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Selection principles and covering properties in Topology

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Abstract. The study of selection principles in mathematics is the study of diagonalization processes. In this survey we restrict ourselves to selection principles which have been considered in the context of topology. As this field is growing very fast I do not intend to be complete or encyclopaedic here. I only give a brief report on some of the activities in this field over the last few years.

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Introduction

The study of selection principles in mathematics is the study of diagonalization processes. In this survey we restrict ourselves to selection principles which have been considered in the context of topology. As this field is growing very fast I do not intend to be complete or encyclopaedic here. I only give a brief report on some of the activities in this field over the last few years. Specifically, no proofs of results are included, and since a separate list of open problems from this area is being prepared for the proceedings of this conference, I also did not explicitly include open problems here. In the interest of getting this already overdue survey to the editors within reasonable time, I had to skip some important developments: I did not discuss at all the so-called star-selection principles, basis properties or measure properties. Basis properties are considered in [8], and measure properties will be discussed in [9]. I also omitted discussing the connection between the theory of filters on the positive integers and the classical selection principles - some information on this can be gleaned from [48]. I also omitted saying anything about preservation of some selection principles by special types of functions, or preservation of selection principles under pre-images of functions and a number of topological constructions. Unattributed results should not be assumed to be my results.

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The first eight sections are intended as an introduction to concepts and notation from this area and contain virtually no results. In the remaining sections I try to give the reader some flavor of the area by describing selected results.

Some points about notation: It is common in set theoretic literature to use the following notation:

- $\forall_n^\infty$: Read "for all but finitely many n"
- $\exists_n^\infty$: Read: "there are infinitely many n"

We will also borrow some notation from model theory, and for a space X and a property P we will write

$$X \models P$$

to denote that "X has property P" (or, equivalently, "X satisfies P").

1 The classical selection principles

Three selection principles have been around for a long time. In 1925 Hurewicz introduced two prototypes in [41]. Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of an infinite set S.

> $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n we have $B_n \subseteq A_n$, and $\cup\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Hurewicz derived this selection principle from a 1924 conjecture of K. Menger. In [50] Menger defined the following basis covering property for metric spaces:

> For each basis $\mathcal{B}$ of the metric space (X, d), there is a sequence $(B_n : n \in \mathbb{N})$ in $\mathcal{B}$ such that $\lim_{n\to\infty} diam_d(B_n) = 0$ and X is covered by $\{B_n : n \in \mathbb{N}\}$.

Menger conjectured that a metric space has this basis property if, and only if, the space is σ-compact. Let $\mathcal{O}_X$ denote the family of open covers of the space X. Hurewicz proved that a metric space has Menger's basis property if, and only if, it has the selection property $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$. Hurewicz did not settle Menger's conjecture, but formulated a related selection principle and made a similar conjecture about it:

> $\mathsf{U}_{fin}(\mathcal{A}, \mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n we have $B_n \subseteq A_n$ and $\{\cup B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

To state Hurewicz's conjecture, we introduce the following two notions: An open cover $\mathcal{U}$ of a space is a γ–cover if it is infinite, and each element of X is in all but finitely many elements of $\mathcal{U}$. Let Γ_X denote the family of all γ-covers of X. For a noncompact topological space let $\mathcal{O}_X^{nc}$ denote the collection of all open covers witnessing that the space is not compact. Hurewicz conjectured that a (non-compact) metric space is σ-compact if, and only if, it satisfies the selection property $\mathsf{U}_{fin}(\mathcal{O}_X^{nc}, \Gamma_X)$.

In 1928 Sierpiński pointed out in [82] that the Continuum Hypothesis, CH, implies that Menger's Conjecture is false: A set of real numbers is said to be a *Lusin* set if it is uncountable, but its intersection with every nowhere dense set of real numbers is countable. In 1914 N. Lusin showed that CH implies the existence of a Lusin set[1]. It is well-known that an uncountable σ-compact set of real numbers must contain an uncountable nowhere dense set. Thus, Lusin sets are not σ-compact. Sierpiński pointed out that Lusin sets have the Menger basis property. K. Gödel later proved the consistency of CH, and thus the consistency that Menger's conjecture is false. In 1988 Fremlin and Miller proved outright in [29] that the Menger Conjecture is false. One can show that Lusin sets do not have the Hurewicz property $\mathsf{U}_{fin}(\mathcal{O}_X^{nc}, \Gamma_X)$, and thus do not violate Hurewicz's Conjecture.

One can see similarly that CH implies that the Hurewicz Conjecture is false. A set of real numbers is said to be a *Sierpiński* set if it is uncountable, but its intersection with each Lebesgue measure zero set of real numbers is countable. In [83] Sierpiński showed that CH implies the existence of a Sierpiński set. Since an uncountable σ-compact set of real numbers contains an uncountable set of Lebesgue measure zero, a Sierpiński set cannot be σ-compact. It can be shown that Sierpiński sets have the Hurewicz selection property $\mathsf{U}_{fin}(\mathcal{O}_X^{nc}, \Gamma_X)$, and thus CH implies that the Hurewicz Conjecture is false. In 1995 it was proved outright in [43] that the Hurewicz Conjecture is false.

In 1938 F. Rothberger introduced a prototype of the following selection principle:

> $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n we have $B_n \in A_n$, and $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Rothberger's motivation for this selection principle was a conjecture of E. Borel. In [20] Borel defined a notion now called strong measure zero: A metric space (X, d) is said to have *strong measure zero* if there is for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers a partition $X = \cup_{n\in\mathbb{N}} X_n$ such that for each n we

[1] The same result was proved in 1913 by Mahlo.

have $diam_d(X_n) < \epsilon_n$. Borel conjectured that only countable sets of real numbers have strong measure zero. Sierpiński pointed out in [81] that Lusin sets have Borel's strong measure zero property, thus showing that the Continuum Hypothesis disproves Borel's Conjecture. Rothberger, in his study of Borel's property, defined the selection principle $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$. He observed that if a metric space has property $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$, then it has Borel's strong measure zero. The converse is not true. It was also noted that Lusin sets have Rothberger's property $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$.

The following selection principle is a natural companion of these classical ones:

> $\mathsf{S}_{ctbl}(\mathcal{A}, \mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n is countable, $B_n \subseteq A_n$ and $\cup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

Though it does not seem as delicate as the three classical selection principles just introduced, its presence gives an aesthetically satisfying way of describing the relationship between the classical selection principles and several classical properties in topology.

2 The Balkan selection principles

I call the following selection principles "Balkan selection principles" because they have recently received much attention from colleagues in the Balkans.

In [1] Addis and Gresham introduced a prototype of the following:

> $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n refines A_n[2] and B_n is pairwise disjoint, and $\cup\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Addis and Gresham specifically considered $\mathsf{S}_c(\mathcal{O}_X, \mathcal{O}_X)$ for topological spaces. One of the nice things about $\mathsf{S}_c(\mathcal{O}_X, \mathcal{O}_X)$ is that it includes some infinite dimensional spaces, but many theorems originally proved for finite dimensional spaces generalize to spaces having $\mathsf{S}_c(\mathcal{O}_X, \mathcal{O}_X)$. Moreover, $\mathsf{S}_c(\mathcal{O}_X, \mathcal{O}_X)$ is a selective version of the much older notion of screenability, introduced in the 1940's by Bing [17]: A space is *screenable* if there is for each open cover $\mathcal{U}$ a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n $\mathcal{V}_n$ is a pairwise disjoint refinement of $\mathcal{U}$ and $\cup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is an open cover of the space. $\mathsf{S}_c(\mathcal{O}_X, \mathcal{O}_X)$ is called *selective screenability*, and has been extensively studied by dimension theorists. The selection principle $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ was introduced in this general form in [3], where the study of $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ for other choices of $\mathcal{A}$ and $\mathcal{B}$ is initiated.

[2] This means: For each $U \in B_n$ there is a $V \in A_n$ with $U \subseteq V$.

Selective versions of several other topological properties seem interesting. In 1944 Dieudonné introduced the notion of paracompactness. The selective version of paracompactness is contained in the following selection principle which is considered in [6] (we now assume there is a topology on S):

> $\mathsf{S}_{lf}(\mathcal{A},\mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n refines A_n and B_n is locally finite, and $\cup\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

In [6] it is pointed out that by a theorem of Michael [51], a T_3–space is paracompact if, and only if, it has property $\mathsf{S}_{lf}(\mathcal{O}_X,\mathcal{O}_X)$.

The following is a selective version of metacompactness:

> $\mathsf{S}_{pf}(\mathcal{A},\mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n refines A_n and B_n is point finite, and $\cup\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

The following is a selective version of mesocompactness:

> $\mathsf{S}_{cf}(\mathcal{A},\mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n refines A_n and B_n is compact finite (meaning every compact set meets only finitely many elements of B_n), and $\cup\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

And the following is a strengthening of selective screenability:

> $\mathsf{S}_{d}(\mathcal{A},\mathcal{B})$ denotes: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n refines A_n and B_n is a discrete family, and $\cup\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Recall that a family $\mathcal{S}$ of subsets of X is *discrete* if, for each $x \in X$ there is a neighborhood U of x such that $|\{S \in \mathcal{S} : S \cap U \neq \emptyset\}| \leq 1$. Also note that if $\mathcal{S}$ is a discrete family of closed subsets of a space X, then for any subset $\mathcal{T}$ of $\mathcal{S}$, also $\cup\mathcal{T}$ is a closed subset of X.

3 The Morelia selection principles

S. Garcia-Ferreira and collaborators extensively studied the following variation on the classical selection principles for open covers. An open cover $\mathcal{U}$ for a space X is said to be an ω-cover if: $X \notin \mathcal{U}$, but for each finite set $F \subset X$ there is a $U \in \mathcal{U}$ with $F \subseteq U$. The symbol Ω_X denotes the collection of ω–covers of X.

A family $\mathcal{F}$ of subsets of the natural numbers $\mathbb{N}$ is a *filter* if:

(1) $\emptyset \not\in \mathcal{F}$;

(2) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq \mathbb{N}$, then $B \in \mathcal{F}$;

(3) $A \cap B \in \mathcal{F}$ whenever A and B are in $\mathcal{F}$.

The filter $\mathcal{F}$ is a *free filter* if $\cap\mathcal{F} = \emptyset$, and it is an *ultrafilter* if whenever $\mathbb{N} = A \cup B$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$. Free ultrafilters on $\mathbb{N}$ have a rich mathematical theory. The following selection properties were studied in [33] and [34]. Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$.

> $\gamma'_{\mathcal{F}}$: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of X there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n we have $U_n \in \mathcal{U}_n$, and for each $x \in X$, $\{n : x \in U_n\} \in \mathcal{F}$.

> $\gamma''_{\mathcal{F}}$: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n we have $\mathcal{V}_n \subset \mathcal{U}_n$ is finite, and there is a sequence $(U_n : n \in \mathbb{N})$ in $\cup_{m\in\mathbb{N}}\mathcal{V}_m$ such that for each $x \in X$ we have $\{n : x \in U_n\} \in \mathcal{F}$.

These are both selective versions of the following property also studied in [33] and [34]:

> $\gamma_{\mathcal{F}}$: For each ω-cover $\mathcal{U}$ of X there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n we have $U_n \in \mathcal{U}$, and for each $x \in X$, $\{n : x \in U_n\} \in \mathcal{F}$.

The latter in turn is related to the classical selection principles $\mathsf{S}_1(\cdot,\cdot)$, as will be seen below in Theorem 8.

4 The Bar-Ilan selection principles

The following type of selection principle has been around even longer than the classical selection principles, and in the spirit of Ramsey-theory asserts that any cover of specified kind has a subset with specified properties. Since the relevance of such principles to the study of other selection principles have been emphasized recently in Tsaban's research, I will refer to these types of selection principles as the Bar-Ilan selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of the infinite set S.

> $\binom{\mathcal{A}}{\mathcal{B}}$ means: For each $A \in \mathcal{A}$ there is a $B \subseteq A$ with $B \in \mathcal{B}$.

Examples of this abound in topology: For example,

- With $\mathsf{Fin}_X = \{\mathcal{U} \in \mathcal{O}_X : \mathcal{U} \text{ finite}\}$, $\binom{\mathcal{O}_X}{\mathsf{Fin}_X}$ denotes that X is compact;

- With $\mathcal{O}_X^{\aleph_0} = \{\mathcal{U} \in \mathcal{O}_X : \mathcal{U} \text{ countable}\}$, $\binom{\mathcal{O}_X}{\mathcal{O}_X^{\aleph_0}}$ denotes that X is Lindelöf;

- With $\Omega_X^{\aleph_0} = \{\mathcal{U} \in \Omega_X : \mathcal{U} \text{ countable}\}$, $\binom{\Omega_X}{\Omega_X^{\aleph_0}}$ denotes that X is (in the terminology of [36]) an ϵ–space;

- $\binom{\Omega_X}{\Gamma_X}$ denotes Gerlits and Nagy's γ-property.

- For $\mathcal{F}$ a free ultrafilter on $\mathbb{N}$, an open cover $\mathcal{U}$ of X is said to be an $\mathcal{F}$-cover if there is an enumeration $(U_n : n \in \mathbb{N})$ of $\mathcal{U}$ such that for each $x \in X$, $\{n : x \in U_n\} \in \mathcal{F}$. Let $\mathcal{O}_{X,\mathcal{F}}$ denote the open $\mathcal{F}$-covers of X. $\binom{\Omega_X}{\mathcal{O}_{X,\mathcal{F}}}$ denotes $\gamma_{\mathcal{F}}$ introduced before.

If instead of containment $\subseteq$ we use refinement $\prec$ as binary relation, as in the Balkan selection principles, we get that for a given cover of some kind there is a refinement with special properties. Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of the infinite set S. Then we define:

$\begin{bmatrix}\mathcal{A}\\ \mathcal{B}\end{bmatrix}$ means: For each $A \in \mathcal{A}$ there is a B refining A with $B \in \mathcal{B}$.

Examples of this also abound in topology:

- Let $\mathcal{O}_X^{\mathsf{lf}}$ denote the locally finite open covers of X. Then $\begin{bmatrix}\mathcal{O}_X\\ \mathcal{O}_X^{\mathsf{lf}}\end{bmatrix}$ denotes paracompactness of X.

- Let $\mathcal{O}_X^{\mathsf{pf}}$ denote the point finite open covers of X. Then $\begin{bmatrix}\mathcal{O}_X\\ \mathcal{O}_X^{\mathsf{pf}}\end{bmatrix}$ denotes metacompactness of X.

Some strong versions of these Bar-Ilan selection principles, namely splittability and countable representability, have been important in developing the theory. Let R be a binary relation on the family of subsets of the infinite set S:

> $\mathsf{Split}_R(\mathcal{A}, \mathcal{B})$ denotes: For each $A \in \mathcal{A}$ there are sets B_1, B_2 such that $B_1 \cap B_2 = \emptyset$, and $B_1 \, R \, A$ and $B_2 \, R \, A$, and B_1, $B_2 \in \mathcal{B}$.

In the special case when $R = \subset$, then $\mathsf{Split}_{\subset}(\mathcal{A},\mathcal{B})$ means that for each $A \in \mathcal{A}$ there are sets B_1, $B_2 \in \mathcal{B}$ with $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 \subset A$. Note that $\mathsf{Split}_{\subset}(\mathcal{A},\mathcal{B})$ implies $\binom{\mathcal{A}}{\mathcal{B}}$.

> $\mathsf{CDR}_R(\mathcal{A},\mathcal{B})$ denotes: For each sequence of elements $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, there is a sequence $B_n, n \in \mathbb{N}$ such that for each n we have $B_n \in \mathcal{B}$ and $B_n \, R \, A_n$ and for all $m \neq n$ we have $B_m \cap B_n = \emptyset$.

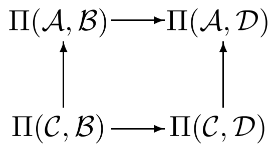
In the special case when $R = \subset$, then $\mathsf{CDR}_{\subset}(\mathcal{A},\mathcal{B})$ means that for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$, each $B_n \in \mathcal{B}$ each $B_n \subseteq A_n$, and for $m \neq n$ we have $B_m \cap B_n = \emptyset$. Note that $\mathsf{CDR}_R(\mathcal{A},\mathcal{B})$ implies $\mathsf{Split}_R(\mathcal{A},\mathcal{B})$.

5 Monotonicity Laws

Let Π be one of our selection principles. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be families of subsets of an infinite set S. The following are some of the most basic observations about these selection principles:

- If $\mathcal{A} \subseteq \mathcal{C}$, then $\Pi(\mathcal{C},\mathcal{B}) \Rightarrow \Pi(\mathcal{A},\mathcal{B})$.
- If $\mathcal{B} \subseteq \mathcal{D}$, then $\Pi(\mathcal{A},\mathcal{B}) \Rightarrow \Pi(\mathcal{A},\mathcal{D})$.

We say that Π is antimonotonic in the first parameter and monotonic in the second. These relationships are indicated by the following diagram, where an arrow denotes an implication in the direction of the arrow:



When more than one selection principle, say Π and Ψ, are considered, it is natural to inquire if they are related. For example, it could be that $\Pi(\mathcal{A},\mathcal{B}) \Rightarrow \Psi(\mathcal{A},\mathcal{B})$. When such relationships exist, the monotonicity diagram can be extended to also indicate these relationships. Typically, the investigation of such relationships between different selection principles is one of the necessary items on the agenda when a new selection principle is introduced. The following implications are examples of this:

$$\mathsf{S}_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\mathcal{B}) \Rightarrow \binom{\mathcal{A}}{\mathcal{B}} \Rightarrow \left[\begin{array}{c}\mathcal{A}\\ \mathcal{B}\end{array}\right]$$

$$\mathsf{S}_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_d(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_c(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_{lf}(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_{cf}(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_{pf}(\mathcal{A},\mathcal{B}) \Rightarrow \binom{\mathcal{A}}{\mathcal{B}} \Rightarrow \left[\begin{array}{c}\mathcal{A}\\ \mathcal{B}\end{array}\right]$$

6 Cancellation Laws

For selection principles Π and Ψ, and families $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the implication

$$\Pi(\mathcal{A},\mathcal{B}) \text{ and } \Psi(\mathcal{B},\mathcal{C}) \Rightarrow \Pi(\mathcal{A},\mathcal{C})$$

is said to be a *cancellation law*. Cancellation laws have played a fundamental role in proving some of the basic results in the study of selection principles in topological spaces. In [89] Tsaban points out the following useful "cancellation laws":

1 Theorem (Cancellation Laws I). *For collections $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of covers of a set S*

(1) $\binom{\mathcal{A}}{\mathcal{B}}$ *and* $\binom{\mathcal{B}}{\mathcal{C}} \Rightarrow \binom{\mathcal{A}}{\mathcal{C}}$

(2) $\mathsf{S}_1(\mathcal{B},\mathcal{C})$ *and* $\binom{\mathcal{A}}{\mathcal{B}} \Rightarrow \mathsf{S}_1(\mathcal{A},\mathcal{C})$.

(3) $\mathsf{S}_{fin}(\mathcal{B},\mathcal{C})$ *and* $\binom{\mathcal{A}}{\mathcal{B}} \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\mathcal{C})$.

(4) $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ *and* $\binom{\mathcal{B}}{\mathcal{C}} \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\mathcal{C})$.

And if $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$, then the converse implications hold in 1 to 4.

Analogous cancellation laws are true for the Balkan selection principles:

2 Theorem (Cancellation Laws II). *For collection $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of covers of a set S and for $\Pi \in \{\mathsf{S}_c, \mathsf{S}_d, \mathsf{S}_{lf}, \mathsf{S}_{pf}, \mathsf{S}_{cf}\}$ we have*

(1) $\left[\begin{array}{c}\mathcal{A}\\ \mathcal{B}\end{array}\right]$ *and* $\left[\begin{array}{c}\mathcal{B}\\ \mathcal{C}\end{array}\right] \Rightarrow \left[\begin{array}{c}\mathcal{A}\\ \mathcal{C}\end{array}\right]$

(2) $\Pi(\mathcal{B},\mathcal{C})$ *and* $\left[\begin{array}{c}\mathcal{A}\\ \mathcal{B}\end{array}\right] \Rightarrow \Pi(\mathcal{A},\mathcal{C})$.

(3) $\Pi(\mathcal{B},\mathcal{C})$ *and* $\left[\begin{array}{c}\mathcal{A}\\ \mathcal{B}\end{array}\right] \Rightarrow \Pi(\mathcal{A},\mathcal{C})$.

(4) $\Pi(\mathcal{A},\mathcal{B})$ *and* $\left[\begin{array}{c}\mathcal{B}\\ \mathcal{C}\end{array}\right] \Rightarrow \Pi(\mathcal{A},\mathcal{C})$.

And if $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$*, then the converse implications hold in 1 to 4.*

7 Classes of open covers for spaces

In this survey we will emphasize the cases where $\mathcal{A}$ or $\mathcal{B}$ are families of open sets which have certain covering properties. Where appropriate, we will briefly mention other examples of topologically significant families $\mathcal{A}$ and $\mathcal{B}$ which have been investigated. So far we mentioned only the following classes of open covers: $\mathcal{O}_X$, $\mathcal{O}_X^{nc}$, Γ_X, Ω_X, $\mathcal{O}_X^{lf}$ and $\mathcal{O}_X^{pf}$. There is a wide variety of important classes of open covers considered in various branches of topology. In this section we catalogue some of these. There are two criteria in use for defining or classifying classes of open covers, namely:

(1) What type of objects are covered by members of the cover and

(2) the manner in which these objects are covered by members of the cover.

The type of objects covered.

We use a general schema introduced in [84]. For space X, a family $\mathcal{K}$ of proper subsets of X is said to be a *Telgársky family* if it has the following three properties:

(1) $A \in \mathcal{K} \Rightarrow A$ is a closed subset of X;

(2) For each $x \in X$ also $\{x\} \in \mathcal{K}$, and

(3) If $A \in \mathcal{K}$ and $B \subset A$ is nonempty and closed, then $B \in \mathcal{K}$.

Let $\mathcal{K}$ be a Telgársky family of subsets of X. The open cover $\mathcal{U}$ of X is said to be a $\mathcal{K}$-cover if there is for each $A \in \mathcal{K}$ a $U \in \mathcal{U}$ such that $A \subseteq U$.

- $\mathcal{DK}$ denotes the collection of subsets of X which can be represented as a union of a discrete family of sets in $\mathcal{K}$.
- $\mathcal{O}(\mathcal{K})$ denotes the collection of $\mathcal{K}$-covers of X.

If $\mathcal{K}$ is a Telgársky family then $\mathcal{K} \subseteq \mathcal{DK}$, and also $\mathcal{DK}$ is a Telgársky family. Here are typical examples of Telgársky families:

- $[X]^1$: The one-element subsets of T_1–space X;
- $[X]^{<\aleph_0}$: The finite subsets of T_1–space X;
- κ: The proper, compact subsets of X;
- č: The closed, Čech-complete proper subsets of X;
- d: The closed, proper, discrete subsets of X;
- dim_n: The closed, proper subsets of X which are normal (in the relative topology) and of covering dimension $\le n$.

The manner in which the objects are covered.

Let $\mathcal{K}$ be a Telgársky family. Then an open cover $\mathcal{U}$ for X is a:

- large $\mathcal{K}$-cover if: For each $C \in \mathcal{K}$ the set $\{U \in \mathcal{U} : C \subset U\}$ is infinite.
- $\tau_{\mathcal{K}}$-cover if it is a large $\mathcal{K}$-cover and for each C and D in $\mathcal{K}$, $\{U \in \mathcal{U} : C \subset U \text{ and } D \not\subset U\}$ is finite, or $\{U \in \mathcal{U} : D \subset U \text{ and } C \not\subset U\}$ is finite.
- $\tau^*_{\mathcal{K}}$-cover if it is a large $\mathcal{K}$-cover and there is for each $C \in \mathcal{K}$ an infinite set $A_C \subset \{U \in \mathcal{U} : C \subset U\}$ such that whenever D and E are in $\mathcal{K}$, then either $A_D \setminus A_E$ is finite, or $A_E \setminus A_D$ is finite.
- $\gamma_{\mathcal{K}}$-cover if it is a large $\mathcal{K}$-cover and for each $C \in \mathcal{K}$ the set $\{U \in \mathcal{U} : C \not\subseteq U\}$ is finite.
- $\omega_{\mathcal{K}}$-cover if it is a large $\mathcal{K}$-cover and for each finite set $\mathcal{F} \subset \mathcal{K}$ there is a set $U \in \mathcal{U}$ with $\cup\mathcal{F} \subset U$.

Observe that every $\gamma_{\mathcal{K}}$-cover is a $\tau_{\mathcal{K}}$-cover and each $\tau_{\mathcal{K}}$-cover is a $\tau^*_{\mathcal{K}}$-cover, and every $\tau^*_{\mathcal{K}}$-cover is an $\omega_{\mathcal{K}}$-cover, and every $\omega_{\mathcal{K}}$-cover is a large $\mathcal{K}$ cover.

In the case when $\mathcal{K} = [X]^1$ it is customary, and we will follow this custom, to omit the subscript $\mathcal{K}$ from the symbols above. If we let T_X denote the τ-covers of X, and T^*_X the τ^*-covers of X, then this observation is, in symbolic form:

$$\Gamma_X \subset \mathsf{T}_X \subset \mathsf{T}^*_X \subset \Omega_X \subset \Lambda_X \subset \mathcal{O}_X.$$

γ-covers were introduced in [65], ω-covers were introduced in [36], and τ-covers were introduced in [88].

Groupability

Some topologically significant families have combinatorial properties induced by the structure of the underlying space. Groupability is one of these combinatorial properties that seems important in the context of selection principles. Let $\mathcal{K}$ be a Telgársky family. Here are four types of groupability for open $\mathcal{K}$ covers:

An open cover $\mathcal{U}$ of a space X is

- $\gamma_{\mathcal{K}}$-groupable if there is a partition $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{F}_n$ such that each $\mathcal{F}_n$ is finite, for $m \neq n$ we have $\mathcal{F}_m \cap \mathcal{F}_n = \emptyset$, and for each $C \in \mathcal{K}$, for all but finitely many n there is an $F \in \mathcal{F}_n$ with $C \subseteq F$.

- $\tau_{\mathcal{K}}$-groupable if there is a partition $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{F}_n$ such that each $\mathcal{F}_n$ is finite, for $m \neq n$ we have $\mathcal{F}_m \cap \mathcal{F}_n = \emptyset$, and for all C and D in $\mathcal{K}$ either $\{n \in \mathbb{N} : (\exists F \in \mathcal{F}_n)(C \subseteq F) \text{ and } (\nexists F \in \mathcal{F}_n)(D \subseteq F)\}$ is finite or else $\{n \in \mathbb{N} : (\exists F \in \mathcal{F}_n)(D \subseteq F) \text{ and } (\nexists F \in \mathcal{F}_n)(C \subseteq F)\}$ is finite.

- $\tau^*_{\mathcal{K}}$-groupable if there is a partition $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{F}_n$ such that each $\mathcal{F}_n$ is finite, for $m \neq n$ we have $\mathcal{F}_m \cap \mathcal{F}_n = \emptyset$, and for each $C \in \mathcal{K}$ there is an infinite set $A_C \subset \{n \in \mathbb{N} : (\exists F \in \mathcal{F}_n)(C \subseteq F)\}$ such that whenever D and E are elements of $\mathcal{K}$, then either $A_D \setminus A_E$ is finite, or else $A_E \setminus A_D$ is finite.

- $\omega_{\mathcal{K}}$-groupable if there is a partition $\mathcal{U} = \cup_{n<\infty}\mathcal{F}_n$ where each $\mathcal{F}_n$ is finite and for $m \neq n$, $\mathcal{F}_m \cap \mathcal{F}_n = \emptyset$, and for each finite subset $\mathcal{E}$ of $\mathcal{K}$ there is an n with $(\forall E \in \mathcal{E})(\exists F \in \mathcal{F}_n)(E \subseteq F)$.

And when $\mathcal{K} = [X]^1$ it is customary, and we will follow this custom here, to leave off the subscript $\mathcal{K}$ in the notation just introduced.

8 Implications among selection principles

Which of these families of open covers of spaces are related through selection principles of the form $\Pi(\mathcal{A}, \mathcal{B})$? We briefly survey some results in connection with the following instances of this general question:

- Scenario 1: Given a space X, and families $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of subsets of X, and selection principles Π and Φ. Is it true that

$$(X \models \Pi(\mathcal{A}, \mathcal{B})) \Rightarrow (X \models \Phi(\mathcal{C}, \mathcal{D}))?$$

- Scenario 2: Given a space X, and families $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of subsets of X, and selection principles Π and Φ. Is it true that

$$(X \models \Pi(\mathcal{A}, \mathcal{B})) \Rightarrow (X \models \Pi(\mathcal{C}, \mathcal{D}))?$$

- Scenario 3: Given a space X and families $\mathcal{A}$ and $\mathcal{B}$, consider a space $\mathsf{S}(X)$ constructed from X in some specific way, as well as families $\mathcal{C}$ and $\mathcal{D}$ of subsets of $\mathsf{S}(X)$, and selection principles Π and Φ. Is either of the following implications true?

$$(X \models \Pi(\mathcal{A},\mathcal{B})) \Rightarrow (\mathsf{S}(X) \models \Phi(\mathcal{C},\mathcal{D}))?$$

$$(\mathsf{S}(X) \models \Phi(\mathcal{C},\mathcal{D})) \Rightarrow (X \models \Pi(\mathcal{A},\mathcal{B}))?$$

Here we will give a small sample of results from Scenario 1 and from Scenario 2, and we will return to Scenario 3 later under the title *Translation to hyperspaces.*

Scenario 1

The following theorem is really just an observation which shows that some classical properties can be naturally formulated as selection principles.

3 Theorem. *For a space X:*

(1) $\binom{\mathcal{O}_X}{\mathcal{O}_X^{\aleph_0}} \Leftrightarrow \mathsf{S}_{ctbl}(\mathcal{O}_X,\mathcal{O}_X)$.

(2) $\binom{\Omega_X}{\Omega_X^{\aleph_0}} \Leftrightarrow \mathsf{S}_{ctbl}(\Omega_X,\Omega_X)$.

But some effort is needed to prove the following two results relating the Bar-Ilan selection principles to classical selection principles. These results appear respectively in [36] and [89]:

4 Theorem. *For topological space X:*

(1) $\binom{\Omega_X}{\Gamma_X} \Leftrightarrow \mathsf{S}_1(\Omega_X,\Gamma_X)$. **(Gerlits-Nagy)**

(2) $\binom{\Omega_X}{\mathsf{T}_X} \Rightarrow \mathsf{S}_{fin}(\Gamma_X,\mathsf{T}_X)$. **(Tsaban)**

There are several useful "decompositions" of selection principles of one type into two selection principles. These results are often useful in obtaining from known results about one of the two composing selection principles, results about the composite selection principle. This is especially the case with obtaining Ramsey-theoretic and game-theoretic characterizations, as will be seen later. Here are some results of this kind:

5 Theorem. *For a space X:*

(1) $\mathsf{S}_1(\mathsf{T}_X,\Gamma_X) \Leftrightarrow \mathsf{S}_{fin}(\Gamma_X,\mathsf{T}_X)$ *and* $\binom{\mathsf{T}_X}{\Gamma_X}$. [89]

(2) $\mathsf{S}_1(\Omega_X,\Gamma_X) \Leftrightarrow \mathsf{S}_{fin}(\Omega_X,\mathsf{T}_X)$ *and* $\binom{\mathsf{T}_X}{\Gamma_X}$. [89]

(3) $\mathsf{S}_{fin}(\Omega_X,\mathsf{T}_X) \Leftrightarrow \mathsf{S}_{fin}(\mathsf{T}_X,\Omega_X)$ *and* $\binom{\Omega_X}{\mathsf{T}_X}$. [89]

Some of the Balkan selection principles are related as follows to classical selection principles:

6 Theorem (Various authors). *For topological space X:*

(1) For $\Pi \in \{\mathsf{S}_{fin}, \mathsf{S}_c, \mathsf{S}_d, \mathsf{S}_{lf}, \mathsf{S}_{pf}, \mathsf{S}_{cf}\}$ *and* $\mathcal{A} \in \{\Omega, \Gamma\}$*:*
$\Pi(\mathcal{A},\Gamma) \Leftrightarrow \mathsf{S}_1(\mathcal{A},\Gamma)$.

(2) For Π *any of* $\{\mathsf{S}_{lf}, \mathsf{S}_{pf}, \mathsf{S}_{cf}\}$*:* $\Pi(\Omega,\Omega) \Leftrightarrow \mathsf{S}_{fin}(\Omega,\Omega)$.

(3) For Π *any of* $\{\mathsf{S}_c, \mathsf{S}_d\}$*:* $\Pi(\Omega,\Omega) \Leftrightarrow \mathsf{S}_1(\Omega,\Omega)$.

Moreover, the classical selection principles are also interrelated. Here are some basic results in this connection.

7 Theorem. *For X a Lindelöf space:*

(1) $\mathsf{U}_{fin}(\mathcal{O}_X^{nc},\Gamma_X) \Leftrightarrow \mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X^{\gamma-gp}) \Leftrightarrow \mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X)$ *and* $\binom{\Lambda_X}{\Lambda_X^{\gamma-gp}}$. [47][3]

(2) $\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\gamma-gp}) \Leftrightarrow \mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X)$ *and* $\binom{\Lambda_X}{\Lambda_X^{\gamma-gp}}$. [47]

(3) $\mathsf{U}_{fin}(\mathcal{O}_X^{nc},\Omega_X) \Leftrightarrow \mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X^{\omega-gp}) \Leftrightarrow \mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X)$ *and* $\binom{\Lambda_X}{\Lambda_X^{\omega-gp}}$. [5]

(4) $\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\omega-gp}) \Leftrightarrow \mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X)$ *and* $\binom{\Lambda_X}{\Lambda_X^{\omega-gp}}$. [5]

(5) $\mathsf{U}_{fin}(\mathcal{O}_X^{nc},\mathcal{O}_X) \Leftrightarrow \mathsf{S}_{fin}(\mathcal{O}_X,\mathcal{O}_X)$. [65]

Also the Morelia selection principles are related to the classical selection principles. Free ultrafilter $\mathcal{F}$ on the natural numbers is said to be a

- Q-point if there is for each partition $\mathbb{N} = \cup_{n\in\mathbb{N}} I_n$ of the natural numbers into finite sets a set $S \in \mathcal{F}$ such that for each n we have $|S \cap I_n| \leq 1$.

[3] In [93] it is shown that this in turn is equivalent to $\binom{\Lambda_X}{\Lambda_X^{\gamma-gp}}$.

- P-point if for each partition $\mathbb{N} = \cup_{n\in\mathbb{N}} A_n$ of the natural numbers either there is an n with $A_n \in \mathcal{F}$, or else there is a $B \in \mathcal{F}$ such that for each n $B \cap A_n$ is finite.
- *selective* if it is both a Q–point and a P–point.

According to [36] a space is an ϵ-space if each ω-cover of it contains a countable subset which is an ω-cover of the space. The following results describe a relation among Morelia selection principles and the classical selection principles.

8 Theorem. *For an ϵ-space X and free ultrafilter $\mathcal{F}$ on $\mathbb{N}$:*

(1) $\gamma''_{\mathcal{F}} \Leftrightarrow \binom{\Omega_X}{\mathcal{O}_{X,\mathcal{F}}}$ *and* $\mathsf{S}_{fin}(\Omega_X, \Omega_X)$ [34] $\Leftrightarrow \mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$.

(2) *If $\mathcal{F}$ is a* P*–point:* $\gamma''_{\mathcal{F}} \Leftrightarrow \binom{\Omega_X}{\mathcal{O}_{X,\mathcal{F}}} \Leftrightarrow \mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$. [35]

(3) *If $\mathcal{F}$ is a* Q*-point:* $\gamma'_{\mathcal{F}} \Leftrightarrow \binom{\Omega_X}{\mathcal{O}_{X,\mathcal{F}}}$ *and* $\mathsf{S}_1(\Omega_X, \Omega_X) \Leftrightarrow \mathsf{S}_1(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$. [35]

(4) *If $\mathcal{F}$ is selective:* $\gamma'_{\mathcal{F}} \Leftrightarrow \binom{\Omega_X}{\mathcal{O}_{X,\mathcal{F}}} \Leftrightarrow \mathsf{S}_1(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$. [35]

Some selection principles can be "factored" into two other selection principles. For example:

9 Theorem. *For a space X:*

(1) *[54]* $(X \models \mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp})) \Leftrightarrow$
$(X \models \mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\gamma-gp}))$ *and* $(X \models \mathsf{S}_1(\Omega_X, \mathcal{O}_X))$

(2) *[91]* $(X \models \mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\omega-gp})) \Leftrightarrow$
$(X \models \mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp}))$ *and* $(X \models \mathsf{S}_1(\Omega_X, \mathcal{O}_X))$

Scenario 2

The following result is fundamental in relating some selection principles to Ramsey-theoretic principles (introduced later below):

10 Theorem. *Let X be a Lindelöf space.*

(1) *For* Π *any of* $\{\mathsf{S}_1, \mathsf{S}_{fin}, \mathsf{S}_c, \mathsf{S}_d, \mathsf{S}_{lf}, \mathsf{S}_{pf}, \mathsf{S}_{cf}\}$*:* $\Pi(\mathcal{O}_X, \mathcal{O}_X) \Leftrightarrow \Pi(\Omega_X, \mathcal{O}_X)$.

11 Theorem. *The following are equivalent for Lindelöf space X:*

$$\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X), \quad \mathsf{S}_{fin}(\Lambda_X, \mathcal{O}_X), \quad \mathsf{S}_{fin}(\Lambda_X, \Lambda_X), \quad \mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X),$$
$$\mathsf{S}_{fin}(\Omega_X, \Lambda_X), \mathsf{S}_{fin}(\Gamma_X, \mathcal{O}_X), \mathsf{S}_{fin}(\Gamma_X, \Lambda_X).$$

12 Theorem. *The following are equivalent for Lindelöf space X:*

$$\mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X),\quad \mathsf{S}_1(\Lambda_X,\mathcal{O}_X),\quad \mathsf{S}_1(\Lambda_X,\Lambda_X),\quad \mathsf{S}_1(\Omega_X,\mathcal{O}_X),$$
$$\mathsf{S}_1(\Omega_X,\Lambda_X),\quad \mathsf{S}_1(\Gamma_X,\mathcal{O}_X),\mathsf{S}_1(\Gamma_X,\Lambda_X).$$

Monotonicity considerations give the following implications:

$$\begin{array}{lcl}
\mathsf{S}_1(\Omega_X,\Gamma_X) & \Rightarrow & \mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\gamma-gp})\\
\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\gamma-gp}) & \Rightarrow & \mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\tau-gp})\\
\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\tau-gp}) & \Rightarrow & \mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\tau^*-gp})\\
\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\tau^*-gp}) & \Rightarrow & \mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\omega-gp})\\
\mathsf{S}_1(\Omega_X,\Omega_X) & \Rightarrow & \mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\omega-gp})\\
\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\omega-gp}) & \Rightarrow & \mathsf{S}_1(\Omega_X,\mathcal{O}_X).
\end{array}$$

It was shown in [54] that the selection principle $\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\gamma-gp})$ is equivalent to the property (*) introduced by Gerlits and Nagy in [36]. By a result of Weiss, $\mathsf{S}_1(\Omega_X,\Gamma_X)\Rightarrow\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\gamma-gp})\Rightarrow\mathsf{S}_1(\Omega_X,\Omega_X)$. And Tsaban proved $\mathsf{S}_1(\Omega_X,\mathcal{O}_X)\not\Rightarrow\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\omega-gp})$.

9 Topological constructions

We now give a brief survey of some results regarding how selection principles are preserved under various topological constructions.

Products in topology

Finite Powers

Here is a small sample of results that certain selection properties are preserved by finite powers. In 8 $\mathcal{F}$ is a free ultrafilter, and in 9 a semi-Q–point.

	Property	Powers	Source
1	$\mathsf{S}_1(\Omega_X,\Gamma_X)$	Yes	[43]
2	$\mathsf{S}_1(\Omega_X,\Omega_X)$	Yes	[64]
3	$\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\gamma-gp})$	Yes	T. Weiss
4	$\mathsf{S}_1(\Omega_X,\Omega_X^{\gamma-gp})$	Yes	[47]
5	$\mathsf{S}_{fin}(\Omega_X,\Omega_X)$	Yes	[43]
6	$\mathsf{S}_{fin}(\Omega_X,\Omega_X^{\gamma-gp})$	Yes	[47]
7	$\binom{\Omega_X}{\mathcal{O}_{X,\mathcal{F}}}$	Yes	[33]
8	$\mathsf{S}_{fin}(\Omega_X,\mathcal{O}_{X,\mathcal{F}})$	Yes	[35]
9	$\mathsf{S}_1(\Omega_X,\mathcal{O}_{X,\mathcal{F}})$	Yes	[35]
10	$\mathsf{S}_c(\mathcal{O}_X,\mathcal{O}_X)$	No	[56]
11	$\mathsf{S}_{lf}(\mathcal{O}_X,\mathcal{O}_X)$	No	[57]
12	$\mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X)$	No	Sierpiński
13	$\mathsf{S}_{fin}(\mathcal{O}_X,\mathcal{O}_X)$	No	[43]
14	$\mathsf{S}_1(\Gamma_X,\Gamma_X)$	No	[43]
15	$\mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X^{\gamma-gp})$	No	[43]

Some negative results about finite powers are more specifically like this:

13 Theorem (Przymusiński). *Let $k \le m$ be positive integers. There is a T_3–space X such that:*

(1) $X^n \models \mathsf{S}_{lf}(\mathcal{O},\mathcal{O})) \Leftrightarrow n < k$ and

(2) X^n is normal $\Leftrightarrow n < m$.

14 Theorem (Pol). *Assume* CH. *Then there is for each n a separable metrizable space X such that:*

- *$X^{n+1} \models \mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ and*
- *$X^n \models \mathsf{S}_c(\mathcal{O},\mathcal{O})$, but*
- *$X^{n+1} \not\models \mathsf{S}_c(\mathcal{O},\mathcal{O})$.*

Finite Products: Preserving factors

Consider some property $\mathtt{P}$. Suppose $\mathtt{P}$ is not preserved by products between two spaces, each having the property $\mathtt{P}$. We shall say that a space X is $\mathtt{P}$-*preserving* (or a $\mathtt{P}$-*preserving factor*) if for each space Y with property $\mathtt{P}$, also $X \times Y$ has property $\mathtt{P}$. It is part of a long tradition in product theory to ask to characterize the P-preserving spaces. This can be a difficult question even for very concrete spaces. A famous example of this is Dowker's Conjecture: The unit interval I is normality preserving. Here are a few results of this kind for selection principles:

	Property	Property - preserving class	Source
1	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	T. Weiss
2	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X)$	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	T. Weiss
3	$\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$	σ-compact	Folklore
4	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	σ-compact	Folklore

Subspaces

Let X be a space and let Y be a subset of X. Then Y inherits a topology from X, the relative topology. It may be that X does not have some selection property, but Y does, or that Y does not have a selection property that X has. It is useful to know under what circumstances subspaces inherit selection properties of their superspaces, and under what circumstances superspaces have properties because appropriate subspaces have these properties. There are usually two types of reasons that influences whether a subspace of a space would have a selection property in the relative topology: topological reasons, or combinatorial reasons.

Topological reasons

There is a standard way to construct from a cover $\mathcal{U}$ of a closed subspace Y of a space X, consisting of sets open in the relative topology of Y, an open cover $\mathcal{V}$ of X such that $\mathcal{U} = \{Y \cap V : V \in \mathcal{V}\}$. Namely, for each $U \in \mathcal{U}$ choose an open subset V_U of X such that $U = Y \cap V_U$. Then put $\mathcal{V} = \{V_U \cup X \setminus Y : U \in \mathcal{U}\}$. It is evident that if $\mathcal{U}$ is a member of

- Γ_Y then $\mathcal{V}$ is a member of Γ_X;
- Ω_Y then $\mathcal{V}$ is a member of Ω_X;
- $\mathcal{O}_Y^{\mathsf{nc}}$ then $\mathcal{V}$ is a member of $\mathcal{O}_X^{\mathsf{nc}}$;
- Λ_Y then $\mathcal{V}$ is a member of Λ_X.

Similarly, groupability properties of $\mathcal{U}$ on Y transfer to the same groupability properties of $\mathcal{V}$ on X.

Thus, if Y is a closed subspace of X and X has one of the following properties, then so does Y:

$$\mathsf{S}_1(\Omega_X, \Gamma_X), \quad \mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp}), \quad \mathsf{S}_1(\Omega_X, \mathcal{O}_X), \\ \mathsf{S}_1(\Gamma_X, \Gamma_X), \quad \mathsf{S}_1(\Gamma_X, \Omega_X), \quad \mathsf{S}_1(\Gamma_X, \mathcal{O}_X).$$

But some local properties like local finiteness, point-finiteness, compact-finiteness of $\mathcal{V}$ on Y may not transfer to the same properties for $\mathcal{V}$ on X.

Combinatorial reasons: Critical cardinals

Several "small" cardinal numbers which codify certain combinatorial principles have been studied by set theorists. These are very useful also in the study of selection principles. It can be expected also that the study of selection principles will lead to new combinatorial principles and their corresponding cardinal numbers. We now give an example of how small cardinals are related to selection principles.

Suppose some, but not all, subspaces of a space have a selection property $\Pi(\mathcal{A}, \mathcal{B})$. Define:

$$\mathsf{non}_X(\Pi(\mathcal{A}, \mathcal{B})) = \min\{|Y| : Y \subset X \text{ and } Y \not\models \Pi(\mathcal{A}, \mathcal{B})\}.$$

This is the *critical cardinal* of the selection principle $\Pi(\mathcal{A}, \mathcal{B})$ for the space X.

Here is a few results in this connection. Here, the symbol $\mathbb{R}$ denotes the space of real numbers with the usual Euclidean topology.

$\mathsf{non}_{\mathbb{R}}$(Property)	Property	Source
$\mathfrak{p}$	$\mathsf{S}_1(\Omega_X,\Gamma_X)$	[30]
	$\mathsf{S}_{fin}(\Omega_X,\mathsf{T}_X)$	[89]
$\max\{\mathfrak{s},\mathfrak{b}\}$	$\mathsf{S}_1(\mathsf{T}_X,\Gamma_X)$	[77]
$\mathfrak{r}$	$\mathsf{U}_{fin}(\Gamma_X,\mathsf{T}_X)$	[89]
	$\mathsf{split}(\Lambda_X,\Lambda_X)$	[43]
$\mathfrak{u}$	$\mathsf{split}(\Omega_X,\Omega_X)$	[43]
$\mathsf{add}(\mathcal{M})$	$\mathsf{S}_1(\Omega_X,\mathcal{O}_X^{\gamma-gp})$	[54]
$\mathsf{cov}(\mathcal{M})$	$\mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X)$	[29]
	$\mathsf{S}_1(\Omega_X,\Omega_X)$	[43]
$\mathfrak{b}$	$\mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X^{\gamma-gp})$	[42]
	$\mathsf{S}_1(\Gamma_X,\Gamma_X)$	[43]
$\mathfrak{d}$	$\mathsf{S}_{fin}(\mathcal{O}_X,\mathcal{O}_X)$	[42]
	$\mathsf{S}_{fin}(\Omega_X,\Omega_X)$	[43]
	$\mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X^{\omega-gp})$	[43]
	$\mathsf{S}_{fin}(\Omega_X,\mathcal{O}_X^{\omega-gp})$	[43]
	$\mathsf{S}_{fin}(\Gamma_X,\Omega_X)$	[43]
	$\mathsf{S}_1(\Gamma_X,\Omega_X)$	[43]
	$\mathsf{S}_1(\Gamma_X,\Lambda_X)$	[43]
	$\mathsf{S}_{fin}(\mathsf{T}_X,\Omega_X)$	[89]

Unions

There are several ways to construct new subspaces from given subspaces of a fixed space. One could take set-theoretic unions or intersections, or other Boolean combinations of these subsets. Several factors can influence whether the resulting subspace would have a selection property because the initial subspaces have some (possibly different) selection property: for example, in the case of unions, the subspaces might form a chain under set-theoretic inclusion, or the sets in the family whose union is being constructed have special topological properties, and so on.

Combinatorial reasons: The additivity cardinal

If not all unions of subspaces of a space, each subspace having property $\Pi(\mathcal{A},\mathcal{B})$, have $\Pi(\mathcal{A},\mathcal{B})$, define $\mathsf{add}_X(\Pi(\mathcal{A},\mathcal{B}))$ to be:

$$\min\{|\mathcal{U}| : (\mathcal{U}\subseteq\mathcal{P}(X))((\cup\mathcal{U}\not\models\Pi(\mathcal{A},\mathcal{B})\&(\forall U\in\mathcal{U})(U\models\Pi(\mathcal{A},\mathcal{B})))\}.$$

This is the *additivity cardinal* of the selection principle $\Pi(\mathcal{A},\mathcal{B})$ in X.

In [76] it was shown for subsets of the real line that the Continuum Hypothesis implies that $\mathsf{add}_{\mathbb{R}}(\mathsf{S}_1(\Omega_X,\Omega_X)) = 2$. Indeed, the example of subsets X and Y of ${}^{\omega}\mathbb{Z}$ constructed there have the property that for $h \in^{\omega} \mathbb{Z}$ there are f, $g \in X \cup Y$ such that for all but finitely many n we have $h(n) \leq \max\{f(n), g(n)\}$. This implies that indeed $X \cup Y$ does not have property $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp})$. In turn, this implies that for each $\Pi(\mathcal{A},\mathcal{B})$ such that $\mathsf{S}_1(\Omega_X,\Omega_X) \Rightarrow \Pi(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp})$, we have under this hypothesis that $\mathsf{add}_{\mathbb{R}}(\Pi(\mathcal{A},\mathcal{B})) = 2$. This result was rediscovered in [12].

But a nice and surprising result from [12] shows that it can be independent of the usual axioms of set theory whether the additivity cardinal is finite. The authors prove

15 Theorem. *For X a set of real numbers:*

(1) **(NCF)** $\Leftrightarrow \max\{\mathfrak{b},\mathfrak{g}\} \leq \mathsf{add}(\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp}))$.

(2) If $\mathfrak{u} < \mathfrak{g}$ *then* $\mathsf{add}(\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp}) = \mathfrak{c}$.

In particular the authors prove that if $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, then the additivity number is 2.

Computing the additivity cardinals for selection principles even for subspaces of the real line seem difficult and mostly upper- and lower-bounds are known. Here is a small sample of some exact results in this connection:

add(Property)	Property	Source
$\mathfrak{t}$	$\mathsf{S}_1(\mathsf{T}_X,\Gamma_X)$	[12], Th. 3.5
$\mathfrak{b}$	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	[12], Th. 2.8

There are several selection principles for which the exact value of the additivity cardinal is not known, but some estimates are known. Here are some results in this connection:

Property	lower bound	upper bound	Source
$\mathsf{S}_{fin}(\mathcal{O}_X,\mathcal{O}_X)$	$\mathfrak{b}$	$\mathsf{cf}(\mathfrak{d})$	[12]
$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp})$		$\mathsf{cf}(\mathfrak{d})$	[12]
$\mathsf{S}_{fin}(\Gamma_X,\Omega_X)$		$\mathsf{cf}(\mathfrak{d})$	[12]
$\mathsf{S}_1(\Gamma_X,\Omega_X)$		$\mathsf{cf}(\mathfrak{d})$	[12]
$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\gamma-gp})$		$\mathfrak{b}$	
$\mathsf{S}_1(\Gamma_X,\Gamma_X)$	$\mathfrak{h}$	$\mathfrak{b}$	
$\mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X)$	$\mathsf{add}(\mathcal{N})$	$\mathsf{cf}(\mathsf{cov}(\mathcal{M}))$	

Topological sums

In [70] it was remarked that:

16 Theorem. *Let $(X_n :< \infty)$ be a sequence of topological spaces. Then:*

(1) $\sum_{n<\infty} X_n \models \mathsf{S}_1(\mathcal{O}_{\sum_{n<\infty} X_n}, \mathcal{O}_{\sum_{n<\infty} X_n}) \Leftrightarrow (\forall n)(X_n \models \mathsf{S}_1(\mathcal{O}_{X_n}, \mathcal{O}_{X_n}))$.

(2) $\sum_{n<\infty} X_n \models \mathsf{S}_{fin}(\mathcal{O}_{\sum_{n<\infty} X_n}, \mathcal{O}_{\sum_{n<\infty} X_n}) \Leftrightarrow (\forall n)(X_n \models \mathsf{S}_{fin}(\mathcal{O}_{X_n}, \mathcal{O}_{X_n}))$.

Similar preservation theorems fail for some of the other selection principles. For example, in [87] it was shown that

17 Theorem. *For each positive integer n there are spaces $(X_j : j \leq n)$ such that for any set $I \subset \{0, \cdots, n\}$ of cardinality at most n, $\sum_{j\in I} X_j \models \mathsf{S}_1(\Omega_{\sum_{j\in I} X_j}, \Gamma_{\sum_{j\in I} X_j})$, but $\sum_{j\leq n} X_j \not\models \mathsf{S}_1(\Omega_{\sum_{j\leq n} X_j}, \Gamma_{\sum_{j\leq n} X_j})$.*

10 Special image spaces

The presence of a selection property can sometimes be tested by analysing the properties of the images of spaces under certain maps, into well-understood spaces. This idea was initiated in [42] by Hurewicz.

The Baire space

Endow the set ω of nonnegative integers with the discrete topology, and let ${}^\omega\omega$ denote the Tychonoff power of countably many copies of this space. There is a natural ordering $\prec$ defined on ${}^\omega\omega$: $f \prec g$ means that $\lim_{n\to\infty}(g(n) - f(n)) = \infty$. A subset D of ${}^\omega\omega$ is said to be *dominating* if there is for each $f \in {}^\omega\omega$ a function $g \in D$ such that $f \prec g$. A subset B of of ${}^\omega\omega$ is said to be *bounded* if there is an $f \in {}^\omega\omega$ such that for each $g \in B$, $g \prec f$.

Hurewicz proved:

- A space X has property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ if, and only if, for each continuous function $f : X \longrightarrow {}^\omega\omega$ the set $f[X] = \{f(x) : x \in X\}$ is not a dominating family in ${}^\omega\omega$.
- A space X has the Hurewicz property $\mathsf{U}_{fin}(\mathcal{O}^{nc}, \Gamma)$ if, and only if, for each continuous function $f : X \longrightarrow {}^\omega\omega$ the set $f[X] = \{f(x) : x \in X\}$ is a bounded family in ${}^\omega\omega$.

For a finite subset F of ${}^\omega\omega$ define $max(F) : {}^\omega\omega \longrightarrow {}^\omega\omega$ so that for each n $max(F)(n) = \max\{f(n) : f \in F\}$. For s subset Z of ${}^\omega\omega$ define $maxfin(Z) = \{max(F) : F \subset Z \text{ finite}\}$. In [90] it is shown:

- A zero-dimensional separable metrizable space X fulfills the property $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp})$ if, and only if, for each continuous mapping $f : X \longrightarrow {}^{\omega}\omega$, the set $maxfin(f[X])$ is not dominating.

The Rothberger space

Consider the discrete two-point space $2 = \{0, 1\}$ and let 2^ω be the Tychonoff power of ω copies of this space. If we associate with each subset of ω its characteristic function, we have a natural bijection between $\mathcal{P}(\omega)$, the powerset of ω, and 2^ω, the Cantor space. Consider the topology on $\mathcal{P}(\omega)$ induced by this bijection, and consider the subspace $[\omega]^{\aleph_0} = \{T \in \mathcal{P}(\omega) : T \text{ infinite}\}$, endowed with the topology inherited. We shall call this space the *Rothberger space* in honor of F. Rothberger, who extensively used this space in his work (see for example [61], [62] and [63]).

Several combinatorial structures in $[\omega]^{\aleph_0}$ play an important role in the study of selection principles. We give just one among many examples of typical results here. For elements A and B of $[\omega]^{\aleph_0}$ the notation $A \subset^* B$ denotes that $B \setminus A$ is infinite while $A \setminus B$ is finite.

Let $\mathcal{F}$ be a subset of the Rothberger space. Then an element A of the Rothberger space is said to be a pseudo-intersection of $\mathcal{F}$ if for each $F \in \mathcal{F}$ we have $A \subset^* F$. $\mathcal{F}$ has the *finite intersection property* if for each finite nonempty set $\mathcal{G} \subset \mathcal{F}$ the set $\cap\mathcal{G}$ is infinite.

- X has property $\mathsf{S}_1(\Omega, \Gamma)$ if, and only if, for each continuous function $f : X \to [\omega]^{\aleph_0}$ such that $f[X]$ has the finite intersection property, there is an A in the Rothberger space such that for each $x \in X$ we have $A \subseteq^* f(x)$ (that is, has a pseudo-intersection) – [59].

11 Game theory

There are some natural infinite two-person games of perfect information associated with the selection principles. In each of these games we are interested in two statements:

(I) Does ONE have a winning strategy?

(II) Does TWO have a winning strategy?

When the answers to both (I) and (II) are "no", then the game is said to be undetermined. Both determined and undetermined games turn out to be extremely important and useful for the study of selection principles.

The rules of the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ are as follows: The players, ONE and TWO, play an inning per positive integer. In the n–th inning first ONE chooses an element $O_n \in \mathcal{A}$, and then TWO responds with a finite set $T_n \subset O_n$. A play $(O_1, T_1, \cdots, O_n, T_n, \cdots)$ is won by TWO if $\cup_{n\in\mathbb{N}} T_n \in \mathcal{B}$: Else, ONE wins.

The game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is played as follows: The players, ONE and TWO, play an inning per positive integer. In the n–th inning first ONE chooses an element $O_n \in \mathcal{A}$, and then TWO responds with a $T_n \in O_n$. A play

$$(O_1, T_1, \cdots, O_n, T_n, \cdots)$$

is won by TWO if $\{T_n : n \in \mathbb{N}\} \in \mathcal{B}$: Else, ONE wins.

The game $\mathsf{G}_c(\mathcal{A},\mathcal{B})$ is played as follows: The players, ONE and TWO, play an inning per positive integer. In the n–th inning first ONE chooses an element $O_n \in \mathcal{A}$, and then TWO responds with a disjoint refinement $T_n \prec O_n$. A play $(O_1, T_1, \cdots, O_n, T_n, \cdots)$ is won by TWO if $\cup\{T_n : n \in \mathbb{N}\} \in \mathcal{B}$: Else, ONE wins.

The games $\mathsf{G}_d(\mathcal{A},\mathcal{B})$, $\mathsf{G}_{pf}(\mathcal{A},\mathcal{B})$, $\mathsf{G}_{cf}(\mathcal{A},\mathcal{B})$ and $\mathsf{G}_{lf}(\mathcal{A},\mathcal{B})$ are played like $\mathsf{G}_c(\mathcal{A},\mathcal{B})$ except that the sets T_n chosen by TWO must be respectively discrete, point finite, compact finite or locally finite.

Does ONE have a winning strategy?

If ONE does not have a winning strategy in a game of the form $\mathsf{G}_a(\mathcal{A},\mathcal{B})$, then the corresponding selection principle $\mathsf{S}_a(\mathcal{A},\mathcal{B})$ holds. The converse of this is not always true. When it is true, then the game is a powerful tool to extract additional information about the families $\mathcal{A}$ and $\mathcal{B}$. The first two fundamental results here are due to Hurewicz and to Pawlikowski.

The game $\mathsf{G}_{fin}(\mathcal{O}_X,\mathcal{O}_X)$ was explicitly defined by Telgársky in [86], but the first fundamental result about it was already proved in 1925 by Hurewicz in Theorem 10 of [41]:

18 Theorem (Hurewicz). *A space X has property $\mathsf{S}_{fin}(\mathcal{O}_X,\mathcal{O}_X)$ if, and only if, ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{O}_X,\mathcal{O}_X)$.*

Galvin explicitly defined the game $\mathsf{G}_1(\mathcal{O}_X,\mathcal{O}_X)$ in [31], and Pawlikowski proved the following fundamental result:

19 Theorem (Pawlikowski). *A space X has property $\mathsf{S}_1(\mathcal{O}_X,\mathcal{O}_X)$ if, and only if, ONE has no winning strategy in $\mathsf{G}_1(\mathcal{O}_X,\mathcal{O}_X)$.*

Here is a small sample of additional results in this connection (In 6 $\mathcal{F}$ is a free ultrafilter on $\mathbb{N}$ and in 12 a Q-point):

	Selection Property)	ONE has no winning strategy in	Source
1	$\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$	$\mathsf{G}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$	[41]
2	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp})$	$\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp})$	[5]
3	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\tau^*-gp})$	$\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_X^{\tau^*-gp})$	[78]
4	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	$\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	[47]
5	$\mathsf{S}_{fin}(\Omega_X, \Omega_X)$	$\mathsf{G}_{fin}(\Omega_X, \Omega_X)$	[66]
6	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$	$\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$	[35]
7	$\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$	$\mathsf{G}_1(\mathcal{O}_X, \mathcal{O}_X)$	[55]
8	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\omega-gp})$	$\mathsf{G}_1(\Omega_X, \mathcal{O}_X^{\omega-gp})$	[5]
9	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\tau^*-gp})$	$\mathsf{G}_1(\Omega_X, \mathcal{O}_X^{\tau^*-gp})$	[78]
10	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	$\mathsf{G}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	[47]
11	$\mathsf{S}_1(\Omega_X, \Omega_X)$	$\mathsf{G}_1(\Omega_X, \Omega_X)$	[66]
12	$\mathsf{S}_1(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$	$\mathsf{G}_1(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$	[35]

Does TWO have a winning strategy?

Consider the spaces for which ONE does not have a winning strategy in one of these games. Now one may ask if TWO has a winning strategy. Not much is known about this question. From the few known results it seems that the spaces where TWO have a winning strategy have important properties in the preservation of selection principles in products. The following result of Telgársky is one of the fundamental theorems regarding this question - [84], [85]:

20 Theorem (Telgársky). *Let X be a $\mathsf{T}_{3\frac{1}{2}}$–space and let $\mathcal{K}$ be a Telgársky family for which each element is a G_δ–subset of X. Then TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}(\mathcal{K}), \mathcal{O}_X)$ if, and only if, X is a union of countably many members of $\mathcal{K}$.*

In particular one has for metrizable spaces X:

TWO has a winning strategy in $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$ if, and only if, TWO has a winning strategy in $\mathsf{G}_{fin}(\Omega, \mathcal{O}^{\gamma-gp})$ if, and only if, X is σ-compact.

And for first-countable spaces X one has:

TWO has a winning strategy in $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ if, and only if, TWO has a winning strategy in $\mathsf{G}_1(\Omega, \mathcal{O}^{\gamma-gp})$ if, and only if, X is countable.

12 Ramsey theory

In 1930 F.P. Ramsey published an influential paper [58] containing combinatorial results now known as Ramsey's theorems. In the hands of Erdös and collaborators these results of Ramsey were generalized and extensively studied, creating a flourishing area of combinatorial mathematics now known as Ramsey Theory. Though the earliest results in Ramsey theory were mostly concentrated on combinatorial properties implied by cardinality, the theory also was developed for some other mathematical structures, most notably in the theory of ultrafilters (for example [14], [18], [19] and [39]) and the theory of linearly ordered sets (for example [24], [25] and [27]). The text [23] contains a nice exposition of Ramsey Theory for cardinal numbers.

There is also a deep connection between Ramsey Theory and the theory of Selection Principles, as will be seen from results mentioned below. The reader should see the material in this section as only the beginning of an exploration of Ramsey Theory in context of selection principles. In particular, a large number of partition relations have not been considered in this context, and need to be considered. Also, no definitive Ramseyan results are yet known for some of the non-classical selection principles. The spaces throughout this discussion of Ramseyan properties are assumed to be ϵ–spaces.

The simplest Ramsey-theoretic statement is as follows: Let $\mathcal{A}$ and $\mathcal{B}_1, \cdots, \mathcal{B}_n$ be families of mathematical structures. Then

$$(\forall n)(\mathcal{A} \rightarrow (\mathcal{B}_1, \cdots, \mathcal{B}_n))$$

denotes the statement that if some structure in A is partitioned into finitely many pieces, then some piece is in $\mathcal{B}_j$ for some j.

As examples of such $\mathcal{A}$ and $\mathcal{B}$ consider open covers of some topological space. If for each n we have $\mathcal{A} \rightarrow (\mathcal{O})_n$, then $\mathcal{A} \subseteq \Omega$. This indicates that in Rasmey-theoretic considerations it is important if a selection principle for open covers is equivalent to one with Ω-covers in the first coordinate. In this connection there are for example the following results:

21 Theorem. *For a Lindelof space X,*

- $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O}) \Leftrightarrow \mathsf{S}_{fin}(\Omega, \mathcal{O})$.
- $\mathsf{S}_1(\mathcal{O}, \mathcal{O}) \Leftrightarrow \mathsf{S}_1(\Omega, \mathcal{O})$.
- $\mathsf{U}_{fin}(\mathcal{O}^{nc}, \Gamma) \Leftrightarrow \mathsf{S}_{fin}(\Omega, \mathcal{O}^{\gamma-gp})$.
- $\mathsf{U}_{fin}(\mathcal{O}^{nc}, \Omega) \Leftrightarrow \mathsf{S}_{fin}(\Omega, \mathcal{O}^{\omega-gp})$.
- $\mathsf{S}_c(\mathcal{O}, \mathcal{O}) \Leftrightarrow \mathsf{S}_c(\Omega, \mathcal{O})$.
- $\mathsf{S}_{lf}(\mathcal{O}, \mathcal{O}) \Leftrightarrow \mathsf{S}_{lf}(\Omega, \mathcal{O})$.

Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are families of open sets with unions dense in X, and if for all n $\mathcal{A} \rightarrow (\mathcal{D})_n$, then $\mathcal{A} \subseteq \mathcal{D}_\Omega$.

The m-element subsets of a set S, that is, $\{X \subseteq S : |X| = m\}$ will be denoted $[S]^m$.

The ordinary partition relation

Let $\mathcal{A}$ and $\mathcal{B}_i$, $i \leq n$ be families of subsets of an infinite set S. Then for each m the symbol

$$\mathcal{A} \rightarrow (\mathcal{B}_1, \cdots, \mathcal{B}_n)^m$$

denotes the statement that for each $A \in \mathcal{A}$ and for each function $f : [A]^m \rightarrow \{1, \cdots, n\}$ there is an $i \in \{1, \cdots, n\}$ and a set $B_i \subseteq A$ such that $B_i \in \mathcal{B}_i$ and $f(X) = i$ for each $X \in [B_i]^m$. In the case where $\mathcal{B}_1 = \cdots = \mathcal{B}_n = \mathcal{B}$, say, it is customary to write

$\mathcal{A} \rightarrow (\mathcal{B})^m_n$ instead of the longer notation above. The symbol is denoting the *ordinary partition relation*

In this notation, Ramsey's famous theorem can be stated as follows: Let S be an infinite set, and let $\mathcal{A}$ be the collection of infinite subsets of S. Then for each n and m we have $\mathcal{A} \rightarrow (\mathcal{A})^m_n$.

Several selection principles of the form $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ have been characterized by the ordinary

partition relation. Here are some such results. In 6 $\mathcal{F}$ is a Q-point ultrafilter. n and k are positive integers.

	Selection Property	Ramseyan partition relation	Source
1	$\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$	$(\forall k)(\Omega_X \rightarrow (\mathcal{O}_X)^2_k)$	[70]
2	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\omega-gp})$	$(\forall k)(\Omega_X \rightarrow (\mathcal{O}_X^{\omega-gp})^2_k)$	[5]
3	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\tau^*-gp})$	$(\forall k)(\Omega_X \rightarrow (\mathcal{O}_X^{\tau^*-gp})^2_k)$	[78]
4	$\mathsf{S}_1(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	$(\forall n,\, k)(\Omega_X \rightarrow (\mathcal{O}_X^{\gamma-gp})^n_k)$	[47]
5	$\mathsf{S}_1(\Omega_X, \Omega_X)$	$(\forall n,\, k)(\Omega_X \rightarrow (\Omega_X)^n_k)$	[66]
6	$\mathsf{S}_1(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$	$(\forall n,\, k)(\Omega_X \rightarrow (\mathcal{O}_{X,\mathcal{F}})^n_k)$	[35]

Moreover, it is consistent that there is a set X of real numbers such that $X \models (\forall n, k)(\Omega \rightarrow (\mathcal{O})^n_k)$, and yet $X \not\models \Omega \rightarrow (\Omega)^2_2$.

The Baumgartner-Taylor relation

Another partition symbol important for the study of selection principles is motivated

by a study of Baumgartner and Taylor in [14]. For each positive integer k, $\mathcal{A} \rightarrow \lceil\mathcal{B}\rceil^2_k$
denotes the following statement:

> For each A in $\mathcal{A}$ and for each function $f : [A]^2 \rightarrow \{1, \cdots, k\}$ there is a set $B \subset A$ and a $j \in \{1, \cdots, k\}$, and a partition $B = \bigcup_{n<\infty} B_n$ of B into pairwise disjoint finite sets such that for each $\{a, b\} \in [B]^2$ for which a and b are not from the same B_n, we have $f(\{a, b\}) = j$, and $B \in \mathcal{B}$.

We say that "B is nearly homogeneous for f". The relation between $\mathcal{A}$ and $\mathcal{B}$ denoted by this partition symbol is called the *Baumgartner-Taylor partition relation*. Several selection principles of the form $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ have been characterized by the Baumgartner-Taylor partition relation. Here are some such results. In 6 $\mathcal{F}$ is a free ultrafilter on $\mathbb{N}$. The symbol k denotes a positive integer.

	Selection Property)	Ramseyan partition relation	Source
1	$\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$	$(\forall k)(\Omega_X \rightarrow \lceil\mathcal{O}_X\rceil^2_k)$	[70]
2	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\omega-gp})$	$(\forall k)(\Omega_X \rightarrow \lceil\mathcal{O}_X^{\omega-gp}\rceil^2_k)$	[5]
3	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\tau^*-gp})$	$(\forall k)(\Omega_X \rightarrow \lceil\mathcal{O}_X^{\tau^*-gp}\rceil^2_k)$	[78]
4	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_X^{\gamma-gp})$	$(\forall k)(\Omega_X \rightarrow \lceil\mathcal{O}_X^{\gamma-gp}\rceil^2_k)$	[47]
5	$\mathsf{S}_{fin}(\Omega_X, \Omega_X)$	$(\forall k)(\Omega_X \rightarrow \lceil\Omega_X\rceil^2_k)$	[66]
6	$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{X,\mathcal{F}})$	$(\forall k)(\Omega_X \rightarrow \lceil\mathcal{O}_{X,\mathcal{F}}\rceil^2_k)$	[35]

The square bracket partition relation

Let $\mathcal{A}$ and $\mathcal{B}_i$, $i \leq n$ be families of subsets of an infinite set S. Then for each m the symbol

$$\mathcal{A} \rightarrow [\mathcal{B}_1, \cdots, \mathcal{B}_n]^m_{<k}$$

In the case where $\mathcal{B}_1 = \cdots = \mathcal{B}_n = \mathcal{B}$, say, it is customary to write

$$\mathcal{A} \rightarrow [\mathcal{B}]^m_{n/<k}$$

instead of the longer notation above. And in the case where $k = n$, it is customary to write

$$\mathcal{A} \rightarrow [\mathcal{B}]^m_n.$$

This partition relation is known as the *square bracket partition relation*. It is evident that for all m, n and k,

$$\mathcal{A} \rightarrow (\mathcal{B}_1, \cdots, \mathcal{B}_n)^m \Rightarrow \mathcal{A} \rightarrow [\mathcal{B}_1, \cdots, \mathcal{B}_n]^m_{<k}$$

The converse is not true. Indeed, even in the case where $\mathcal{A}$ is an ultrafilter on the set of positive integers, it may happen that $\mathcal{A} \rightarrow [\mathcal{A}]^2_3$ and yet $\mathcal{A} \not\rightarrow [\mathcal{A}]^2_2$. Much still needs to be considered in connection with the square bracket partition relation in the context of selection principles. The following was proved in [71]:

22 Theorem. *For X an infinite separable metric space, the following are equivalent:*

(1) $\Omega \rightarrow [\Omega]^2_2$;

(2) For each $k > 2$, $\Omega \rightarrow [\Omega]^2_{k/\leq 2}$.

Since $\Omega \rightarrow [\Omega]^2_2$ is equivalent to $\Omega \rightarrow (\Omega)^2_2$, it follows that the square bracket partition relation $\Omega \rightarrow [\Omega]^2_3$ characterizes $\mathsf{S}_1(\Omega, \Omega)$.

Polarized partition relations

Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be families of subsets of an infinite set S, $i = 1, 2$.

$$\begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{pmatrix} \rightarrow \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix}^{1,1}_{k/<\ell}$$

denotes the statement that for each $A_1 \in \mathcal{A}_1$ and for each $A_2 \in \mathcal{A}_2$ and for each function $f : A_1 \times A_2 \rightarrow \{1, \cdots, k\}$ there exist sets $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ and a set $J \subseteq \{1, \cdots, k\}$ such that $|J| \leq \ell$ and $\{f(x, y) : (x, y) \in B_1 \times B_2\} \subseteq J$. This partition relation is the *polarized square bracket partition relation*. Its relation to the selection principles still needs much investigation. The following result from [72] seems to be the only known for polarized partition relations and selection principles.

23 Theorem. *Let X be an ϵ–space.*

(1) $(X \models \mathsf{S}_1(\Omega, \Omega)) \Rightarrow (X \models \begin{pmatrix} \Omega \\ \Omega \end{pmatrix} \rightarrow \begin{bmatrix} \Omega \\ \Omega \end{bmatrix}^{1,1}_{k/<3})$.

(2) $(X \models \begin{pmatrix} \Omega \\ \Omega \end{pmatrix} \rightarrow \begin{bmatrix} \Omega \\ \Omega \end{bmatrix}^{1,1}_{k/<3}) \Rightarrow (X \models \mathsf{S}_{fin}(\Omega, \Omega))$.

(3) $(X \models \begin{pmatrix} \Omega \\ \Omega \end{pmatrix} \rightarrow \begin{bmatrix} \Omega \\ \Omega \end{bmatrix}^{1,1}_{k/<3}) \Rightarrow (X \models \mathsf{Split}(\Omega, \Omega))$.

It is not known if any of these implications is reversible.

13 Absolute and relative properties

Most of what we discussed so far assumed that we have a fixed space X, and all considerations were in terms of this space. Let Y be a subspace of X. The covers of Y by sets open in X will be denoted $\mathcal{O}_{XY}$. For specific choices of $\mathcal{A}$ the symbols $\mathcal{A}_X$ and $\mathcal{A}_{XY}$ have the obvious meaning. For example, Ω_X denotes the ω–covers of X and Ω_{XY} denotes the ω–covers of Y by sets open in X.

When Y is a proper subspace of X then a selection principle of the form $\Pi(\mathcal{A}_X, \mathcal{B}_{XY})$ is said to be a *relative* selection principle. When $Y = X$ it is common to leave off the subscripts X and XY, and to speak of *absolute* selection principles.

Only fairly recently it became clear that the relative versions of selection principles are not trivial generalizations of the absolute versions. Often the relative versions of results for the absolute case are harder to prove. And what is more, the relative versions give new insights about old concepts. We will only give a few results now to illustrate the remarks just made about relative selection principles.

In [75] the relative version of the Rothberger property was shown to characterize Borel's strong measure zero sets. Specifically:

24 Theorem. *Let X be a σ-compact metrizable space. Then for each subspace Y of X the following are equivalent:*

(1) $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_{XY})$ *holds;*

(2) ONE has no winning strategy in the game $\mathsf{G}_1(\Omega_X, \mathcal{O}_{XY})$*;*

(3) For each positive integer k, $\Omega_X \to (\mathcal{O}_{XY})^2_k$;

(4) Y has strong measure zero with respect to each metrization of X.

In [3] the relative Menger property was considered and the following theorem proved:

25 Theorem. *Let X be a Lindelöf space. Then for each subspace Y of X the following are equivalent:*

(1) $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$.

(2) ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$.

(3) For each k, $\Omega_X \to \lceil \mathcal{O}_{XY} \rceil^2_k$

Results of [4] and [5] prove the following theorem regarding the relative Hurewicz property:

26 Theorem. *Let X be a Lindelöf space. Then for each subspace Y of X the following are equivalent:*

(1) $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{XY}^{\gamma-gp})$.

(2) ONE has no winning strategy in $\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_{XY}^{\gamma-gp})$.

(3) For each k, $\Omega_X \rightarrow \lceil \mathcal{O}_{XY}^{\gamma-gp} \rceil_k^2$

The results of [9] prove the following:

27 Theorem. *Let X be a space with the property* $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$. *Then for each subspace Y of X the following are equivalent:*

(1) $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{XY}^{\omega-gp})$

(2) ONE has no winning strategy in the game $\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_{XY}^{\omega-gp})$

(3) For each k, $\Omega_X \rightarrow \lceil \mathcal{O}_{XY}^{\omega-gp} \rceil_k^2$.

In [44] the notion of a relative γ-set, that is, a subset Y of $\mathbb{R}$ for which $\mathsf{S}_1(\Omega_X, \Gamma_{XY})$ holds, was introduced and studied. In this paper the authors point out that the notion of a relative γ set does not coincide with that of a γ-set. Indeed, in [5] it was shown that it is consistent that there are subsets Y of the real line such that $\mathsf{S}_1(\Omega_{\mathbb{R}}, \Gamma_{\mathbb{R}Y})$ holds, but $\mathsf{S}_{fin}(\mathcal{O}_Y, \mathcal{O}_Y)$ fails. More recently A.W. Miller informed the author of the following very interesting results: Consider the notions of relative γ–sets in both $\mathbb{R}$, the real line with its usual topology, and in 2^ω, the Cantor set. Define:

- $\mathfrak{p}_0 = \min\{|X| : X \subset \mathbb{R}$ and X does not satisfy $\mathsf{S}_1(\Omega_X, \Gamma_X)\}$.
- $\mathfrak{p}_1 = \min\{|X| : X \subset 2^\omega$ and X does not satisfy $\mathsf{S}_1(\Omega_{2^\omega}, \Gamma_X)\}$.
- $\mathfrak{p}_2 = \min\{|X| : X \subset \mathbb{R}$ and X does not satisfy $\mathsf{S}_1(\Omega_{\mathbb{R}}, \Gamma_X)\}$.

28 Theorem (Miller). *Each of the following statements is consistent, relative to the consistency of classical mathematics:*

(1) $\aleph_1 = \mathfrak{p}_0 = \mathfrak{p}_1 < \mathfrak{p}_2$.

(2) $\aleph_1 = \mathfrak{p}_0 < \mathfrak{p}_1 = \mathfrak{p}_2$.

These results indicate that for a subspace Y of X, satisfying a relative selection property may depend very strongly on the properties of the space X. We expect that the usual equivalences which hold between absolute selection properties and Ramseyan properties and game theoretic properties will, in case of relative selection properties of subspace Y of X, also depend on the properties of the space X. For example, it is expected that the hypotheses in the following two results from [9] cannot be merely omitted (but perhaps could be weakened somewhat):

29 Theorem. *Let X be an infinite σ-compact metrizable space and let Y be a subspace of X. The following statements are equivalent:*

(1) $\mathsf{S}_1(\Lambda_X, \mathcal{O}_{XY}^{\omega-gp})$.

(2) ONE has no winning strategy in the game $\mathsf{G}_1(\Omega_X, \mathcal{O}_{XY}^{\omega-gp})$.

(3) For each positive integer k, $\Omega_X \to (\mathcal{O}_{XY}^{\omega-gp})^2_k$.

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Much still needs to be investigated regarding the relative selection principles, also in connection with the Balkan, Bar-Ilan and the Morelia selection principles.

14 Translation to hyperspaces

There are many constructions of new spaces from given spaces. Often these examples are constructed to illustrate the relationship among topological properties. Often these new spaces can be used to give an analysis of some of the properties of the originating spaces. Here we mention only one example of such "hyperspace" constructions, and how it preserves selection principles. For a Tychonoff space X the set $\mathsf{C}(X)$ of real-valued continuous functions on X can be endowed with a variety of topologies. We use $\mathsf{C}_p(X)$ to denote this space endowed with the topology of pointwise convergence.

For a space X and a point $x \in X$ define $\Omega_x = \{A \subset X \setminus \{x\} : x \in \overline{A}\}$. Also define $\Gamma_x = \{A \subset X \setminus \{x\} :$ for each neighborhood U of X, $A \setminus U$ is finite$\}$. We say that X has *countable tightness* at x if there is for each $A \in \Omega_x$ a countable subset $B \subset A$ with $B \in \Omega_x$. According to Arkhangel'skii [2] X has *countable fan tightness* at x if $\mathsf{S}_{fin}(\Omega_x, \Omega_x)$ holds. According to Sakai [64] X has *countable strong fan tightness* at x if $\mathsf{S}_1(\Omega_x, \Omega_x)$ holds. And X has the Fréchet-Urysohn property at x if there is for each $X \in \Omega_x$ a subset $B \subset A$ with $B \in \Gamma_x$ - in terms of earlier notation: $\binom{\Omega_x}{\Gamma_x}$ holds. These properties are all related to the closure operator on X, and consequently the closure structure of X.

A beautiful theory emerges when considering these closure properties on $\mathsf{C}_p(X)$ for Tychonoff spaces X. We briefly describe a small selection of results from this branch of investigation. First, observe that since $\mathsf{C}_p(X)$ is a topological group we may confine our attention to any specific element f of $\mathsf{C}_p(X)$ when considering these closure structures. It is computationally convenient to consider the zero function $\mathbf{0}$. According to Gerlits and Nagy a space X is an ϵ-space if each ω-cover of X contains a countable subset which is an ω-cover of X.

Here are some typical results:

$\mathsf{C}_p(X) \models$	$X \models$	$(\forall n)(X^n \models)$
Countable tightness	ϵ space	Lindelöf
$\mathsf{S}_1(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$	$\mathsf{S}_1(\Omega_X, \Omega_X)$	$\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$
$\mathsf{S}_{fin}(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$	$\mathsf{S}_{fin}(\Omega_X, \Omega_X)$	$\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$
$\binom{\Omega_{\mathbf{0}}}{\Gamma_{\mathbf{0}}}$	$\mathsf{S}_1(\Omega_X, \Gamma_X)$	$\mathsf{S}_1(\Omega_X, \Gamma_X)$

In $\mathsf{C}_p(X)$ we may also define M_f to be the set of sequences $(f_n : n < \infty)$ such that for each x, $(f_n(x) : n < \infty)$ converges monotonically to 0. Then in [74] is proven that for X a perfectly normal $\mathsf{T}_{3\frac{1}{2}}$–space: $(X \models \mathsf{S}_{fin}(\Omega, \mathcal{O}^{\gamma-gp})) \Leftrightarrow (\mathsf{C}_p(X) \models \mathsf{S}_1(\mathsf{M}_0, \Gamma_0))$

In [66] it is also proved that for Tychonoff ϵ–spaces:

- $\mathsf{S}_1(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}}) \Leftrightarrow (\forall n, k)(\Omega_{\mathbf{0}} \to (\Omega_{\mathbf{0}})^n_k)$.
- $\mathsf{S}_{fin}(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}}) \Leftrightarrow (\forall k)(\Omega_{\mathbf{0}} \to \lceil \Omega_{\mathbf{0}} \rceil^2_k)$.

15 Topological groups

A topological group is a group $(G, *)$ which carries a topology τ such that the group operation and inverse operation are continuous. When an algebraic structure which interacts well with the topology is present, then special types of open covers can be defined. Let U be an open neighborhood of the identity element 1_G topological group $(G, *)$. Then for each element x of G the set $x * U = \{x * u : u \in U\}$ is an open neighborhood of x. Thus the set

$$\mathcal{O}(U) = \{x * U : x \in G\}$$

is an open cover of G. We shall use the notation $\mathcal{O}_{\sf nbd}$ to denote the set $\{\mathcal{O}(U) : U$ a neighborhood of $1_G\}$.

For each finite subset F of G the set $F * U = \cup_{x\in F} x * U$ is an open set containing F. The group $(G, *)$ is said to be *totally bounded* if there is for each open neighborhood U of 1_G a finite set F such that $G = F * U$. When $(G, *)$ is not totally bounded then for each open neighborhood U of 1_G

$$\Omega(U) = \{F * U : F \subset G \text{ finite}\}$$

is an ω–cover of G. We will use the symbol $\Omega_{\sf nbd}$ to denote the set of such ω-covers of G.

Selection principles have also made their appearance in the study of boundedness properties of topological groups. The selection principle $\mathsf{S}_1(\Omega_{\sf nbd}, \mathcal{O})$ for a topological group $(G, *)$ is exactly a boundedness notion introduced by Okunev, and generally called *o-boundedness*. In unpublished work Kočinac independently introduced this notion and called groups with this selection property Menger groups. Tkačenko also introduced a game equivalent to the game $\mathsf{G}_1(\Omega_{\sf nbd}, \mathcal{O})$, and calls groups where TWO has a winning strategy in this game *strictly o-bounded groups*.

Evidently there are several other selection principles and corresponding games that ought to be studied in the context of topological groups, and indeed such a study is underway. Here are some results from [7]:

31 Theorem. *Let $(G, *)$ be a topological group.*

(1) $(G \models \mathsf{S}_1(\Omega_{nbd}, \mathcal{O}_G)) \Leftrightarrow (G \models \mathsf{S}_{fin}(\Omega_{nbd}, \mathcal{O}_G))$.

(2) $(G \models \mathsf{S}_1(\Omega_{nbd}, \mathcal{O}_G^{wgp})) \Leftrightarrow (G \models \mathsf{S}_1(\Omega_{nbd}, \Omega)) \Leftrightarrow ((\forall n)(G^n \models \mathsf{S}_1(\Omega_{nbd}, \mathcal{O}_G))$.

(3) $(G \models \mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}_G^{wgp})) \Leftrightarrow ((\forall n)(G^n \models \mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}_G))$.

(4) $(G \models \mathsf{S}_1(\Omega_{nbd}, \mathcal{O}_G^{gp})) \Leftrightarrow (G \models \mathsf{S}_1(\Omega_{nbd}, \Gamma))$.

32 Theorem. *Let $(G, *)$ be a topological group satisfying $\mathsf{S}_1(\Omega_{nbd}, \Gamma)$ and let $(H, *)$ be any topological group. If for an $\mathcal{A}$ in $\{\Gamma, \Omega, \mathcal{O}\}$ the group H satisfies $\mathsf{S}_1(\Omega_{nbd}, \mathcal{A})$, then so does $G \times H$.*

33 Theorem. *Let $(G, *)$ be a topological group satisfying $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}^{gp})$ and let $(H, *)$ be any topological group. If for an $\mathcal{A}$ in $\{\mathcal{O}, \mathcal{O}^{wgp}, \mathcal{O}^{gp}\}$ the group H satisfies $\mathsf{S}_1(\mathcal{O}_{nbd}, \mathcal{A})$, then so does $G \times H$.*

34 Theorem. *Let $(H, *)$ be a zerodimensional metrizable group and let G be a subgroup of H. The following are equivalent:*

(1) $G \models \mathsf{S}_1(\Omega_{nbd}(G), \mathcal{O}_G)$.

(2) $H \models \mathsf{S}_{fin}(\Omega_H, \mathcal{O}_G)$*.*

(3) ONE has no winning strategy in $\mathsf{G}_1(\Omega_{nbd}(G), \mathcal{O}_G)$*.*

It follows that a zerodimensional metrizable group has property $\mathsf{S}_1(\Omega_{nbd}, \mathcal{O}_G)$ if, and only if, all its subgroups have this property.

Several results characterizing subgroups of $({}^{\omega}\mathbb{Z}, +)$ and of $(\mathbb{R}, +)$ defined by these selection principles Ramsey-theoretically can also be found in [7]. A series of papers by Taras Banakh is a very good source for more information on topological groups satisfying $\mathsf{S}_1(\Omega_{nbd}, \mathcal{O})$ - see for example [10] and [11].

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