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On certain modified Szasz-Mirakyan operators in polynomial weighted spaces

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Abstract. We consider certain modified Szasz-Mirakyan operators $A_n(f;r)$ in polynomial weighted spaces of functions of one variable and we study approximation properties of these operators.

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Introduction

In the paper [1] M. Becker studied approximation problems for functions $f \in C_p$ and Szasz-Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1}$$

$x \in R_0 = [0, +\infty)$, $n \in N := \{1, 2, \dots\}$, where C_p with fixed $p \in N_0 := \{0, 1, 2, \dots\}$ is polynomial weighted space generated by the weighted function

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \qquad \text{if} \qquad p \geq 1, \tag{2}$$

i.e. C_p is the set of all real-valued functions f, continuous on R_0 and such that $w_p f$ is uniformly continuous and bounded on R_0. The norm in C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x)\,|f(x)|. \tag{3}$$

In [1] theorems on the degree of approximation of $f \in C_p$ by the operators S_n were proved. From these theorems it was deduced that

$$\lim_{n \to \infty} S_n(f;x) = f(x), \tag{4}$$

for every $f \in C_p$, $p \in N_0$ and $x \in R_0$. Moreover the convergence (4) is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

In this paper we shall modify the formula (1) and we shall study certain approximation properties of introduced operators.

Let C_p be the space given above and let $f \in C_p^1 := \{f \in C_p : f' \in C_p\}$, where f' is the first derivative of f.

For $f \in C_p$ we define the modulus of continuity $\omega_1(f;\cdot)$ as usual ([2]) by formula

$$\omega_1(f;C_p;t) := \sup_{0\leq h\leq t} \|\Delta_h f(\cdot)\|_p, \qquad t \in R_0, \tag{5}$$

where $\Delta_h f(x) := f(x+h) - f(x)$, for $x, h \in R_0$. From the above it follows that

$$\lim_{t\to 0+} \omega_1(f;C_p;t) = 0, \tag{6}$$

for every $f \in C_p$. Moreover if $f \in C_p^1$ then there exists $M_1 = const. > 0$ such that

$$\omega_1(f;C_p;t) \leq M_1 \cdot t \qquad \text{for} \quad t \in R_0. \tag{7}$$

We introduce the following

1 Definition. Let $R_2 := [2, +\infty)$ and let $r \in R_2$ and $p \in N_0$ be fixed numbers. For functions $f \in C_p$ we define the operators

$$A_n(f;r;x) := e^{-(nx+1)^r} \sum_{k=0}^{\infty} \frac{(nx+1)^{rk}}{k!} f\left(\frac{k}{n(nx+1)^{r-1}}\right), \tag{8}$$

$x \in R_0$, $n \in N$.

Similarly as S_n, the operator A_n is linear and positive. In § 2 we shall prove that A_n is an operator from the space C_p into itself for every fixed $p \in N_0$.

From (8) we easily derive the following formulas

$$A_n(1;r;x) = 1, \tag{9}$$

$$A_n(t;r;x) = x + \frac{1}{n}, \qquad A_n(t^2;r;x) = \left(x+\frac{1}{n}\right)^2 \left[1 + \frac{1}{(nx+1)^r}\right]$$

$$A_n(t^3;r;x) = \left(x+\frac{1}{n}\right)^3 \left[1 + \frac{3}{(nx+1)^r} + \frac{1}{(nx+1)^{2r}}\right],$$

for every fixed $r \in R_2$ and for all $n \in N$ and $x \in R_0$.

1 Main results

From formulas (8), (9) and $A_n(t^k;r;x)$, $1 \le k \le 3$, given above we obtain

2 Lemma. *Let $r \in R_2$ be a fixed number. Then for all $x \in R_0$ and $n \in N$ we have*

$$A_n\left(t-x;r;x\right) = \frac{1}{n},$$

$$A_n\left((t-x)^2;r;x\right) = \frac{1}{n^2}\left[1+\frac{1}{(nx+1)^{r-2}}\right],$$

$$A_n\left((t-x)^3;r;x\right) = \frac{1}{n^3}\left[1+\frac{3}{(nx+1)^{r-2}}+\frac{1}{(nx+1)^{2r-3}}\right].$$

Next we shall prove

3 Lemma. *Let $s \in N$ and $r \in R_2$ be fixed numbers. Then there exist positive numbers $\lambda_{s,j}$, $1 \le j \le s$, depending only on j and s, such that*

$$A_n(t^s;r;x) = \left(x+\frac{1}{n}\right)^s \sum_{j=1}^{s} \frac{\lambda_{s,j}}{(nx+1)^{(j-1)r}} \qquad (10)$$

for all $n \in N$ and $x \in R_0$. Moreover $\lambda_{s,1} = \lambda_{s,s} = 1$.

Proof. We shall use the mathematical induction on s.

The formula (10) for $s = 1, 2, 3$ is given above.

Let (10) holds for $f(x) = x^j$, $1 \le j \le s$, with fixed $s \in N$. We shall prove (10) for $f(x) = x^{s+1}$. From (8) it follows that

$$A_n(t^{s+1};r;x) = e^{-(nx+1)^r}\sum_{k=1}^{\infty}\frac{(nx+1)^{rk}}{(k-1)!}\frac{k^s}{(n(nx+1)^{r-1})^{s+1}} =$$

$$= \frac{(nx+1)^r}{(n(nx+1)^{r-1})^{s+1}}e^{-(nx+1)^r}\sum_{k=0}^{\infty}\frac{(nx+1)^{rk}}{k!}(k+1)^s =$$

$$= \frac{(nx+1)^r}{(n(nx+1)^{r-1})^{s+1}}e^{-(nx+1)^r}\sum_{k=0}^{\infty}\frac{(nx+1)^{rk}}{k!}\sum_{\mu=0}^{s}\binom{s}{\mu}k^\mu =$$

$$= \frac{(nx+1)^r}{(n(nx+1)^{r-1})^{s+1}}\sum_{\mu=0}^{s}\binom{s}{\mu}(n(nx+1)^{r-1})^\mu A_n(t^\mu;r;x).$$

By our assumption we get

$$A_n(t^{s+1};r;x) = \frac{(nx+1)^r}{(n(nx+1)^{r-1})^{s+1}}\cdot$$

$$\cdot\left\{1+\sum_{\mu=1}^{s}\binom{s}{\mu}(nx+1)^{r\mu}\sum_{j=1}^{\mu}\frac{\lambda_{\mu,j}}{(nx+1)^{(j-1)r}}\right\}=$$

$$=\left(x+\frac{1}{n}\right)^{s+1}\left\{\frac{1}{(nx+1)^{rs}}+\sum_{j=1}^{s}\sum_{\mu=j}^{s}\binom{s}{\mu}\frac{\lambda_{\mu,j}}{(nx+1)^{(s+j-\mu-1)r}}\right\}=$$

$$=\left(x+\frac{1}{n}\right)^{s+1}\left\{\frac{1}{(nx+1)^{rs}}+\sum_{j=1}^{s}\frac{1}{(nx+1)^{(j-1)r}}\sum_{\mu=s-j+1}^{s}\binom{s}{\mu}\lambda_{\mu,\mu+j-s}\right\}=$$

$$=\left(x+\frac{1}{n}\right)^{s+1}\sum_{j=1}^{s+1}\frac{\lambda_{s+1,j}}{(nx+1)^{(j-1)r}}$$

and $\lambda_{s+1,1}=\lambda_{s+1,s+1}=1$, which proves (10) for $f(x)=x^{s+1}$. Hence the proof of (10) is completed. QED

4 Lemma. *Let $p\in N_0$ and $r\in R_2$ be fixed numbers. Then there exists a positive constant $M_2\equiv M_2(p,r)$, depending only on the parameters p and r such that*

$$\|A_n(1/w_p(t);r;\cdot)\|_p\le M_2,\qquad n\in N. \tag{11}$$

Moreover for every $f\in C_p$ we have

$$\|A_n(f;r;\cdot)\|_p\le M_2\|f\|_p,\qquad n\in N. \tag{12}$$

The formula (8) *and inequality* (12) *show that A_n, $n\in N$, is a positive linear operator from the space C_p into C_p, for every $p\in N_0$.*

Proof. The inequality (11) is obvious for $p=0$ by (2), (3) and (9). Let $p\in N$. Then by (2) and (8)-(10) we have

$$w_p(x)A_n(1/w_p(t);r;x)=w_p(x)\{1+A_n(t^p;r;x)\}=$$

$$=\frac{1}{1+x^p}++\frac{(x+1/n)^p}{1+x^p}\sum_{j=1}^{p}\frac{\lambda_{p,j}}{(nx+1)^{(j-1)r}}\le$$

$$\le 1+\sum_{\mu=0}^{p}\binom{p}{\mu}\frac{x^\mu}{1+x^p}\sum_{j=1}^{p}\frac{\lambda_{p,j}}{(nx+1)^{(j-1)r}}\le M_2(p,r),$$

for $x\in R_0$, $n\in N$ and $r\in R_2$, where $M_2(p,r)$ is a positive constant depending only p and r. From this follows (11).

The formula (8) and (3) imply

$$\|A_n(f(t);r;\cdot)\|_p\le\|f\|_p\|A_n(1/w_p(t);r;\cdot)\|_p,\qquad n\in N,\quad r\in R_2,$$

for every $f\in C_p$. By applying (11), we obtain (12). QED

5 Lemma. *Let $p \in N_0$ and $r \in R_2$ be fixed numbers. Then there exists a positive constant $M_3 \equiv M_3(p, r)$ such that*

$$\left\| A_n\left(\frac{(t-\cdot)^2}{w_p(t)}; r; \cdot\right)\right\|_p \leq \frac{M_3}{n^2} \qquad \text{for all} \quad n \in N. \tag{13}$$

Proof. The formulas given in 2 Lemma and (2), (3) imply (13) for $p = 0$.
By (2) and (9) we have

$$A_n\left((t-x)^2/w_p(t); r; x\right) = A_n\left((t-x)^2; r; x\right) + A_n\left(t^p(t-x)^2; r; x\right),$$

for $p, n \in N$ and $r \in R_2$. If $p = 1$ then by the equality we get

$$A_n\left((t-x)^2/w_1(t); r; x\right) = A_n\left((t-x)^2; r; x\right) + A_n\left(t(t-x)^2; r; x\right) =$$

$$= A_n\left((t-x)^3; r; x\right) + (1+x)A_n\left((t-x)^2; r; x\right),$$

which by (2) and (3) and 2 Lemma yields (13) for $p = 1$.
Let $p \geq 2$. By applying (10), we get

$$w_p(x)A_n\left(t^p(t-x)^2; r; x\right) = w_p(x)\left\{A_n\left(t^{p+2}; r; x\right) - 2xA_n\left(t^{p+1}; r; x\right) + \right.$$

$$\left. + x^2 A_n\left(t^p; r; x\right)\right\} = w_p(x)\left\{\left(x+\frac{1}{n}\right)^{p+2}\sum_{j=1}^{p+2}\frac{\lambda_{p+2,j}}{(nx+1)^{(j-1)r}} + \right.$$

$$-2x\left(x+\frac{1}{n}\right)^{p+1}\sum_{j=1}^{p+1}\frac{\lambda_{p+1,j}}{(nx+1)^{(j-1)r}} +$$

$$\left. + x^2\left(x+\frac{1}{n}\right)^{p}\sum_{j=1}^{p}\frac{\lambda_{p,j}}{(nx+1)^{(j-1)r}}\right\} =$$

$$= w_p(x)\left(x+\frac{1}{n}\right)^p\left\{\frac{1}{n^2} + \left(x+\frac{1}{n}\right)^2\sum_{j=2}^{p+2}\frac{\lambda_{p+2,j}}{(nx+1)^{(j-1)r}} + \right.$$

$$\left. -2x\left(x+\frac{1}{n}\right)\sum_{j=2}^{p+1}\frac{\lambda_{p+1,j}}{(nx+1)^{(j-1)r}} + x^2\sum_{j=2}^{p}\frac{\lambda_{p,j}}{(nx+1)^{(j-1)r}}\right\}$$

which implies

$$w_p(x)A_n\left(t^p(t-x)^2; r; x\right) \leq \frac{1}{n^2}\frac{(1+x)^p}{1+x^p}\left\{1 + \frac{1}{(nx+1)^{r-2}}\left(\sum_{j=2}^{p+2}\lambda_{p+2,j} + \right.\right.$$

$$+\ 2\sum_{j=2}^{p+1}\lambda_{p+1,j}+\sum_{j=2}^{p}\lambda_{p,j}\Bigg)\Bigg\}\le\frac{M_3(p,r)}{n^2}$$

for $x\in R_0$, $n\in N$ and $r\in R_2$. Thus the proof is completed. QED

Now we shall give approximation theorems for A_n.

6 Theorem. *Let $p\in N_0$ and $r\in R_2$ be fixed numbers. Then there exists a positive constant $M_4\equiv M_4(p,r)$ such that for every $f\in C_p^1$ we have*

$$\|A_n(f;r;\cdot)-f(\cdot)\|_p\le\frac{M_4}{n}\|f'\|_p,\qquad n\in N. \tag{14}$$

PROOF. Let $x\in R_0$ be a fixed point. Then for $f\in C_p^1$ we have

$$f(t)-f(x)=\int_x^t f'(u)du,\qquad t\in R_0.$$

From this and by (8) and (9) we get

$$A_n\left(f(t);r;x\right)-f(x)=A_n\left(\int_x^t f'(u)du;r;x\right),\qquad n\in N.$$

But by (2) and (3) we have

$$\left|\int_x^t f'(u)du\right|\le\|f'\|_p\left(\frac{1}{w_p(t)}+\frac{1}{w_p(x)}\right)|t-x|,\qquad t\in R_0,$$

which implies

$$w_p(x)|A_n(f;r;x)-f(x)|\le \tag{15}$$
$$\le\|f'\|_p\left\{A_n\left(|t-x|;r;x\right)+w_p(x)A_n\left(\frac{|t-x|}{w_p(t)};r;x\right)\right\}$$

for $n\in N$. By the Hölder inequality and by (9) and 2, 4, 5 Lemmas it follows that

$$A_n\left(|t-x|;r;x\right)\le\left\{A_n\left((t-x)^2;r;x\right)A_n\left(1;r;x\right)\right\}^{1/2}\le\frac{\sqrt{2}}{n},$$

$$w_p(x)A_n\left(\frac{|t-x|}{w_p(t)};r;x\right)\le$$

$$\le w_p(x)\left\{A_n\left(\frac{(t-x)^2}{w_p(t)};r;x\right)\right\}^{1/2}\left\{A_n\left(\frac{1}{w_p(t)};r;x\right)\right\}^{1/2}\le$$

$$\le\frac{M_4}{n}$$

for $n\in N$. From this and by (15) we immediately obtain (14). QED

7 Theorem. *Let $p \in N_0$ and $r \in R_2$ be fixed numbers. Then there exists $M_6 \equiv M_6(p,r)$ such that for every $f \in C_p$ and $n \in N$ we have*

$$\|A_n(f;r;\cdot) - f(\cdot)\|_p \le M_6\omega_1\left(f;C_p;\frac{1}{n}\right). \tag{16}$$

Proof. We use Steklov function f_h of $f \in C_p$

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)dt, \qquad x \in R_0, \quad h > 0. \tag{17}$$

From (17) we get

$$f_h(x) - f(x) = \frac{1}{h}\int_0^h \Delta_t f(x)dt,$$

$$f_h'(x) = \frac{1}{h}\Delta_h f(x), \quad x \in R_0, \quad h > 0,$$

which imply

$$\|f_h - f\|_p \le \omega_1(f;C_p;h), \tag{18}$$

$$\|f_h'\|_p \le h^{-1}\omega(f;C_p;h), \tag{19}$$

for $h > 0$. From this we deduce that $f_h \in C_p^1$ if $f \in C_p$ and $h > 0$.
Hence we can write

$$w_p(x)|A_n(f;x) - f(x)| \le w_p(x)\{|A_n(f - f_h;x)| +$$

$$+ |A_n(f_h;x) - f_h(x)| + |f_h(x) - f(x)|\} := L_1(x) + L_2(x) + L_3(x),$$

for $n \in N$, $h > 0$ and $x \in R_0$. From (12) and (18) we get

$$\|L_1\|_p \le M_2\|f_h - f\|_p \le M_2\omega_1(f;C_p;h),$$

$$\|L_3\|_p \le \omega_1(f;C_p;h).$$

By 6 Theorem and (19) it follows that

$$\|L_2\|_p \le \frac{M_4}{n}\|f_h'\|_p \le \frac{M_4}{nh}\omega_1(f;C_p;h).$$

Consequently

$$\|A_n(f;r;\cdot) - f(\cdot)\|_p \le \left(1 + M_2 + \frac{M_4}{nh}\right)\omega_1(f;C_p;h).$$

Now, for fixed $n \in N$, setting $h = \frac{1}{n}$, we obtain

$$\|A_n(f;r;\cdot) - f(\cdot)\|_p \le M_6(p,r)\omega_1\left(f;C_p;\frac{1}{n}\right)$$

and we complete the proof. QED

From 6 Theorem and 7 Theorem we derive the following two corollaries:

8 Corollary. *For every fixed* $r \in R_2$ *and* $f \in C_p$, $p \in N_0$, *we have*

$$\lim_{n\to\infty} \|A_n(f;r;\cdot) - f(\cdot)\|_p = 0.$$

9 Corollary. *If* $f \in C_p^1$, $p \in N_0$ *and* $r \in R_2$, *then*

$$\|A_n(f;r;\cdot) - f(\cdot)\|_p = O(1/n).$$

Finally, we shall give the Voronovskaya type theorem for A_n.

10 Theorem. *Let* $f \in C_p^1$ *and let* $r \in R_2$ *be fixed number. Then,*

$$\lim_{n\to\infty} n\left\{A_n\left(f;r;x\right) - f(x)\right\} = f'(x) \tag{20}$$

for every $x \in R_0$.

Proof. Let $x \in R_0$ be a fixed point. Then by the Taylor formula we have

$$f(t) = f(x) + f'(x)(t-x) + \varepsilon(t;x)(t-x)$$

for $t \in R_0$, where $\varepsilon(t) \equiv \varepsilon(t;x)$ is a function belonging to C_p and $\varepsilon(x) = 0$. Hence by (8) and (9) we get

$$A_n(f;r;x) = f(x) + f'(x)A_n(t-x;r;x) + A_n(\varepsilon(t)(t-x);r;x), \qquad n \in N, \tag{21}$$

and by Hölder inequality

$$|A_n(\varepsilon(t)(t-x);r;x)| \le \left\{A_n\left(\varepsilon^2(t);x\right)\right\}^{1/2}\left\{A_n\left((t-x)^2;x\right)\right\}^{1/2}.$$

By 8 Corollary we deduce that

$$\lim_{n\to\infty} A_n\left(\varepsilon^2(t);r;x\right) = \varepsilon^2(x) = 0.$$

From this and by 2 Lemma we get

$$\lim_{n\to\infty} nA_n(\varepsilon(t)(t-x);r;x) = 0. \tag{22}$$

Using (22) and 2 Lemma to (21), we obtain the desired assertion (20). QED

11 Remark. It is easily verified that the operators

$$\overline{A}_n(f;r;x) := e^{-(nx+1)^r}\sum_{k=0}^{\infty}\frac{(nx+1)^{rk}}{k!}n(nx+1)^{r-1}\int_{(k+r)/(n(nx+1)^{r-1})}^{(k+1+r)/(n(nx+1)^{r-1})} f(t)dt,$$

$p \in N_0$, $x \in R_0$, $n \in N$ and $r \in R_2$, have analogous approximation properties in the space C_p.

12 Remark. In [1] it was proved that if $f \in C_p$, $p \in N_0$, then for the Szasz-Mirakyan operators S_n (defined by (1)) is satisfied the following inequality

$$w_p(x)|S_n(f;x) - f(x)| \leq M_9\omega_2\left(f; C_p; \sqrt{\frac{x}{n}}\right), \qquad x \in R_0, \quad n \in N_0,$$

where $M_9 = const. > 0$ and $\omega_2(f;\cdot)$ is the modulus of smoothness defined by the formula

$$\omega_2(f; C_p; t) := \sup_{0\leq h\leq t} \|\Delta_h^2 f(\cdot)\|_p, \qquad t \in R_0,$$

where $\Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h)$. In particular, if $f \in C_p^1$, $p \in N_0$, then

$$w_p(x)|S_n(f;x) - f(x)| \leq M_{10}\sqrt{\frac{x}{n}},$$

for $x \in R_0$ and $n \in N$ ($M_{10} = const. > 0$).

7 Theorem and 10 Theorem and 9 Corollary in our paper show that operators A_n, $n \in N$, give better degree of approximation of functions $f \in C_p$ and $f \in C_p^1$ than S_n.

References

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