3^h-*p-base* if for every point $x\in Q$ and for every neighbourhood $U\in B$ of x there exists an element $V\in B$ such that $x\in V\subset U$ and

(a) $\mathrm{Cl}_X(V)\subset U$,
(b) $\mathrm{Cl}_X(V)\cap Q\subset U$,
(c) $\mathrm{Cl}_X(V\cap Q)\subset U$, and
(d) $\mathrm{Cl}_Q(V\cap Q)\subset U$, respectively.

We denote by $\mathbb{P}(\mathrm{T}_3\text{-p-base})$ (respectively, by $\mathbb{P}(\mathrm{T}_3^s\text{-p-base})$, $\mathbb{P}(\mathrm{T}_3^c\text{-p-base})$, and by $\mathbb{P}(\mathrm{T}_3^h\text{-p-base})$) the class of p-bases consisting of all triads $(Q,B,X)\in I\!P(\text{p-base})$ such that Q is a subset of a space X and B is a T_3-p-base (respectively, a T_3^s-p-base, a T_3^c-p-base, and a T_3^h-p-base).

32 Proposition. *The classes*

$$\begin{aligned}&\mathbb{P}(\mathrm{T}_3\text{-}p\text{-}base),\\ &\mathbb{P}(\mathrm{T}_3^s\text{-}p\text{-}base),\\ &\mathbb{P}(\mathrm{T}_3^c\text{-}p\text{-}base),\end{aligned}$$

and

$$\mathbb{P}(\mathrm{T}_3^h\text{-}p\text{-}base)$$

are saturated classes of p-bases.

Proof. We prove the proposition for the class $\mathbb{P} \equiv \mathbb{P}(\mathrm{T}_3^s\text{-p-base})$. The proofs for the other classes are similar.

Let $\mathbf{S}$ be an indexed collection of spaces, $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ a restriction of $\mathbf{S}$, and $\mathbf{B}$ a $\mathbb{P}$-co-p-base for $\mathbf{Q}$ in $\mathbf{S}$. Denote by

$$\mathbf{N} \equiv \{\{V_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

an arbitrary co-indication of $\mathbf{B}$. Let ϑ_0 and ϑ_1 be two one-to-one mappings of τ into itself such that $\vartheta_0(\tau) \cap \vartheta_1(\tau) = \emptyset$. Let

$$\mathbf{M}^+ \equiv \{\{W_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

be a co-mark of $\mathbf{S}$ such that $V_\delta^X = W_{\vartheta_0(\delta)}^X$ and $X \backslash \mathrm{Cl}_X(V_\delta^X) = W_{\vartheta_1(\delta)}^X$. Obviously, $\mathbf{M}^+$ is a co-extension of $\mathbf{N}$. Let also

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

be an arbitrary co-extension of $\mathbf{M}^+$ and θ an inditial mapping from $\mathbf{M}^+$ to $\mathbf{M}$. Set $\theta_0 = \theta \circ \vartheta_0$ and $\theta_1 = \theta \circ \vartheta_1$. Then, $\mathbf{M}$ is also a co-extension of $\mathbf{N}$ and θ_0 is an indicial mapping from $\mathbf{N}$ to $\mathbf{M}$.

Now, we consider the elements of $\mathbf{S}$ as marked spaces. The chosen mark of an element $X \in \mathbf{S}$ is considered to be the mark $\mathbf{M}(X)$. For every $s \in \mathcal{F} \setminus \{\emptyset\}$ and for every $X \in \mathbf{S}$ denote by $\widetilde{\mathrm{A}}_s^X$ the minimal algebra of X containing the set $\mathrm{A}_s^X \cup \{Q^X \cap A : A \in \mathrm{A}_s^X\}$, where A_s^X is the s-algebra of the marked space X.

For every $s \in \mathcal{F}$ we define an equivalence relation $\sim_+^s$ on $\mathbf{S}$ considering that two elements X and Y of $\mathbf{S}$ are $\sim_+^s$-equivalent if either $s = \emptyset$ or $s \neq \emptyset$ and: (a) $X \sim_{\mathbf{M}}^s Y$ and $d_s^X(Q^X) = d_s^Y(Q^Y)$ (therefore, we can consider the natural isomorphism i of A_s^X onto A_s^Y), and (b) there exists an isomorphism $\widetilde{i}$ of $\widetilde{\mathrm{A}}_s^X$ onto $\widetilde{\mathrm{A}}_s^Y$ such that $\widetilde{i}(A) = i(A)$ and $\widetilde{i}(Q^X \cap A) = Q^Y \cap i(A)$ for every $A \in \mathrm{A}_s^X$. It is easy to verify that $\sim_+^s$ is really an equivalence relation on $\mathbf{S}$. Moreover, the family $\mathrm{R}^+ \equiv \{\sim_+^s : s \in \mathcal{F}\}$ is an $(\mathbf{M}, \mathbf{Q})$-admissible family of equivalence relations on $\mathbf{S}$.

We prove that $\mathbf{M}^+$ is an initial co-mark corresponding to the restriction $\mathbf{Q}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $\mathbb{P}$, and R^+ is an initial family corresponding to the co-mark $\mathbf{M}$, the restriction $\mathbf{Q}$, the co-indication $\mathbf{N}$ of $\mathbf{B}$, and the class $\mathbb{P}$. Indeed, let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an arbitrary admissible family of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+. Consider elements $\mathbf{L}$, $\mathbf{H}$, and $\mathbf{E}$ of $\mathrm{C}^\diamondsuit(\mathrm{R})$ such that $\mathbf{E} \subset \mathbf{H} \subset \mathbf{L}$. We need to prove that the p-base $\mathrm{B}_{\theta_0(\tau)}^{\mathbf{H}}$ for $\mathrm{T}(\mathbf{E}|_{\mathbf{Q}})$ in $\mathrm{T}(\mathbf{L})$ is a T_3^s-p-base. For this it suffices to prove that the p-base $\mathrm{B}_{\theta_0(\tau)}^{\mathrm{T}}$ for $\mathrm{T}|_{\mathbf{Q}}$ in T is a T_3^s-p-base.

Let $\mathbf{a}$ be a point of $\mathrm{T}|_{\mathbf{Q}}$ and $U \in \mathrm{B}_{\theta_0(\tau)}^{\mathrm{T}}$ an open neighbourhood of $\mathbf{a}$. We need to find an element V of $\mathrm{B}_{\theta_0(\tau)}^{\mathrm{T}}$ such that $\mathbf{a} \in V \subset \mathrm{Cl}_{\mathrm{T}}(V) \cap \mathrm{T}|_{\mathbf{Q}} \subset U$.

Without loss of generality we can suppose that $U = U^{\mathrm{T}}_{\theta_0(\delta)}(\mathbf{K}) \in \mathrm{B}^{\mathrm{T}}_{\theta_0(\tau)}$, where $\delta \in \tau$ and $\mathbf{K} \in \mathrm{C}(\sim^t)$ for some $t \in \mathcal{F}$. Hence, $U^X_{\theta_0(\delta)} = V^X_\delta \in \mathbf{N}(X)$ for every $X \in \mathbf{K}$. There exists an element (x, X) of $\mathbf{a}$ such that $x \in Q^X \cap V^X_\delta$ and $X \in \mathbf{K}$ (see Lemma 2.7 of [2]). Since $\mathbf{B}(X)$ is a T^s_3-base for Q^X in X there exists an element ε of τ such that $x \in V^X_\varepsilon$ and

$$\mathrm{Cl}_X(V^X_\varepsilon) \cap Q^X \subset V^X_\delta$$

or, equivalently, $x \in U^X_{\theta_0(\varepsilon)}$ and

$$\mathrm{Cl}_X(U^X_{\theta_0(\varepsilon)}) \cap Q^X \subset U^X_{\theta_0(\delta)}.$$

Let $s = \{\theta_0(\delta), \theta_0(\varepsilon), \theta_1(\varepsilon)\}$. Since the family R is a final refinement of R^+ there exists an element q of $\mathcal{F}$ such that $\sim^q \subset \sim^s_+$ and $t \subset q$. Let $\mathbf{F}$ be the $\sim^q$-equivalence class of X. Then, $\mathbf{F} \subset \mathbf{K}$. We prove that the set $V \equiv U^{\mathrm{T}}_{\theta_0(\varepsilon)}(\mathbf{F})$ satisfies the above mentioned conditions, that is, $\mathbf{a} \in U^{\mathrm{T}}_{\theta_0(\varepsilon)}(\mathbf{F})$ and

$$\mathrm{Cl}_{\mathrm{T}}(U^{\mathrm{T}}_{\theta_0(\varepsilon)}(\mathbf{F})) \cap \mathrm{T}|_{\mathbf{Q}} \subset U^{\mathrm{T}}_{\theta_0(\delta)}(\mathbf{F}) \subset U^{\mathrm{T}}_{\theta_0(\delta)}(\mathbf{K}).$$

It is suffices to prove the first inclusion. Let $\mathbf{b} \in \mathrm{Cl}_{\mathrm{T}}(U^{\mathrm{T}}_{\theta_0(\varepsilon)}(\mathbf{F})) \cap \mathrm{T}|_{\mathbf{Q}}$. There exists an element (y, Y) of $\mathbf{b}$ such that $y \in Q^Y$ and $Y \in \mathbf{F}$. By Lemma 2.7 of [2], $y \in \mathrm{Cl}_Y(U^Y_{\theta_0(\varepsilon)})$ and, therefore, $y \in \mathrm{Cl}_Y(U^Y_{\theta_0(\varepsilon)}) \cap Q^Y$. By construction, the sets $U^Y_{\theta_0(\delta)}$ and $\mathrm{Cl}_Y(U^Y_{\theta_0(\varepsilon)}) = Y \setminus U^Y_{\theta_1(\varepsilon)}$ are elements of A^Y_s. Moreover,

$$i(U^X_{\theta_0(\delta)}) = U^Y_{\theta_0(\delta)}$$

and

$$i(\mathrm{Cl}_X(U^X_{\theta_0(\varepsilon)}))i(X \setminus U^X_{\theta_1(\varepsilon)})Y \setminus U^Y_{\theta_1(\varepsilon)}\mathrm{Cl}_Y(U^Y_{\theta_0(\varepsilon)}),$$

where i is the natural isomorphism of A^X_s onto A^Y_s. Therefore,

$$\widetilde{i}(\mathrm{Cl}_X(U^X_{\theta_0(\varepsilon)}) \cap Q^X)\mathrm{Cl}_Y(U^Y_{\theta_0(\varepsilon)}) \cap Q^Y.$$

Since $\widetilde{i}$ is an isomorphism, by Lemma 1.1 of [2] relation $(1*)$ implies that

$$\mathrm{Cl}_Y(U^Y_{\theta_0(\varepsilon)}) \cap Q^Y \subset U^Y_{\theta_0(\delta)},$$

which means that $y \in U^Y_{\theta_0(\delta)}$. Therefore, $\mathbf{b} \in U^{\mathrm{T}}_{\theta_0(\delta)}(\mathbf{F})$, which proves relation $(2*)$.

The above show that $\mathrm{B}^{\mathrm{T}}_{\theta_0(\tau)}$ is a T^s_3-p-base. Hence, R^+ is an initial family and, therefore, $\mathbf{M}^+$ is an initial co-mark. Thus, $\mathbb{P}$ is a saturation class of p-bases. $\boxed{QED}$

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