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Monochromatic configurations for finite colourings of the plane

Sukumar Das Adhikari
Harish-Chandra Research Institute
(Formerly, Mehta Research Institute)
Chhatnag Road, Jhusi, Allahabad 211 019, INDIA
adhikari@mri.ernet.in

Purusottam Rath
Harish-Chandra Research Institute
(Formerly, Mehta Research Institute)
Chhatnag Road, Jhusi, Allahabad 211 019, INDIA
rath@mri.ernet.in

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Abstract. A strengthened form of Gurevich's conjecture was proved by R. L. Graham ([2], [3]), which says that for any $\alpha > 0$ and any pair of non-parallel lines L_1 and L_2, in any partition of the plane into finitely many classes, some class contains the vertices of a triangle which has area α and two sides parallel to the lines L_i. Later, a shorter proof, using the main idea of Graham, was presented in [1]. Following some questions raised by Graham [2] and by suitable modifications of methods therein, here we establish a similar result in the case of vertices of a trapezium.

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Introduction

Gurevich had asked whether for any finite colouring of the Euclidean plane E^2 there always exists a triangle of area 1 such that all three vertices of the triangle receive the same colour. The following very interesting result of R. L. Graham ([2], [3]) not only settles the question, it is stronger and as he mentions, his proof can have natural generalizations to yield higher dimensional analogues.

1 Theorem (Graham). *For a positive integer r, there is a positive integer $T(r)$ such that given any r-colouring of the set of integer lattice points in E^2, there is always a right triangle of area $T(r)$ such that its all three vertices are lattice points of the same colour.*

2 Remark. From the proof presented in [2] and [3], one observes that the triangle in Theorem 1 can be made to satisfy further constraints of having two of its sides parallel to the axes and by suitable scaling, the following corollary is easily obtained.

3 Corollary. *For any $\alpha > 0$ and any pair of non-parallel lines L_1 and L_2, in any partition of the plane into finitely many classes, some class contains the vertices of a triangle which has area α and two sides parallel to the lines L_i.*

Apart from mentioning several interesting known results related to the above theorem, the above mentioned paper [2] of Graham raised the question of establishing results similar to Theorem 1 for other configurations which are not simplexes, such as vertices of a parallelogram (or of a rhombus or of a rectangle) in the plane.

In the present article, following a method suggested by Graham [2], we establish the corresponding result in the case of vertices of a trapezium.

More precisely, we prove the following theorem.

4 Theorem. *Let E^2 denote the Euclidean plane endowed with rectangular Cartesian co-ordinates. For a positive integer r, there is a positive integer T_r such that given any r-colouring of the set of integer lattice points in E^2, there is always a trapezium of area T_r such that all of its four vertices are lattice points of the same colour and three of its sides are parallel to the co-ordinate axes.*

As we shall be mentioning in our remarks after the proof of the theorem, our proof can be modified to establish similar results for some other polygons. However, the cases of important configurations like parallelograms and rectangles etc., continue to remain elusive.

Notations. In what follows, $\mathbf{Z}^+$ will denote the set of positive integers and $\mathbf{Z}_{\geq \mathbf{0}}$ the set of non-negative integers.

Proof. (of Theorem 2) We shall need the following two dimensional case of a result due to G. Grünwald (Gallai) which is a higher dimensional generalization of van der Waerden's theorem.

5 Lemma (Grünwald). *Let $r \in \mathbf{Z}^+$. Then, given any finite set $S \subset (\mathbf{Z}_{\geq \mathbf{0}})^2$, and an r-colouring of $(\mathbf{Z}_{\geq \mathbf{0}})^2$, there exist positive integers $N = N(S, r)$ and 'd' and a point 'v' in $(\mathbf{Z}_{\geq \mathbf{0}})^2$ such that the set $dS + v$ is monochromatic where $dS + v$ is a subset of $B_N \stackrel{\text{def}}{=} \{ (a, b) : a, b \in \mathbf{Z}_{\geq \mathbf{0}}, 0 \leq a, b \leq N \}$.*

Our proof of Theorem 2 is by induction on r.

When $r = 1$, considering the trapezium with vertices $(0, 2)$, $(0, 0)$, $(2, 0)$ and $(2, 1)$, we get a trapezium with area 3 trivially satisfying the condition of having its vertices of the same colour and we observe that it has two sides parallel to the Y-axis and one side parallel to the X-axis.

For $r \geq 2$, we assume that there are positive integers $C_{r-1} (\geq 3)$ and T_{r-1} such that for any $(r-1)$-colouring of the set of integer lattice points in the square $\{ (a,b) : 0 \leq a, b \leq C_{r-1} - 1 \}$, there is a trapezium of area T_{r-1} with vertices in this set, two sides parallel to the Y-axis and one side parallel to the X-axis such that the set of vertices is monochromatic.

Let

$$n_r = ((C_{r-1})! C_{r-1} + 1)! \; T_{r-1}$$

Now, suppose an r-colouring of the set of integer lattice points in E^2 is given. Considering the finite set $S = \{ (iT_{r-1}, 0), (iT_{r-1}, T_{r-1}) | 0 \leq i \leq n_r^2 \}$ of lattice points in E^2, by Lemma 1 there is a monochromatic (say with colour r) set of the form $dS + v$ which is a subset of $B_N = \{ (a,b) : a, b \in \mathbf{Z}_{\geq \mathbf{0}}, 0 \leq a, b \leq N \}$ for some $N = N(S, r)$.

Thus writing $v = (v_1, v_2)$, the set

$$\{(v_1 + idT_{r-1}, v_2), (v_1 + idT_{r-1}, v_2 + dT_{r-1}) | 0 \leq i \leq n_r^2\}$$

contained in B_N is monochromatic with colour r.

Now, consider the lattice points

$$Q_{ij} = (v_1 + idT_{r-1}, \quad v_2 + dT_{r-1} + j2dT_{r-1}(C_{r-1}!)) ,$$

$$1 \leq i \leq \frac{C_{r-1} n_r}{C_{r-1}! \; T_{r-1}}, \; 1 \leq j \leq C_{r-1}.$$

Case I. (One of the Q_{ij}'s have colour r)

If a Q_{ij} from the above collection has colour r, then we consider the trapezium with vertices

$$\begin{aligned}
Q_{ij} &= (v_1 + idT_{r-1}, \quad v_2 + dT_{r-1} + j2dT_{r-1}C_{r-1}!) , \\
B_{ij} &= (v_1 + idT_{r-1}, v_2) , \\
C_{ij} &= \left(v_1 + idT_{r-1} + dT_{r-1} \left(\frac{2n_r}{1 + C_{r-1}! \; j} \right), v_2 \right) \\
\text{and } D_{ij} &= \left(v_1 + idT_{r-1} + dT_{r-1} \left(\frac{2n_r}{1 + C_{r-1}! \; j} \right), v_2 + dT_{r-1} \right) ,
\end{aligned}$$

where the last three points belong to the monochromatic set (with colour r) $dS + v$.

Thus, the points Q_{ij}, B_{ij}, C_{ij} and D_{ij} are of the same colour and are vertices of a trapezium of area

$$\begin{aligned} & dT_{r-1}\left(\frac{2n_r}{1+C_{r-1}!\ j}\right)dT_{r-1}+\frac{1}{2}dT_{r-1}\left(\frac{2n_r}{1+C_{r-1}!\ j}\right)j2dT_{r-1}C_{r-1}! \\ = \; & T_{r-1}(2d^2T_{r-1}n_r). \end{aligned}$$

Case II. (None of the Q_{ij}'s have colour r)

In this case, the collection of Q_{ij}'s receive an $(r-1)$-colouring. We consider the following sub-collection of $\{Q_{ij}\}$:

$$\left(v_1+idT_{r-1}\frac{n_r}{C_{r-1}!\ T_{r-1}},\quad v_2+dT_{r-1}+j2dT_{r-1}C_{r-1}!\right),$$
$$1\le i\le C_{r-1}, 1\le j\le C_{r-1}.$$

By the induction hypothesis, considering the gaps between the points in this array of points, there is a trapezium with two sides parallel to the Y-axis and one side parallel to the X-axis such that all of its vertices are of the same colour and its area equals

$$T_{r-1}\cdot dT_{r-1}\frac{n_r}{C_{r-1}!\ T_{r-1}}\cdot 2dT_{r-1}C_{r-1}! = T_{r-1}(2d^2T_{r-1}n_r).$$

Therefore, from Case I and Case II, the result holds in the case of r-colouring, with

$$T_r = T_{r-1}(2d^2T_{r-1}n_r).$$

$\boxed{QED}$

6 Remark. It is clear that by suitable scaling, as a corollary to Theorem 2, we obtain that for any $\alpha > 0$, in any partition of the plane, endowed with rectangular Cartesian co-ordinates, into finitely many classes, some class contains the vertices of a trapezium which has area α and two sides parallel to the Y-axis and one side parallel to the X-axis. It is also clear that suitable modifications of our proof can yield corresponding results for vertices of other type of polygons. For instance, in Case I, if we replace Q_{ij} by $Q_{ij}+(1,0)$ (by $Q_{ij}+(dT_{r-1}\left(\frac{n_r}{1+C_{r-1}!\ j}\right),0)$), thus working with a translation of the set $\{Q_{ij}\}$ in Case II, the same argument will yield the result in the case of the vertices of a pentagon with two sides equal and parallel to the Y-axis and one side parallel to the X-axis (and other two sides equal).

References

[1] S. D. Adhikari: *On a theorem of Graham*, Expo. Math., **19**, no. 4, p. 365 – 367 (2001).

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