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More on δ-semiopen sets

M. Caldas
Departamento de Matematica Aplicada,
Universidade Federal Fluminense,
Rua Mario Santos Braga, s/n 24020-140, Niteroi, RJ Brasil
gmamccs@vm.uff.br

D. N. Georgiou
Department of Mathematics,
University of Patras,
26500 Patras, Greece
georgiou@math.upatras.gr

S. Jafari
Department of Mathematics and Physics,
Roskilde University,
Postbox 260, 4000 Roskilde, Denmark
sjafari@ruc.dk

T. Noiri
Department of Mathematics,
Yatsushiro College of Technology,
Kumamoto, 866-8501 Japan
noiri@as.yatsushiro-nct.ac.jp

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Abstract. In 1963, Levine [2] introduced the notion of semi-open sets which is weaker than the notion of open sets in topological spaces. Since then several interesting generalized open sets came to existence. In 1968, Veličko [5] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces. In 1997, Park et al. [4] have offered a new notion called δ-semiopen sets which are stronger than semi-open sets but weaker than δ-open sets. They also studied the relationships between these sets and several other types of open sets.

It is the aim of this paper to offer some weak separation axioms by utilizing δ-semiopen sets and the δ-semi-closure operator.

Keywords: δ-semiopen set, δ-semi$D$-set, δ-semiD_0, δ-semiD_1, δ-semiD_2, sober δ-semiR_0, δ-semiR_0, δ-semiR_1.

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Introduction and preliminaries

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$ respectively.

Levine [2] defined semiopen sets which are weaker than open sets in topological spaces. After Levines semiopen set, mathematicians gave in several papers different and interesting new open sets as well as generalized open sets. In 1968, Veličko [5] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces. In 1997, Park et al. [4] have introduced the notion of δ-semiopen sets which are stronger than semiopen sets but weaker than δ-open sets and investigated the relationships between several types of these open sets. Also for the notions of δ-T_i spaces, $i = 0, 1, 2$, R_0 spaces, weakly R_0 spaces, δ-R_0 spaces and weakly δ-R_0 spaces see [1] and [3].

Let A be a subset of X. A point $x \in X$ is called the δ-*cluster* point of A if $A \cap Int(Cl(U)) \neq \emptyset$ for every open set U of X containing x. The set of all δ-cluster points of A is called the δ-*closure* of A, denoted by $Cl_\delta(A)$. A subset A of X is called δ-*closed* if $A = Cl_\delta(A)$. The complement of a δ-closed set is called δ-*open*.

A subset A of X is called *semi-open* if $A \subset Cl(Int(A))$. Also, a subset A of a topological space X is said to be δ-semiopen set [4] if there exists a δ-open set U of X such that $U \subset A \subset Cl(U)$. The complement of a δ-semiopen set is called a δ-semiclosed set. A point $x \in X$ is called the δ-semicluster point of A if $A \cap U \neq \emptyset$ for every δ-semiopen set U of X containing x. The set of all δ-semicluster points of A is called the δ-semiclosure of A, denoted by $sCl_\delta(A)$. We denote the collection of all δ-semiopen (resp. δ-semiclosed) sets by $\delta\mathrm{SO}(X, \tau)$ (resp. $\delta\mathrm{SC}(X, \tau)$). We say that a set U in a topological space (X, τ) is a δ-semineighborhood of a point x if U contains a δ-semiopen set to which x belongs.

1 Lemma. *The intersection of arbitrary collection of δ-semiclosed sets in (X, τ) is δ-semiclosed*

2 Corollary. *Let A be a subset of a topological space (X, τ), $sCl_\delta(A) = \cap\{\, F \in \delta SC(X, \tau) \mid A \subset F \,\}$.*

3 Corollary. *$sCl_\delta(A)$ is δ-semiclosed, that is $sCl_\delta(sCl_\delta(A)) = sCl_\delta(A)$.*

4 Lemma. *For subsets A and A_i ($i \in I$) of a space (X, τ), the following hold:*

(1) $A \subset sCl_\delta(A)$.
(2) If $A \subset B$, then $sCl_\delta(A) \subset sCl_\delta(B)$.
(3) $sCl_\delta(\cap\{\, A_i \mid i \in I \,\}) \subset \cap\{\, sCl_\delta(A_i) \mid i \in I \,\}$.
(4) $sCl_\delta(\cup\{\, A_i \mid i \in I \,\}) = \cup\{\, sCl_\delta(A_i) \mid i \in I \,\}$.
(5) A is δ-semiclosed if and only $A = sCl_\delta(A)$.

5 Example. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Clearly the family of all δ-open sets is the family τ and $\delta\mathrm{SO}(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Let $A = \{a, b\}$ and $B =$

$\{b, c\}$. Then, we have $sCl_\delta(A \cap B) = \{b\}$ and $sCl_\delta(A) \cap sCl_\delta(B) = X \cap \{b, c\} = \{b, c\}$. So $sCl_\delta(A \cap B) \neq sCl_\delta(A) \cap sCl_\delta(B)$. Also $sCl_\delta(A) = X \neq A$.

1 δ-semiD-sets and associated separation axioms

6 Definition. A subset A of a topological space X is called a δ-semiD-set if there are two $U, V \in \delta SO(X, \tau)$ such that $U \neq X$ and $A{=}U \setminus V$.

It is true that every δ-semiopen set U different from X is a δ-semiD-set if $A{=}U$ and $V{=}\emptyset$.

7 Example. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\delta\mathrm{SO}(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ and the family of all δ-semiD-sets is the family $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. So, the set $\{c\}$ is a δ-semiD-set which is not δ-semiopen.

8 Definition. A topological space (X, τ) is called δ-semiD_0 if for any distinct pair of points x and y of X there exists a δ-semiD-set of X containing x but not y or a δ-semiD-set of X containing y but not x.

9 Definition. A topological space (X, τ) is called δ-*semi*D_1 if for any distinct pair of points x and y of X there exists a δ-semiD-set of X containing x but not y and a δ-semiD-set of X containing y but not x.

10 Definition. A topological space (X, τ) is called δ-semiD_2 if for any distinct pair of points x and y of X there exists disjoint δ-semiD-sets G and E of X containing x and y, respectively.

11 Definition. A topological space (X, τ) is called δ-semiT_0 if for any distinct pair of points in X, there is a δ-semiopen set containing one of the points but not the other.

12 Definition. A topological space (X, τ) is called δ-semiT_1 if for any distinct pair of points x and y in X, there is a δ-semiopen U in X containing x but not y and a δ-semiopen set V in X containing y but not x.

13 Definition. A topological space (X, τ) is called δ-semiT_2 if for any distinct pair of points x and y in X, there exist δ-semiopen sets U and V in X containing x and y, respectively, such that $U \cap V = \emptyset$.

14 Remark. Obviously, we have:
(i) If (X, τ) is δ-semiT_i , then it is δ-semiT_{i-1}, $i = 1, 2$.
(ii) If (X, τ) is δ-semiT_i , then it is δ-semiD_i , $i = 0, 1, 2$.
(iii) If (X, τ) is δ-semiD_i , then it is δ-semiD_{i-1} , $i = 1, 2$.

15 Example. Let (X, τ) be a topological space such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Clearly the family of all δ-open sets is the family τ, $\delta\mathrm{SO}(X, \tau) = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\},$

$\{a,c,d\}, \{b,c,d\}, \emptyset, X\}$ and the family of all δ-semiD-sets is the family $\{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \emptyset\}$. The space (X,τ) is not δ-T_i space and T_i space, $i = 0,1,2$. But the space (X,τ) is δ-semiT_i and δ-semiD_i space, $i = 0,1,2$.

16 Theorem. *For a topological space (X,τ) the following statements are true:*

(1) (X,τ) is δ-semiD_0 if and only if it is δ-semiT_0.

(2) (X,τ) is δ-semiD_1 if and only if it is δ-semiD_2.

Proof. (1) The sufficiency is Remark 14(ii). To prove necessity. Let (X,τ) be δ-semiD_0. Then for each distinct pair $x,y \in X$, at least one of x,y, say x, belongs to a δ-semiD-set G but $y \notin G$. Let $G = U_1 \setminus U_2$ where $U_1 \neq X$ and U_1,$U_2 \in \delta SO(X,\tau)$. Then $x \in U_1$, and for $y \notin G$ we have two cases: (a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$.

In case (a), U_1 contains x but does not contain y ;

In case (b), U_2 contains y but does not contain x. Hence X is δ-semiT_0.

(2) Sufficiency. Remark 14(iii).

Necessity. Suppose X δ-semiD_1. Then for each distinct pair $x,y \in X$, we have δ-semiD-sets G_1,G_2 such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \backslash U_2$ and $G_2 = U_3 \backslash U_4$. From $x \notin G_2$ we have either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(1) $x \notin U_3$. From $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. From $x \in U_1 \backslash U_2$ we have $x \in U_1 \backslash (U_2 \cup U_3)$ and from $y \in U_3 \backslash U_4$ we have $y \in U_3 \backslash (U_1 \cup U_4)$. It is easy to see that $(U_1 \backslash (U_2 \cup U_3)) \cap (U_3 \backslash (U_1 \cup U_4) = \emptyset$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \backslash U_2$, $y \in U_2$. $(U_1 \backslash U_2) \cap U_2 = \emptyset$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \backslash U_4$, $x \in U_4$. $(U_3 \backslash U_4) \cap U_4 = \emptyset$.
From the discussion above we know that the space X is δ-semiD_2. $\boxed{QED}$

17 Theorem. *If (X,τ) is δ-semiD_1, then it is δ-semiT_0.*

Proof. The proof of this theorem follows by Remark 14 and Theorem 16. $\boxed{QED}$

Questions 1.

(i) Does there exist a space which is δ-semiD_1 and it is not δ-semiT_1?

(ii) Does there exist a space which is δ-semiD_1 and it is not δ-semiT_2?

(iii) Does there exist a space which is δ-semiT_0 and it is not δ-semiD_1?

18 Theorem. *A topological space (X,τ) is δ-semiT_0 if and only if for each pair of distinct points x, y of X, $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$.*

Proof. Sufficiency: Suppose that $x,y \in X$, $x \neq y$ and $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. Let z is a point of X such that $z \in sCl_\delta(\{x\})$ but $z \notin sCl_\delta(\{y\})$. We claim that

$x \notin sCl_\delta(\{y\})$. For, if $x \in sCl_\delta(\{y\})$ then $sCl_\delta(\{x\}) \subset sCl_\delta(\{y\})$. And this contradicts the fact that $z \notin sCl_\delta(\{y\})$. Consequently x belongs to the δ-semiopen set $[sCl_\delta(\{y\})]^c$ to which y does not belong.

Necessity: Let (X,τ) be a δ-semiT_0 space and x, y be any two distinct points of X. There exists a δ-semiopen set G containing x or y, say x but not y. Then G^c is a δ-semiclosed set which does not contain x but contains y. Since $sCl_\delta(\{y\})$ is the smallest δ-semiclosed set containing y (Corollary 2), $sCl_\delta(\{y\}) \subset G^c$, and so $x \notin sCl_\delta(\{y\})$. Consequently $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. $\boxed{QED}$

19 Theorem. *A topological space (X,τ) is δ-semiT_1 if and only if the singletons are δ-semiclosed sets.*

Proof. Suppose (X,τ) is δ-semiT_1 and x be any point of X. Let $y \in \{x\}^c$. Then $x \neq y$ and so there exists a δ-semiopen set U_y such that $y \in U_y$ but x$\notin U_y$. Consequently $y \in U_y \subset \{x\}^c$ i.e., $\{x\}^c = \bigcup\{\, U_y \mid y \in \{x\}^c\,\}$ which is δ-semiopen.

Conversely. Suppose $\{p\}$ is δ-semiclosed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in \{x\}^c$. Hence $\{x\}^c$ is a δ-semiopen set containing y but not containing x. Similarly $\{y\}^c$ is a δ-semiopen set containing x but not containing y. Accordingly X is a δ-semiT_1 space. $\boxed{QED}$

20 Definition. A point $x \in X$ which has only X as the δ-semineighborhood is called a δ-semineat point.

21 Theorem. *For a δ-semiT_0 topological space (X,τ) the following are equivalent:*

(1) (X,τ) is δ-semiD_1;

(2) (X,τ) has no δ-semineat point.

Proof. (1) $\rightarrow$ (2). Since (X,τ) is δ-semiD_1, so each point x of X is contained in a δ-semiD-set $O{=}U\backslash V$ and thus in U. By definition $U \neq X$. This implies that x is not a δ-semineat point.

(2) $\rightarrow$ (1). If X is δ-semiT_0, then for each distinct pair of points $x, y \in X$, at least one of them, x(say) has a δ-semineighborhood U containing x and not y. Thus U which is different from X is a δ-semiD-set. If X has no δ-semineat point, then y is not a δ-semineat point. This means that there exists a δ-semineighborhood V of y such that $V \neq X$. Thus $y \in (V\backslash U)$ but not x and $V\backslash U$ is a δ-semiD-set. Hence X is δ-semiD_1. $\boxed{QED}$

22 Remark. It is clear that a δ-semiT_0 topological space (X,τ) is not δ-semiD_1 if and only if there is a unique δ-semineat point in X. It is unique because if x and y are both δ-semineat point in X, then at least one of them say x has a δ-semineighborhood U containing x but not y. But this is a contradiction since $U \neq X$.

23 Definition. A topological space (X, τ) is δ-semisymmetric if for x and y in X, $x \in sCl_\delta(\{y\})$ implies $y \in sCl_\delta(\{x\})$.

24 Definition. A subset A of a topological space (X, τ) is called a (δ, δ)-semigeneralized-closed set (briefly (δ, δ)-sg-closed) if $sCl_\delta(A) \subset U$ whenever $A \subset U$ and U is δ-semiopen in (X, τ).

25 Lemma. *Every δ-semiclosed set is (δ, δ)-sg-closed.*

26 Theorem. *A topological space (X, τ) is δ-semisymmetric if and only if $\{x\}$ is (δ, δ)-sg-closed for each $x \in X$.*

Proof. Assume that $x \in sCl_\delta(\{y\})$ but $y \notin sCl_\delta(\{x\})$. This means that the complement of $sCl_\delta(\{x\})$ contains y. Therefore the set $\{y\}$ is a subset of the complement of $sCl_\delta(\{x\})$. This implies that $sCl_\delta(\{y\})$ is a subset of the complement of $sCl_\delta(\{x\})$. Now the complement of $sCl_\delta(\{x\})$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset E \in \delta SO(X, \tau)$, but $sCl_\delta(\{x\})$ is not a subset of E. This means that $sCl_\delta(\{x\})$ and the complement of E are not disjoint. Let y belongs to their intersection. Now we have $x \in sCl_\delta(\{y\})$ which is a subset of the complement of E and $x \notin E$. But this is a contradiction. $\boxed{QED}$

27 Corollary. *If a topological space (X, τ) is a δ-semiT_1 space, then it is δ-semisymmetric.*

Proof. In a δ-semiT_1 space singleton sets are δ-semiclosed (Theorem 19) and therefore (δ, δ)-sg-closed (Lemma 25). By Theorem 26, the space is δ-semisymmetric. $\boxed{QED}$

28 Corollary. *For a topological space (X, τ) the following are equivalent:*
(1) (X, τ) is δ-semisymmetric and δ-semiT_0;
(2) (X, τ) is δ-semiT_1.

Proof. By Corollary 27 and Remark 14 it suffices to prove only (1) $\to$ (2). Let, then $x \neq y$ and by δ-semiT_0, we may assume that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in \delta SO(X, \tau)$. Then $x \notin sCl_\delta(\{y\})$ and hence $y \notin sCl_\delta(\{x\})$. There exists a $G_2 \in \delta SO(X, \tau)$ such that $y \in G_2 \subset \{x\}^c$ and (X, τ) is a δ-semiT_1 space. $\boxed{QED}$

29 Theorem. *For a δ-semisymmetric topological space (X, τ) the following are equivalent:*
(1) (X, τ) is δ-semiT_0;
(2) (X, τ) is δ-semiD_1;
(3) (X, τ) is δ-semiT_1.

Proof. (1) $\to$ (3) : Corollary 28.
(3) $\to$ (2) $\to$ (1) : Remark 14 and Theorems 16 and 17. $\boxed{QED}$

30 Definition. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be δ-semicontinuous if for each $x \in X$ and each δ-semiopen set V containing $f(x)$, there is a δ-semiopen set U in X containing x such that $f(U) \subset V$.

31 Lemma. *A function $f : (X, \tau) \to (Y, \sigma)$ is δ-semicontinuous if and only if the inverse image of each δ-semiopen set is δ-semiopen.*

32 Definition. Let (X, τ) be a topological space, $x \in X$ and $\{x_s, s \in S\}$ be a net of X. We say that the net $\{x_s, s \in S\}$ δ-semiconverges to x if for each δ-semiopen set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.

33 Definition. A filterbase F is called δ-semiconvergent to a point x in X, if for any δ-semiopen set U of X containing x, there exists B in F such that B is a subset of U.

34 Theorem. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) f is δ-semicontinuous;

(2) For each $x \in X$ and each filterbase $\mathcal{F}$ which δ-semiconverges to x, $f(\mathcal{F})$ δ-semiconverges to $f(x)$.

(3) For each $x \in X$ and each net $\{x_s, s \in S\}$ in X which δ-semiconverges to x, we have that the net $\{f(x_s), s \in S\}$ of Y δ-semiconverges to $f(x) \in Y$.

PROOF. Obvious. $\boxed{QED}$

35 Definition. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be δ-semiD-continuous if for each $x \in X$ and each δ-semiD-set V containing $f(x)$, there is a δ-semiD-set U in X containing x such that $f(U) \subset V$.

36 Lemma. *function $f : (X, \tau) \to (Y, \sigma)$ is δ-semiD-continuous if and only if the inverse image of each δ-semiD-set is δ-semiD-set.*

37 Definition. Let (X, τ) be a topological space, $x \in X$ and $\{x_s, s \in S\}$ be a net of X. We say that the net $\{x_s, s \in S\}$ δ-semiD-converges to x if for each δ-semiD-set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.

38 Definition. A filterbase F is called δ-semiD-convergent to a point x in X, if for any δ-semiD-set U of X containing x, there exists B in F such that B is a subset of U.

39 Theorem. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) f is δ-semiD-continuous;

(2) For each $x \in X$ and each filterbase $\mathcal{F}$ which δ-semiD-converges to x, $f(\mathcal{F})$ δ-semiD-converges to $f(x)$.

(3) For each $x \in X$ and each net $\{x_s, s \in S\}$ in X which δ-semiD-converges to x, we have that the net $\{f(x_s), s \in S\}$ of Y δ-semiD-converges to $f(x) \in Y$.

PROOF. Obvious. $\boxed{QED}$

40 Example. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and let $f : X \to X$ be a map such that $f(a) = b$, $f(b) = c$ and $f(c) = a$. Clearly the map f is not δ-semicontinuous. But the map f is δ-semiD-continuous.

41 Theorem. *If $f : (X, \tau) \to (Y, \sigma)$ is a δ-semicontinuous surjective function and E is a δ semiD-set in Y, then the inverse image of E is a δ-semiD-set in X.*

PROOF. Let E be a δ semiD-set in Y. Then there are δ-semiopen sets U_1 and U_2 in Y such that $S = U_1 \backslash U_2$ and $U_1 \neq Y$. By the δ-semicontinuity of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are δ-semiopen in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1) \backslash f^{-1}(U_2)$ is a δ-semiD-set. $\boxed{QED}$

42 Theorem. *If (Y, σ) is δ-semiD_1 and $f : (X, \tau) \to (Y, \sigma)$ is δ-semicontinuous and bijective, then (X, τ) is δ-semiD_1.*

PROOF. Suppose that Y is a δ-semiD_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is δ-semiD_1, there exist δ-semiD-sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 41, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are δ-semiD-sets in X containing x and y respectively. This implies that X is a δ-semiD_1 space. $\boxed{QED}$

43 Theorem. *A topological space (X, τ) is δ-semiD_1 if and only if for each pair of distinct points $x, y \in X$, there exists a δ-semicontinuous surjective function $f : (X, \tau) \to (Y, \sigma)$, where Y is a δ-semiD_1 space such that $f(x)$ and $f(y)$ are distinct.*

PROOF. Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a δ-semicontinuous, surjective function f of a space X onto a δ-semiD_1 space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint δ-semiD-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is δ-semicontinuous and surjective, by Theorem 41, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint δ-semiD-sets in X containing x and y, respectively. Hence by Theorem 16, X is δ-semiD_1 space. $\boxed{QED}$

2 Sober δ-semiR_0 spaces

44 Definition. Let A be a subset of a space X. The δ-semikernel of A, denoted by $sKer_\delta(A)$, is defined to be the set $sKer_\delta(A) = \cap\{O \in \delta SO(X,\tau) : A \subset O\}$.

45 Lemma. *Let (X,τ) be a topological space and $x \in X$. Then* $sKer_\delta(A) = \{x \in X : sCl_\delta(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in sKer_\delta(A)$ and suppose $sCl_\delta(\{x\}) \cap A = \emptyset$. Hence $x \notin X \backslash sCl_\delta(\{x\})$ which is a δ-semiopen set containing A. This is absurd, since $x \in sKer_\delta(A)$. Consequently, $sCl_\delta(\{x\}) \cap A \neq \emptyset$. Next, let x such that $sCl_\delta(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin sKer_\delta(A)$. Then, there exists a δ-semiopen set D containing A and $x \notin D$. Let $y \in sCl_\delta(\{x\}) \cap A$. Hence, D is a δ-semineighborhood of y which does not containing x. By this contradiction $x \in sKer_\delta(A)$ and the claim. $\boxed{QED}$

46 Definition. A topological space (X,τ) is said to be sober δ-semiR_0 if

$$\cap_{x\in X} sCl_\delta(\{x\}) = \emptyset.$$

47 Example. Let (X,τ) be a topological space such that $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Clearly the space X is not weakly R_0 and weakly δ-R_0. Also $\cap_{x\in X} sCl_\delta(\{x\}) = \emptyset$. Thus the space X is sober δ-semiR_0.

48 Theorem. *A topological space (X,τ) is sober δ-semiR_0 if and only if $sKer_\delta(\{x\}) \neq X$ for every $x \in X$.*

Proof. Suppose that the space (X,τ) be sober δ-semiR_0. Assume that there is a point y in X such that $sKer_\delta(\{y\}) = X$. Then $y \notin O$ which O is some proper δ-semiopen subset of X. This implies that $y \in \cap_{x\in X} sCl_\delta(\{x\})$. But this is a contradiction.

Now assume that $sKer_\delta(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap_{x\in X} sCl_\delta(\{x\})$, then every δ-semiopen set containing y must contain every point of X. This implies that the space X is the unique δ-semiopen set containing y. Hence $sKer_\delta(\{y\}) = X$ which is a contradiction. Therefore (X,τ) is sober δ-semiR_0. $\boxed{QED}$

49 Theorem. *If the topological space X is sober δ-semiR_0 and Y is any topological space, then the product $X \times Y$ is sober δ-semiR_0.*

Proof. By showing that $\cap_{(x,y)\in X\times Y} sCl_\delta(\{x,y\}) = \emptyset$ we are done. We have:

$$\begin{aligned} &\cap_{(x,y)\in X\times Y}\, sCl_\delta(\{x,y\}) \subset \cap_{(x,y)\in X\times Y}(sCl_\delta(\{x\}) \times sCl_\delta(\{y\})) \\ &= \cap_{x\in X}\, sCl_\delta(\{x\}) \times \cap_{y\in Y} sCl_\delta(\{y\}) \subset \emptyset \times Y = \emptyset. \end{aligned}$$

$\boxed{QED}$

3 δ-semiR_0 and δ-semiR_1 spaces

50 Definition. A topological space (X, τ) is said to be a δ-semiR_0 space if every δ-semiopen set contains the δ-semiclosure of each of its singletons.

51 Definition. A topological space (X, τ) is said to be δ-semiR_1 if for x, y in X with $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$, there exist disjoint δ-semiopen sets U and V such that $sCl_\delta(\{x\})$ is a subset of U and $sCl_\delta(\{y\})$ is a subset of V.

52 Example. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Clearly the space X is not R_0, δ-R_0, R_1 and δ-R_1 space. Also the space (X, τ) is δ-semiR_0 and δ-semiR_1.

53 Lemma. *Let (X, τ) be a topological space and $x \in X$. Then*

$$y \in sKer_\delta(\{x\}) \text{ if and only if } x \in sCl_\delta(\{y\}).$$

Proof. Suppose that $y \notin sKer_\delta(\{x\})$. Then there exists a δ-semiopen set V containing x such that $y \notin V$. Therefore we have $x \notin sCl_\delta(\{y\})$. The converse is similarly shown. $\boxed{QED}$

54 Lemma. *The following statements are equivalent for any points x and y in a topological space (X, τ) :*
(1) $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$;
(2) $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$.

Proof. (1) $\rightarrow$ (2) : Suppose that $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$, then there exists a point z in X such that $z \in sKer_\delta(\{x\})$ and $z \notin sKer_\delta(\{y\})$. From $z \in sKer_\delta(\{x\})$ it follows that $\{x\} \cap sCl_\delta(\{z\}) \neq \emptyset$ which implies $x \in sCl_\delta(\{z\})$. By $z \notin sKer_\delta(\{y\})$, we have $\{y\} \cap sCl_\delta(\{z\}) = \emptyset$. Since $x \in sCl_\delta(\{z\})$, $sCl_\delta(\{x\}) \subset sCl_\delta(\{z\})$ and $\{y\} \cap sCl_\delta(\{z\}) = \emptyset$. Therefore it follows that $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. Now $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$ implies that $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$.
(2) $\rightarrow$ (1) : Suppose that $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. Then there exists a point z in X such that $z \in sCl_\delta(\{x\})$ and $z \notin sCl_\delta(\{y\})$. It follows that there exists a δ-semiopen set containing z and therefore x but not y, namely, $y \notin sKer_\delta(\{x\})$ and thus $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$. $\boxed{QED}$

55 Theorem. *If (X, τ) is δ-semiR_1, then (X, τ) is δ-semiR_0.*

Proof. Let U be δ-semiopen and $x \in U$. If $y \notin U$, then since $x \notin sCl_\delta(\{y\})$, $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. Hence, there exists a δ-semiopen V_y such that $sCl_\delta(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin sCl_\delta(\{x\})$. Thus $sCl_\delta(\{x\}) \subset U$. Therefore (X, τ) is δ-semiR_0. $\boxed{QED}$

Question 2. Does there exist a space which is δ-semiR_0 and it is not δ-semiR_1?

56 Theorem. *A topological space (X,τ) is δ-semiR_1 if and only if for $x, y \in X$, $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$, there exist disjoint δ-semiopen sets U and V such that $sCl_\delta(\{x\}) \subset U$ and $sCl_\delta(\{y\}) \subset V$.*

PROOF. It follows from Lemma 54. $\boxed{QED}$

57 Theorem. *A topological space (X,τ) is a δ-semiR_0 space if and only for any x and y in X, $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$ implies $sCl_\delta(\{x\}) \cap sCl_\delta(\{y\}) = \emptyset$.*

PROOF. Neces. Assume (X,τ) δ-semiR_0 and $x, y \in X$ such that $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. Then, there exist $z \in sCl_\delta(\{x\})$ such that $z \notin sCl_\delta(\{y\})$ (or $z \in sCl_\delta(\{y\})$ such that $z \notin sCl_\delta(\{x\})$). There exists $V \in \delta SO(X,\tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin sCl_\delta(\{y\})$. Thus $x \in X \backslash sCl_\delta(\{y\}) \in \delta SO(X,\tau)$, which implies $sCl_\delta(\{x\}) \subset X \backslash sCl_\delta(\{y\})$ and $sCl_\delta(\{x\}) \cap sCl_\delta(\{y\}) = \emptyset$. The proof for otherwise is similar.

Sufficiency. Let $V \in \delta SO(X,\tau)$ and let $x \in V$. We will show that $sCl_\delta(\{x\}) \subset V$. Really, let $y \notin V$, i.e., $y \in X \backslash V$. Then $x \neq y$ and $x \notin sCl_\delta(\{y\})$. This shows that $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. By assumption, $sCl_\delta(\{x\}) \cap sCl_\delta(\{y\}) = \emptyset$. Hence $y \notin sCl_\delta(\{x\})$. Therefore $sCl_\delta(\{x\}) \subset V$. $\boxed{QED}$

58 Theorem. *A topological space (X,τ) is a δ-semiR_0 space if and only if for any points x and y in X, $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$ implies $sKer_\delta(\{x\}) \cap sKer_\delta(\{y\}) = \emptyset$.*

PROOF. Suppose that (X,τ) is a δ-semiR_0 space. Thus by Lemma 54, for any points x and y in X if $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$ then $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$. Now we prove that $sKer_\delta(\{x\}) \cap sKer_\delta(\{y\}) = \emptyset$. Assume that $z \in sKer_\delta(\{x\}) \cap sKer_\delta(\{y\})$. By $z \in sKer_\delta(\{x\})$ and Lemma 53, it follows that $x \in sCl_\delta(\{z\})$. Since $x \in sCl_\delta(\{x\})$, by Theorem 57 $sCl_\delta(\{x\}) = sCl_\delta(\{z\})$. Similarly, we have $sCl_\delta(\{y\}) = sCl_\delta(\{z\}) = sCl_\delta(\{x\})$. This is a contradiction. Therefore, we have $sKer_\delta(\{x\}) \cap sKer_\delta(\{y\}) = \emptyset$.

Conversely , let (X,τ) be a topological space such that for any points x and y in X, $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$ implies $sKer_\delta(\{x\}) \cap sKer_\delta(\{y\}) = \emptyset$. If $sCl_\delta(\{x\}) \neq sCl_\delta(\{y\})$, then by Lemma 54, $sKer_\delta(\{x\}) \neq sKer_\delta(\{y\})$. Therefore $sKer_\delta(\{x\}) \cap sKer_\delta(\{y\}) = \emptyset$ which implies $sCl_\delta(\{x\}) \cap sCl_\delta(\{y\}) = \emptyset$. Because $z \in sCl_\delta(\{x\})$ implies that $x \in sKer_\delta(\{z\})$ and therefore $sKer_\delta(\{x\}) \cap sKer_\delta(\{z\}) \neq \emptyset$. By hypothesis, we therefore have $sKer_\delta(\{x\}) = sKer_\delta(\{z\})$. Then $z \in sCl_\delta(\{x\}) \cap sCl_\delta(\{y\})$ implies that $sKer_\delta(\{x\}) = sKer_\delta(\{z\}) = sKer_\delta(\{y\})$. This is a contradiction. Therefore, $sCl_\delta(\{x\}) \cap sCl_\delta(\{y\}) = \emptyset$ and by Theorem 57 (X,τ) is a δ-semiR_0 space. $\boxed{QED}$

59 Theorem. *For a topological space (X,τ), the following properties are equivalent:*
(1) (X,τ) is a δ-semiR_0 space;
(2) For any nonempty set A and $G \in \delta SO(X,\tau)$ such that $A \cap G \neq \emptyset$, there

exists $F \in \delta SC(X,\tau)$ *such that* $A \cap F \neq \emptyset$ *and* $F \subset G$;
(3) Any $G \in \delta SO(X,\tau)$, $G = \cup\{\, F \in \delta SC(X,\tau) \mid F \subset G\,\}$;
(4) Any $F \in \delta SC(X,\tau)$, $F = \cap\{\, G \in \delta SO(X,\tau) \mid F \subset G\,\}$;
(5) For any $x \in X, sCl_\delta(\{x\}) \subset sKer_\delta(\{x\})$.

Proof. (1) $\to$ (2) : Let A be a nonempty set of X and $G \in \delta SO(X,\tau)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \in \delta SO(X,\tau), sCl_\delta(\{x\}) \subset G$. Set $F = sCl_\delta(\{x\})$, then $F \in \delta SC(X,\tau), F \subset G$ and $A \cap F \neq \emptyset$.
(2) $\to$ (3) : Let $G \in \delta SO(X,\tau)$, then $G \supset \cup\{F \in \delta SC(X,\tau) : F \subset G\}$. Let x be any point of G. There exists $F \in \delta SC(X,\tau)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{\, F \in \delta SC(X,\tau) \mid F \subset G\,\}$ and hence $G = \cup\{\, F \in \delta SC(X,\tau) \mid F \subset G\,\}$.
(3) $\to$ (4) : This is obvious.
(4) $\to$ (5) : Let x be any point of X and $y \notin sKer_\delta(\{x\})$. There exists $V \in \delta SO(X,\tau)$ such that $x \in V$ and $y \notin V$; hence $sCl_\delta(\{y\}) \cap V = \emptyset$. By (4) $(\cap\{\, G \in \delta SO(X,\tau) \mid sCl_\delta(\{y\}) \subset G\,\}) \cap V = \emptyset$ and there exists $G \in \delta SO(X,\tau)$ such that $x \notin G$ and $sCl_\delta(\{y\}) \subset G$. Therefore, $sCl_\delta(\{x\}) \cap G = \emptyset$ and $y \notin sCl_\delta(\{x\})$. Consequently, we obtain $sCl_\delta(\{x\}) \subset sKer_\delta(\{x\})$.
(5) $\to$ (1) : Let $G \in \delta SO(X,\tau)$ and $x \in G$. Let $y \in sKer_\delta(\{x\})$, then $x \in sCl_\delta(\{y\})$ and $y \in G$. This implies that $sKer_\delta(\{x\}) \subset G$. Therefore, we obtain $x \in sCl_\delta(\{x\}) \subset sKer_\delta(\{x\}) \subset G$. This shows that (X,τ) is a δ-semiR_0 space. $\boxed{QED}$

60 Corollary. *For a topological space* (X,τ)*, the following properties are equivalent:*
(1) (X,τ) *is a* δ*-semi*R_0 *space;*
(2) $sCl_\delta(\{x\}) = sKer_\delta(\{x\})$ *for all* $x \in X$.

Proof. (1) $\to$ (2) : Suppose that (X,τ) is a δ-semiR_0 space. By Theorem 59, $sCl_\delta(\{x\}) \subset sKer_\delta(\{x\})$ for each $x \in X$. Let $y \in sKer_\delta(\{x\})$, then $x \in sCl_\delta(\{y\})$ and by Theorem 57 $sCl_\delta(\{x\}) = sCl_\delta(\{y\})$. Therefore, $y \in sCl_\delta(\{x\})$ and hence $sKer_\delta(\{x\}) \subset sCl_\delta(\{x\})$. This shows that $sCl_\delta(\{x\}) = sKer_\delta(\{x\})$.
(2) $\to$ (1) : This is obvious by Theorem 59. $\boxed{QED}$

61 Theorem. *For a topological space* (X,τ)*, the following properties are equivalent:*
(1) (X,τ) *is a* δ*-semi*R_0 *space;*
(2) $x \in sCl_\delta(\{y\})$ *if and only if* $y \in sCl_\delta(\{x\})$.

Proof. (1) $\to$ (2) : Assume X is δ-semiR_0. Let $x \in sCl_\delta(\{y\})$ and D be any δ-semiopen set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every δ-semiopen set which contains y contains x. Hence $y \in Cl_\delta(\{x\})$.
(2) $\to$ (1) : Let U be a δ-semiopen set and $x \in U$. If $y \notin U$, then $x \notin sCl_\delta(\{y\})$ and hence $y \notin sCl_\delta(\{x\})$. This implies that $sCl_\delta(\{x\}) \subset U$. Hence (X,τ) is

δ-semiR_0.

QED

By Theorem 61 and Definition 23, we have:

62 Remark. For a topological space (X, τ), the following properties are equivalent:
(1) (X, τ) is a δ-semiR_0 space;
(2) (X, τ) is a δ-semisymmetric.

63 Theorem. *For a topological space (X, τ), the following properties are equivalent:*
(1) (X, τ) is a δ-semiR_0 space;
(2) If F is δ-semiclosed, then $F = sKer_\delta(F)$;
(3) If F is δ-semiclosed and $x \in F$, then $sKer_\delta(\{x\}) \subset F$;
(4) If $x \in X$, then $sKer_\delta(\{x\}) \subset sCl_\delta(\{x\})$.

Proof. (1) $\to$ (2) : This obviously follows from Theorem 59.
(2) $\to$ (3) : In general, $A \subset B$ implies $sKer_\delta(A) \subset sKer_\delta(B)$. Therefore, it follows from (2) that $sKer_\delta(\{x\}) \subset sKer_\delta(F) = F$.
(3) $\to$ (4) : Since $x \in sCl_\delta(\{x\})$ and $sCl_\delta(\{x\})$ is δ-semiclosed, by (3) $sKer_\delta(\{x\}) \subset sCl_\delta(\{x\})$.
(4) $\to$ (1) : We show the implication by using Theorem 61. Let $x \in sCl_\delta(\{y\})$. Then by Lemma 53 $y \in sKer_\delta(\{x\})$. Since $x \in sCl_\delta(\{x\})$ and $sCl_\delta(\{x\})$ is δ-semiclosed, by (4) we obtain $y \in sKer_\delta(\{x\}) \subset sCl_\delta(\{x\})$. Therefore $x \in sCl_\delta(\{y\})$ implies $y \in sCl_\delta(\{x\})$. The converse is obvious and (X, τ) is δ-semiR_0. QED

64 Lemma. *Let (X, τ) be a topological space and let x and y be any two points in X such that every net in X δ-semiconverging to y δ-semiconverges to x. Then $x \in sCl_\delta(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ is a net in $sCl_\delta(\{y\})$. By the fact that $\{x_n\}_{n\in\mathbf{N}}$ δ-semiconverges to y, then $\{x_n\}_{n\in\mathbf{N}}$ δ-semiconverges to x and this means that $x \in sCl_\delta(\{y\})$. QED

65 Theorem. *For a topological space (X, τ), the following statements are equivalent :*
(1) (X, τ) is a δ-semiR_0 space;
(2) If $x, y \in X$, then $y \in sCl_\delta(\{x\})$ if and only if every net in X δ-semiconverging to y δ-semiconverges to x.

Proof. (1) $\to$ (2) : Let $x, y \in X$ such that $y \in sCl_\delta(\{x\})$. Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net in X such that $\{x_\alpha\}_{\alpha\in\Lambda}$ δ-semiconverges to y. Since $y \in sCl_\delta(\{x\})$, by Theorem 57 we have $sCl_\delta(\{x\}) = sCl_\delta(\{y\})$. Therefore $x \in sCl_\delta(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in\Lambda}$ δ-semiconverges to x. Conversely, let $x, y \in X$ such that every net in X δ-semiconverging to y δ-semiconverges to x. Then $x \in sCl_\delta(\{y\})$

by Lemma 64. By Theorem 57, we have $sCl_\delta(\{x\}) = sCl_\delta(\{y\})$. Therefore $y \in sCl_\delta(\{x\})$.
(2) $\rightarrow$ (1) : Assume that x and y are any two points of X such that $sCl_\delta(\{x\}) \cap sCl_\delta(\{y\}) \neq \emptyset$. Let $z \in sCl_\delta(\{x\}) \cap sCl_\delta(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $sCl_\delta(\{x\})$ such that $\{x_\alpha\}_{\alpha\in\Lambda}$ δ-semiconverges to z. Since $z \in sCl_\delta(\{y\})$, then $\{x_\alpha\}_{\alpha\in\Lambda}$ δ-semiconverges to y. It follows that $y \in sCl_\delta(\{x\})$. By the same token we obtain $x \in sCl_\delta(\{y\})$. Therefore $sCl_\delta(\{x\}) = sCl_\delta(\{y\})$ and by Theorem 57 (X, τ) is δ-semiR_0. $\boxed{QED}$

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