(LB)-spaces and quasi-reflexivity

Manuel Valdivia
Departamento de Análisis Matemático,
Universidad de Valencia,
Dr. Moliner, 50,
46100 Burjasot (Valencia), Spain
manuel.valdivia@uv.es

Received: 03/06/2010; accepted: 24/09/2010.

Abstract. Let \((X_n)\) be a sequence of infinite-dimensional Banach spaces. For \(E\) being the space \(\bigoplus_{n=1}^{\infty} X_n\), the following equivalences are shown: 1. Every closed subspace \(Y\) of \(E\), with the Mackey topology \(\mu(Y, Y')\), is an (LB)-space. 2. Every separated quotient of \(E' [\mu(E', E)]\) is locally complete. 3. \(X_n\) is quasi-reflexive, \(n \in \mathbb{N}\). Besides this, the following two properties are seen to be equivalent: 1. \(E' [\mu(E', E)]\) has the Krein-Šmulian property. 2. \(X_n\) is reflexive, \(n \in \mathbb{N}\).

MSC 2000 classification: primary 46A13, secondary 46A04

Dedicated to the memory of V.B. Moscatelli

1 Introduction and notation

The linear spaces that we shall use here are assumed to be defined over the field \(\mathbb{K}\) of real or complex numbers, and the topologies on them will all be Hausdorff. As usual, \(\mathbb{N}\) represents the set of positive integers. If \((E, F)\) is a dual pair, then \(\sigma(E, F)\), \(\mu(E, F)\) and \(\beta(E, F)\) denote the weak, Mackey and strong topologies on \(E\), respectively; we shall write \(\langle \cdot, \cdot \rangle\) for the bilinear functional associated to \((E, F)\). If \(E\) is a locally convex space, \(E'\) is its topological dual and \((E, E')\), and also \((E', E)\), denote the standard duality. \(E''\) stands for the dual of \(E' [\beta(E', E)]\). We identify, in the usual manner, \(E\) with a linear subspace of \(E''\). If \(A\) is a bounded absolutely convex subset of a locally convex space \(E\), then \(E|_A\) means the linear span of \(A\) with the norm provided by the gauge of \(A\); the space \(E\) is said to be locally complete whenever \(E|_A\) is complete for each closed bounded absolutely convex subset \(A\) of \(E\). If \(E\) is sequentially complete, in particular if \(E\) is complete, then \(E\) is locally complete.

We write \(\omega\) to denote the space \(\mathbb{K}^\mathbb{N}\) with the product topology.

\(\text{Supported in part by MICINN and FEDER Project MTM2008-03211}
\text{http://siba-ese.unisalento.it/} \text{© 2011 Università del Salento}
Following [6] (see also [1, p. 299]), we say that a locally convex space $E$ is $B$-complete if each subspace $F$ of $E'$ [$\sigma(E', E)$] is closed provided it intersects every closed absolutely convex subset which is equicontinuous in a closed set.

A locally convex space $E$ is said to be an (LB)-space if it is the inductive limit of a sequence of Banach spaces, or, equivalently, if $E$ is the separated quotient of the topological direct sum of a sequence of Banach spaces.

In [7], we obtain the following result: a) Let $(X_n)$ be a sequence of infinite-dimensional Banach spaces. If $E := \bigoplus_{n=1}^{\infty} X_n$, then the following are equivalent: 1. $E'$ [$\mu(E', E)$] is $B$-complete. 2. Every separated quotient of $E'$ [$\mu(E', E)$] is complete. 3. $X_n$ is quasi-reflexive, $n \in \mathbb{N}$.

In Section 2 of this paper, we obtain a theorem which adds new equivalences to the three before stated.

A locally convex space $E$ is said to have the Krein-Šmulian Property whenever a convex subset $A$ of $E'$ is $\sigma(E', E)$-closed provided that, for each absolutely convex $\sigma(E', E)$-closed and equicontinuous subset $M$ of $E'$, the set $A \cap M$ is $\sigma(E', E)$-closed. Krein-Šmulian’s theorem asserts that every Fréchet space has the Krein-Šmulian Property [1, p. 246].

In this paper, we characterize when $E'$ [$\mu(E', E)$] has the Krein-Šmulian Property, when $E := \bigoplus_{n=1}^{\infty} X_n$, with $X_n$, $n \in \mathbb{N}$, being a Banach space of infinite dimension.

2 (LB)-spaces and quasi-reflexivity

**Theorem 1.** Let $(X_n)$ be a sequence of infinite-dimensional Banach spaces. For $E$ being $\bigoplus_{n=1}^{\infty} X_n$, the following are equivalent:

1. Every separated quotient of $E'$ [$\mu(E', E)$] is locally complete.
2. Every closed subspace $Y$ of $E$, with the Mackey topology $\mu(Y', Y)$, is an (LB)-space.
3. $X_n$ is quasi-reflexive, $n \in \mathbb{N}$.

**Proof.** For each $n \in \mathbb{N}$, we write $E_n := \bigoplus_{j=1}^{n} X_j$ and we consider, in the usual way, that $E_n$ is a subspace of $E$. We set $B_n := n \bigoplus_{j=1}^{n} B(X_j), \; n \in \mathbb{N}$.

$1 \Rightarrow 2.$ Let us assume that 2. is not satisfied for a certain closed subspace $Y$ of $E$. We put $Y_n := E_n \cap Y$ and $A_n := B_n \cap Y, \; n \in \mathbb{N}$. We find a linear functional $v$ on $Y$ such that it is not continuous although its restriction to each subspace $Y_n$ is continuous. After Hahn-Banach’s extension theorem we obtain, for each $n \in \mathbb{N}$, an element $u_n$ of $Y'$ such that $u_n|_{Y_n} = v|_{Y_n}$.

For an arbitrary $x$ of $Y$, we find $n_0 \in \mathbb{N}$ such that $x \in Y_{n_0}$. Then $\langle x, u_n \rangle = \langle x, v \rangle, \; n \geq n_0$, and thus $\{u_n : n \in \mathbb{N}\}$ is a bounded subset of $Y'$ [$\sigma(Y', Y)$]. If $T$ denotes the polar subset in $Y$ of $\{u_n : n \in \mathbb{N}\}$, we have that $T$ is a barrel in $Y$ that absorbs each of the sets $A_n, \; n \in \mathbb{N}$.

We now find a sequence of positive integers $(j_n)$ such that $\frac{1}{j_n} A_n \subseteq T, \; n \in \mathbb{N}$. 

(LB)-spaces and quasi-reflexivity

Let $A$ be the convex hull of
$$
\bigcup \left\{ \frac{1}{j_n 2^n} A_n : n \in \mathbb{N} \right\}.
$$
Since $A$ is absorbing in $Y$, we have that $A^\circ$ is a closed bounded absolutely convex subset of $Y'[\sigma(Y', Y)]$. We prove next that $Y'_A$ is not a Banach space. Let $\| \cdot \|$ denote the norm in $Y_A$. Given $\varepsilon > 0$, we find $n_0 \in \mathbb{N}$ such that $\frac{1}{j_n 2^n} < \frac{\varepsilon}{4}$.

We take two integers $p, q$ such that $p > q > n_0$. We can find an element $z$ of $A$ for which
$$
\| u_p - u_q \| \leq 4 \left| \langle z, u_p - u_q \rangle \right|.
$$

$z$ may be written in the form
$$
\sum_{n=1}^{\infty} \alpha_n z_n, \quad \alpha_n \geq 0, \quad z_n \in \frac{1}{j_n 2^n}, \quad n \in \mathbb{N}, \quad \sum_{n=1}^{\infty} \alpha_n = 1,
$$
where the terms of the sequence $(\alpha_n)$ are all zero from a certain subindex on. Then

$$
\| u_p - u_q \| \leq 4 \left| \langle z, u_p - u_q \rangle \right| \leq 4 \sum_{n=1}^{\infty} \left| \langle \alpha_n z_n, u_p - u_q \rangle \right| = 4 \sum_{n=n_0+1}^{\infty} \alpha_n \left| \langle z_n, u_p \rangle \right| + \left| \langle z_n, u_q \rangle \right|
$$

$$
\leq 8 \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} = \frac{8}{2^{n_0}} < \varepsilon.
$$

Consequently, $(u_n)$ is a Cauchy sequence in $Y'_A$. If this were a Banach space, this sequence would converge to a certain element $u$ of $Y'_A$. Clearly, $u$ should coincide with $v$, which is a contradiction. 2 \Rightarrow 3. Assuming that 3 does not hold, after result a), there is a closed subspace $Z$ of $E'[\mu(E', E)]$ such that $E'[\mu(E', E)]/Z$ is not complete. Let $Y$ represent the subspace of $E$ orthogonal to $Z$. Let $w$ be a linear functional on $Y$ which belongs to the completion of $E'[\mu(E', E)]/Z$ but does not belong to $E'[\mu(E', E)]/Z$. From the theorem of Ptak-Collins, [4, p. 271], $w^{-1}(0)$ intersects every weakly compact absolutely convex subset of $Y$ in a closed subset, hence $w$ is bounded in every bounded subset of $Y$. Since $w$ is not continuous in $Y$, we deduce from above that $Y'[\mu(Y, Y')]$ is not an (LB)-space. 3 \Rightarrow 1. After result a), every separated quotient of $E'[\mu(E', E)]$ is complete and thus it is locally complete. $\square$

In the previous theorem, we have considered closed subspaces $Y$ of $E = \bigoplus_{n=1}^{\infty} X_n$ endowed with the Mackey topology $\mu(Y, Y')$. It may happen that for some closed subspace $Y$ of $E$, $Y$ is not an (LB)-space and nevertheless $Y'[\mu(Y, Y')]$ is indeed an (LB)-space. In Theorem 2, this property is considered when $X_n$ is a reflexive Banach space, $n \in \mathbb{N}$.

We shall then use the following result that we obtained in [8]: b) Let $F$ be a Fréchet space such that for each closed subspace $G$ of $F$ and each bounded subset $B$ of $F/G$ there is a bounded subset $A$ of $F$ for which $\varphi(A) = B$, where $\varphi$ is the canonical projection from $F$ onto $F/G$, then one of the following assertions holds: 1. $F$ is a Banach space. 2. $F$ is a Schwartz space. 3. $F$ is the product of a Banach space by $\omega$.

**Theorem 2.** Let $(X_n)$ be a sequence of reflexive Banach spaces of infinite dimension. Then, there is a closed subspace $Y$ of $E := \bigoplus_{n=1}^{\infty} X_n$ whose topology is not that of Mackey’s $\mu(Y, Y')$.

**Proof.** By applying result b) we obtain a closed subspace $G$ of $F := \prod_{n=1}^{\infty} X_n$ and a closed bounded absolutely convex subset $B$ of $F/G$ such that there is no bounded subset $A$ of $F$ with $\varphi(A) = B$, where $\varphi$ is the canonical projection from $F$ onto $F/G$. Clearly, $F/G$ is
reflexive and so $B$ is weakly compact. We have that $F'[\mu(F', F)] = E$. We identify, in the usual manner, $(F/G)'$ with the subspace $Y$ of $E$ orthogonal to $G$. If $B^o$ is the polar set of $B$ in $Y$, then $B^o$ is a zero-neighborhood in $Y[\mu(Y, Y')]$. Now, given that $B$ is not the image by $\varphi$ of any bounded subset of $F$, there is no zero-neighborhood $U$ in $E$ for which $U \cap Y \subset B^o$. Therefore, the subspace $Y$ of $E$ does not have the Mackey topology. $Y[\mu(Y, Y')]$ is an (LB)-space in light of our former theorem.

3 The Krein–Smulian Property

**Theorem 3.** Let $(X_n)$ be a sequence of Banach spaces of infinite dimension. If $E$ is $\bigoplus_{n=1}^{\infty}X_n$, then $E'[\mu(E', E)]$ has the Krein–Smulian property if and only if $X_n$ is reflexive, $n \in \mathbb{N}$.

Before giving the proof of this theorem, we shall obtain some previous results. For the next four propositions, we shall consider the sequence $(Z_n)$ of infinite-dimensional separable Banach spaces such that $Z_1$ is quasi-reflexive non-reflexive and $Z_n$ is reflexive, for $n = 2, 3, \ldots$ we put $F := \bigoplus_{n=1}^{\infty}Z_n$ and $F_n := \bigoplus_{j=1}^{n}Z_j$, and identify, in the usual fashion, $F_n$ with a subspace of $F$ and $F_n$ with $F_n^{\sigma}$, $n \in \mathbb{N}$. We take a vector $y$ in $Z_1^{\ast} \setminus Z_1$. We fix now $j$ in $\mathbb{N}$. In $F_{j+1}[\sigma(F_{j+1}, F'_{j+1})]$, $F_j + B(Z_{j+1})$ is a closed subset whose intersection with $Z_{j+1}$ coincides with $B(Z_{j+1})$ and, since $B(Z_{j+1})$ is not a weak neighborhood of zero in $Z_{j+1}$, we have that $F_j + B(Z_{j+1})$ has no interior points. On the other hand,

$$\frac{1}{j}y \in F_1 \subset F_j + B(Z_{j+1})$$

and $F_{j+1}[\beta(F_{j+1}, F'_{j+1})]$ is separable, so there is a sequence $(z_n)$ in $F_{j+1} \setminus (F_j + B(Z_{j+1}))$ which converges to $\frac{1}{j}y$ in $F_{j+1}[\sigma(F_{j+1}, F'_{j+1})]$. We may now find a subsequence $(z_{j_n})$ of $(z_n)$ which is basic in $F_{j+1}$, [5]. Let $T_{j+1} \in F_j$ be the projection from $F_{j+1}$ onto $Z_{j+1}$ along $F_j$. Then, $T_{j+1}z_{j_n} \notin B(Z_{j+1})$, $n \in \mathbb{N}$, and the sequence $(T_{j+1}z_{j_n})$ converges weakly to the origin in $Z_{j+1}$. Consequently, meaning we may find in $(z_{j_n})$ a subsequence $(y_{j_n})$ such that $(T_{j+1}y_{j_n})$ is basic in $Z_{j+1}$, [3, p. 334]. In $F_{j+1}[\sigma(F_{j+1}, F'_{j+1})]$, we put $A_j$ for the closed convex hull of $\{j_{j_n} : n \in \mathbb{N}\}$. We have that $\{y_{j_n} : n \in \mathbb{N}\} \cup \{\frac{1}{j}y\}$ is compact and hence $A_j$ is also compact. We choose in $F_{j+1}$ a sequence $(u_{j_n})$ such that $\langle y_{j_n}, u_{j_n} \rangle = 1$, $\langle y_{j_m}, u_{j_n} \rangle = 0$, $m \neq n$, $m, n \in \mathbb{N}$.

**Proposition 1.** An element $z$ of $F_{j+1}$ is in $A_j$ if and only if it can be represented as

$$z = \sum_{n=1}^{\infty}a_ny_{j_n} + \frac{1}{j}ay, \quad a \geq 0, a_n \geq 0, n \in \mathbb{N}, \sum_{n=1}^{\infty}a_n + a = 1,$$

where the coefficients $a$ and $a_n$, $n \in \mathbb{N}$, are univocally determined by $z$.

**Proof.** Clearly, if an element $z$ of $F_{j+1}$ has the representation above given, then it belongs to $A_j$.

An arbitrary element of the convex hull $M_j$ of $\{y_{j_n} : n \in \mathbb{N}\} \cup \{\frac{1}{j}y\}$ has the form

$$\sum_{n=1}^{\infty}a_ny_{j_n} + \frac{1}{j}ay, \quad a \geq 0, a_n \geq 0, n \in \mathbb{N}, \sum_{n=1}^{\infty}a_n + a = 1,$$
where the terms of the sequence \((a_n)\) are all zero except for a finite number of them. Given \(z\) in \(A_j\), we find a net 
\[
\left\{\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y : \ i \in I, \ \geq \right\}
\]
in \(M_j\) such that it \(\sigma(F_{j+1}, F'_{j+1})\)-converges to \(z\). Given \(r\) in \(\mathbb{N}\), we have that 
\[
\left(\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y, u_r\right) = a_n^{(i)} + \frac{1}{j} a^{(i)} (y, u_r)
\]
thus, in \(\mathbb{R}\),
\[
\lim_{n \to \infty} a_n^{(i)} = \langle z, u_r \rangle =: a_r.
\]
Clearly, \(\sum_{r=1}^{\infty} a_r \leq 1\). Let \(a := 1 - \sum_{r=1}^{\infty} a_r\). We consider the vector 
\[
\sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j} a y
\]
of \(F_{j+1}[\sigma(F_{j+1}, F'_{j+1})]\) and we proceed to show that it coincides with \(z\). Given \(u\) in \(F'_{j+1}\), having in mind that \(\{y_{jn} : n \in \mathbb{N}\}\) is bounded in \(F_{j+1}\), we find \(\lambda_j > 0\) such that 
\[
|\langle y_{jn}, u \rangle| < \lambda_j, \ n \in \mathbb{N}, \ |\langle y, u \rangle| < \lambda_j.
\]
Given \(\varepsilon > 0\), we find \(s \in \mathbb{N}\) such that 
\[
|\langle y_{jn} - \frac{1}{j} y, u \rangle| < \frac{\varepsilon}{6}, \ n \geq s.
\]
We now determine \(i_0\) in \(I\) such that, for \(i \geq i_0\),
\[
|a_n - a_n^{(i)}| < \frac{\varepsilon}{6\lambda_j s}, \ n = 1, 2, \ldots, s,
\]
\[
|\langle z - \left(\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y, u \right), u \rangle| < \frac{\varepsilon}{3}
\]
Then, for such values of \(i\),
\[
|\langle z - \left(\sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j} a y, u \right), u \rangle| \leq |\langle z - \left(\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y, u \right), u \rangle| \]

+ \(\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y - \left(\sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j} a y, u \right) < \frac{\varepsilon}{3}
\]

+ \(\sum_{n=1}^{\infty} (a_n^{(i)} - a_n) y_{jn} + \frac{1}{j} (a^{(i)} - a) y, u \rangle| = \frac{\varepsilon}{3}
\]

+ \(\sum_{n=1}^{\infty} (a_n^{(i)} - a_n) y_{jn} + \frac{1}{j} (1 - \sum_{n=1}^{\infty} a_n^{(i)} - (1 - \sum_{n=1}^{\infty} a_n) y, u \rangle| \]

\[
= \frac{\varepsilon}{3} + |\sum_{n=1}^{\infty} (a_n^{(i)} - a_n) y_{jn} + \frac{1}{j} \sum_{n=1}^{\infty} (a_n - a_n^{(i)}) y, u \rangle|.
\]
\[ \leq \frac{\varepsilon}{3} + | \left( \sum_{n=1}^{s} (a_n^{(i)} - a_n)y_{jn} + \frac{1}{j} \sum_{n=1}^{s} (a_n - a_n^{(i)})y, u) \right| \\
+ | \left( \sum_{n=s+1}^{\infty} (a_n^{(i)} - a_n)y_{jn} + \frac{1}{j} \sum_{n=s+1}^{\infty} (a_n - a_n^{(i)})y, u) \right| \\
\leq \frac{\varepsilon}{3} + 2\lambda j \sum_{n=1}^{s} | a_n - a_n^{(i)} | + \sum_{n=s+1}^{\infty} | a_n - a_n^{(i)} | \cdot | y_{jn} - \frac{1}{j} y, u) | \\
\leq \frac{\varepsilon}{3} + 2\lambda j s \frac{\varepsilon}{6\lambda j s} + 2\frac{\varepsilon}{6} = \varepsilon, \]

from where we deduce that, in $\tilde{F}_{j+1} [\sigma(\tilde{F}_{j+1}, F'_{j+1})]$, 

\[ z = \sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j} a y. \]

Besides, it is plain that 

\[ a_n = \left< z, u_n \right>, \quad n \in \mathbb{N}, \quad a = 1 - \sum_{n=1}^{\infty} \left< z, u_n \right>. \]

**Corollary 1.** We have that 

\[ A_j \cap F_{j+1} = \{ \sum_{n=1}^{\infty} a_n y_{jn} : a_n \geq 0, n \in \mathbb{N}, \sum_{n=1}^{\infty} a_n = 1 \}. \]

**Corollary 2.** If $z \in A_j$, then $z$ may be univocally expressed as 

\[ z = b z_1 + \frac{1}{j} c y, \quad z_1 \in A_j \cap F_{j+1}, \quad b \geq 0, \quad c \geq 0, \quad b + c = 1. \]

In the sequel, we put $D$ for the convex hull of 

\[ \cup \{ A_j \cap F_{j+1} : j \in \mathbb{N} \} \]

and $D_r$ for the convex hull of 

\[ \cup \{ A_j \cap F_{j+1} : j = 1, 2, \ldots, r \}, \quad r \in \mathbb{N}. \]

**Proposition 2.** For each $r \in \mathbb{N}$, we have that 

\[ D_r = D \cap F_{r+1}. \]

**Proof.** Given a positive integer $s$, we take an element $z$ of $D_{s+1}$. Then, $z$ may be written in the form 

\[ z = \sum_{j=1}^{s+1} \alpha_j z_j, \quad z_j \in A_j \cap F_{j+1}, \quad \alpha_j \geq 0, \quad j = 1, 2, \ldots, s + 1, \sum_{j=1}^{s+1} \alpha_j = 1. \]

Let us first assume that $\alpha_{s+1} \neq 0$. After Corollary 1, $z_{j+1}$ can be written as 

\[ \sum_{n=1}^{\infty} a_n y_{(j+1)n}, a_n \geq 0, n \in \mathbb{N}, \sum_{n=1}^{\infty} a_n = 1. \]
We have that \((T_{s+2}(y_{(s+1)n}))\) is a basic sequence in \(Z_{s+2}\) and thus the vector of \(Z_{s+2}\)

\[
T_{s+2}(z_{s+1}) = T_{s+2}(\sum_{n=1}^{\infty} a_n y_{(s+1)n}) = \sum_{n=1}^{\infty} a_n T_{s+2}(y_{(s+1)n})
\]

is non-zero. Then

\[
\alpha_{s+1} z_{s+1} \notin F_{s+1}
\]

and since

\[
\sum_{j=1}^{s} \alpha_j z_j \in F_{s+1},
\]

it follows that

\[
z = \sum_{j=1}^{s+1} \alpha_j z_j \notin F_{s+1}.
\]

On the other hand, if \(\alpha_{s+1} = 0\), we have that \(z\) belongs to \(D_s\).

We deduce from above that

\[
D_{s+1} \cap F_{s+1} \subset D_s
\]

and, since \(D_s\) is clearly contained in \(D_{s+1} \cap F_{s+1}\), it follows that

\[
D_s = D_{s+1} \cap F_{s+1}.
\]

Finally, given \(r \in \mathbb{N}\), we have that

\[
D_r = D_{r+1} \cap F_{r+1} = D_{r+2} \cap F_{r+2} \cap F_{r+1} = D_{r+2} \cap F_{r+1}
\]

and, proceeding recurrently, we have that, for each \(m \in \mathbb{N}\),

\[
D_r = D_{r+m} \cap F_{r+1},
\]

from where we conclude that

\[
D_r = (\bigcup_{m=1}^{\infty} D_{r+m}) \cap F_{r+1} = D \cap F_{r+1}
\]

Proposition 3. For each \(r \in \mathbb{N}\), \(D_r\) is closed in \(F_{r+1}\).

Proof. We write \(C_r\) for the convex hull of \(\cup\{A_j : j = 1, 2, \ldots, r\}\). Clearly, \(C_r\) is \(\sigma(F_{r+1}, F'_{r+1})\)-compact and so it suffices to show that \(D_r\) coincides with \(C_r \cap F_{r+1}\). We take \(z\) in \(C_r\). After Corollary 2, \(z\) may be written in the form

\[
\sum_{j=1}^{r} \alpha_j (a_j z_j + \frac{1}{j} b_j y_j), \quad a_j \geq 0, \quad b_j \geq 0, \quad \alpha_j \geq 0,
\]

\[
a_j + b_j = 1, \quad z_j \in A_j \cap F_{j+1}, \quad j = 1, 2, \ldots, r, \quad \sum_{j=1}^{r} \alpha_j = 1.
\]

If \(z\) belongs to \(F_{r+1}\), then \(\sum_{j=1}^{r} \frac{1}{j} \alpha_j b_j = 0\) and thus \(\alpha_j b_j = 0, j = 1, 2, \ldots, r\). Then

\[
z = \sum_{j=1}^{r} \alpha_j a_j z_j = \sum_{j=1}^{r} \alpha_j (1 - b_j) z_j = \sum_{j=1}^{r} \alpha_j z_j,
\]

from where we deduce that \(z\) is in \(D_r\). Therefore

\[
C_r \cap F_{r+1} \subset D_r.
\]

On the other hand, it is immediate that \(D_r\) is contained in \(C_r \cap F_{r+1}\) and the result follows.
Proposition 4. In $F$, each weakly compact absolutely convex subset intersects $D$ in a closed set. Besides, $D$ is not closed in $F$.

Proof. Let $M$ be a weakly compact absolutely convex subset of $F$. Then there is $r \in \mathbb{N}$ such that $M$ is contained in $F_{r+1}$. Then

$$M \cap D = M \cap F_{r+1} \cap D = M \cap D_r$$

and, after the previous proposition, we have that $M \cap D_r$ is closed in $F_{r+1}$, from where we get that $M \cap D$ is closed in $F$. On the other hand, the origin of $F$ is not in $D$. We consider a weak neighborhood $U$ of the origin in $F$. We find an open neighborhood $V$ of the origin in $F''$ such that $V \cap F \subset U$. We find $s \in \mathbb{N}$ so that $\frac{1}{2^s}y \in V$. Now, since $V$ is a neighborhood of $\frac{1}{2^s}y$ in $F''$, there is $m \in \mathbb{N}$ for which $y_m \in V$. Consequently, $U \cap D \neq \emptyset$, thus the weak closure of $D$ in $E$ contains the origin and hence $D$ is not closed in $F$.

Finally, we give the proof of Theorem 3, but for that we shall need the following result to be found in [9]: (c) Let $X$ be an infinite-dimensional Banach space such that $X''$ is separable. Let $T$ be a closed subspace of $X''$ containing $X$. Then there is an infinite-dimensional closed subspace $Y$ of $X$ such that $X + Y = T$.

Proof. If $X_n$ is reflexive, $n \in \mathbb{N}$, then $E$ is the Mackey dual of the space $E' [\mu(E', E)]$ and so this space has the Krein-Smulian Property. If some of the spaces $X_n$, $n \in \mathbb{N}$, is not quasi-reflexive, then we apply result a) to obtain that $E'' [\mu(E', E)]$ is not B-complete and so it does not have the Krein-Smulian Property. It remains to consider the case in which all the spaces $X_n$, $n \in \mathbb{N}$, are quasi-reflexive and there is at least one of them which is not reflexive. More precisely, let us assume that $X_1$ is not reflexive. From Eberlein's theorem, $B(X_1)$ is not weakly countably compact and so there is a sequence $(x_n)$ in $B(X_1)$ with no weak cluster points in $X_1$. Let $Z_1$ be the closed linear span in $X_1$ of $\{x_n : n \in \mathbb{N}\}$. Then, $Z_1$ is a separable Banach space which is quasi-reflexive but not reflexive. For each $n \in \mathbb{N}$, $n > 1$, we find in $X_n$ a separable closed subspace $Y_n$ of infinite dimension. Since $Y_n$ is quasi-reflexive, it follows that $Y_n''$ is separable, from where, applying result c) for the case $T = X = Y_n$, we have that there is a separable closed subspace $Z_0$ of $Y_n$, with infinite dimension, such that $Y_n + Z_0 = Y_n$, that is, $Z_0 \subset Y_n$ and so $Z_0$ is reflexive. We have that $F := \bigoplus_{n=1}^{\infty} Y_n$ is a closed subspace of $E = \bigoplus_{n=1}^{\infty} X_n$. On the other hand, after Proposition 4, there is a convex subset $D$ of $F$, not closed, which meets each weakly compact absolutely convex subset of $F$ in a closed set. Then $D$ is a convex non-closed subset of $E$ that meets each weakly compact absolutely convex subset of $E$ in a closed subset of $E$. Consequently, $E'' [\mu(E', E)]$ does not have the Krein-Smulian Property.

References


(LB)-spaces and quasi-reflexivity


