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Time-Frequency Analysis: Function Spaces and Applications

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Abstract. We give a survey on recent results concerning modulation spaces, with emphasis on applications to boundedness of localization operators and Fourier Integral operators. We also recall the basic notions of the Time-Frequency Analysis, which provides language and motivation to these results.

Keywords: modulation spaces, localization operators, Fourier integral operators, Wigner distribution, short-time Fourier transform, Schatten class

MSC 2000 classification: primary 47G30, secondary 35S05

Dedicated to the memory of V.B. Moscatelli

1 Introduction and Definitions

The objective of this paper is to report on the modulation spaces, introduced by Feichtinger in 1983, and nowadays used in many fields of Analysis. Namely, our aim is to explain the important role of these spaces in the study of the properties of the so-called *localization operators*, see [11], and of a certain class of Fourier Integral operators (in short, FIOs), see [17].

To this end, here and in Section 2, we begin to give some basic elements of Time-Frequency Analysis, explaining the meaning of the localization operators as filters in Signal Theory, cf. [31]. This will prepare to the definition and the study of the modulation spaces, which we postpone to Sections 3 and 4, and to the applications in Sections 5 and 6.

At the basis of the Time-Frequency Analysis there are the linear operators of translation and modulation (so-called time-frequency shifts) given by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t)\,. \tag{1}$$

These occur in the following time-frequency representation. Let g be a non-zero window function in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, then the short-time Fourier transform (STFT) of a signal

$f \in L^2(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i \omega t}\,dt\,. \tag{2}$$

We have $V_g f \in L^2(\mathbb{R}^{2d})$. This definition can be extended to every pair of dual topological vector spaces, whose duality, denoted by $\langle \cdot,\cdot \rangle$, extends the inner product on $L^2(\mathbb{R}^d)$. For instance, it may be suited to the framework of tempered distributions.

Just few words to explain the meaning of the previous "time-frequency" representation. If $f(t)$ represents a signal varying in time, its Fourier transform $\hat{f}(\omega)$ shows the distribution of its frequency ω, without any additional information about "when" these frequencies appear. To overcome this problem, one may choose a non-negative window function g well localized around the origin. Then, the information of the signal f at the instant x can be obtained by shifting the window g till the instant x under consideration, and by computing the Fourier transform of the product $f(x)g(t-x)$, that localizes f around the instant time x.

Once the analysis of the signal f is terminated, we can reconstruct the original signal f by a suitable inversion procedure. Namely, the reproducing formula related to the STFT, for every pairs of windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ with $\langle \varphi_1, \varphi_2 \rangle \neq 0$, reads as follows

$$\int_{\mathbb{R}^{2d}} V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2 \, dx d\omega = \langle \varphi_2, \varphi_1 \rangle f\,. \tag{3}$$

The function φ_1 is called the *analysis* window, because the STFT $V_{\varphi_1} f$ gives the time-frequency distribution of the signal f, whereas the window φ_2 permits to come back to the original f and, consequently, is called the *synthesis* window.

Let us know fix attention on localization operators. The name is due to Daubechies [19], who first used these operators as a mathematical tool to localize a signal on the time-frequency plane. Localization operators with Gaussian windows were already known in physics: they were introduced as a quantization rule by Berezin [4] in 1971 and called anti-Wick operators. Since their first appearance, they have been extensively studied as an important mathematical tool in signal analysis and other applications (see [42, 51] and references therein). Beyond signal analysis and the anti-Wick quantization procedure [4, 43], we recall their employment as approximation of pseudodifferential operators ("wave packets") [18, 29].

Namely, the signal analysis often requires to highlight some features of the time-frequency distribution of f. This is achieved by first multiplying the STFT $V_{\varphi_1} f$ by a suitable function $a(x,\omega)$ and secondly by constructing $\tilde{f}$ from the product $aV_{\varphi_2} f$. In other words, we recover a filtered version of the original signal f which we denote by $A_a^{\varphi_1,\varphi_2}$. This intuition motivates the definition of time-frequency localization operators.

Definition 1. The localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol $a \in \mathcal{S}(\mathbb{R}^{2d})$ and windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ is defined to be

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega\,, \quad f \in L^2(\mathbb{R}^d). \tag{4}$$

The preceding definition makes sense also if we assume $a \in L^\infty(\mathbb{R}^{2d})$, see below. In particular, if $a = \chi_\Omega$ for some compact set $\Omega \subseteq \mathbb{R}^{2d}$ and $\varphi_1 = \varphi_2$, then $A_a^{\varphi_1,\varphi_2}$ is interpreted as the part of f that "lives on the set Ω" in the time-frequency plane. This is why $A_a^{\varphi_1,\varphi_2}$ is called a *localization* operator.

Often it is more convenient to interpret the definition of $A_a^{\varphi_1,\varphi_2}$ in a weak sense, then (4) can be recast as

$$\langle A_a^{\varphi_1,\varphi_2} f, g \rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g \rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d)\,. \tag{5}$$

If we enlarge the class of symbols to the tempered distributions, i.e., we take $a \in \mathcal{S}'(\mathbb{R}^{2d})$ whereas $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then (4) is a well-defined continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. The previous assertion can be proven directly using the weak definition. For every window $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$ the STFT V_{φ_1} is a continuous mapping from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$ (see, e.g., [31, Theorem 11.2.5]). Since also $V_{\varphi_2} g \in \mathcal{S}(\mathbb{R}^{2d})$, the brackets $\langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle$ are well-defined in the duality between $\mathcal{S}'(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, the left-hand side of (5) can be interpreted in the duality between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ and shows that $A_a^{\varphi_1,\varphi_2}$ is a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. The continuity of the mapping $A_a^{\varphi_1,\varphi_2}$ is achieved by using the continuity of both the STFT and the brackets $\langle \cdot, \cdot \rangle$. Similar arguments can be applied for tempered ultra-distributions, cf. [35].

If $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, then $A_a = A_a^{\varphi_1,\varphi_2}$ is the classical anti-Wick operator and the mapping $a \to A_a^{\varphi_1,\varphi_2}$ is interpreted as a quantization rule [4, 43, 51].

Localization operators can be viewed as a multilinear mapping

$$(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}, \tag{6}$$

acting on products of symbol and window spaces. The dependence of the localization operator $A_a^{\varphi_1,\varphi_2}$ on all three parameters has been widely studied in different functional frameworks. The start was given by subspaces of the tempered distributions. The basic subspace is $L^2(\mathbb{R}^d)$, but many other Banach and Hilbert spaces, as well as topological vector spaces, have been considered. We mention L^p spaces [7, 51], potential and Sobolev spaces [8], modulation spaces [11, 27, 48, 49] and Gelfand-Shilov spaces [14] (the last ones in the ultra-distribution environment) as samples of spaces either for choosing symbol and windows or for defining the action of the related localization operator. The outcomes are manifold. The continuity of the mapping in (6) can be expressed by an inequality of the form

$$\|A_a^{\varphi_1,\varphi_2}\|_{op} \leq C\|a\|_{B_1}\, \|\varphi_1\|_{B_2}\, \|\varphi_2\|_{B_3}\,, \tag{7}$$

where B_1, B_2, B_3 are suitable spaces of symbols and windows. For example, this is the case when $a \in L^\infty(\mathbb{R}^d)$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Thus for this particular choice of symbol classes and window spaces we obtain the L^2 boundedness. The previous easy proof gives just a flavour of the boundedness results for localization operators, we shall see that the symbol class L^∞ can be enlarged significantly. Even a tempered distribution like δ may give the boundedness of the corresponding localization operator. Apart from continuity, estimates of the type (7) also supply Hilbert-Schmidt, Trace class and Schatten class properties for $A_a^{\varphi_1,\varphi_2}$ [12, 14].

Among the many function/distribution spaces employed, modulation spaces reveal to be the *optimal choice* for handling localization operators, see Section 3 below. As special case we mention Feichtinger's algebra $M^1(\mathbb{R}^d)$ defined by the norm

$$\|f\|_{M^1} := \|V_g f\|_{L^1(\mathbb{R}^{2d})}$$

for some (hence all) non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ [24, 31]. Its dual space $M^\infty(\mathbb{R}^{2d})$ is a very useful subspace of tempered distributions and possesses the norm

$$\|f\|_{M^\infty} := \sup_{(x,\omega)\in\mathbb{R}^{2d}} |V_g f(x,\omega)|\,.$$

With these spaces the estimate (7) reads as follows, cf. [11]:

Theorem 1. *If $a \in M^\infty(\mathbb{R}^{2d})$, and $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$, with operator norm at most*

$$\|A_a^{\varphi_1,\varphi_2}\|_{B(L^2)} \leq C\|a\|_{M^\infty}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}\,.$$

The striking fact is the converse of the preceding result.

Theorem 2. *If $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to* all *windows $\varphi_1, \varphi_2 \in M^1$, i.e., if there exists a constant $C > 0$ depending only on the symbol a such that, for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|A_a^{\varphi_1,\varphi_2}\|_{B(L^2)} \leq C\|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}\,, \tag{8}$$

then $a \in M^\infty$.

Similar statements hold true for Schatten class properties [12] and for weighted ultra-distributional modulation spaces [14]. A recent result in the study of localization operators [28] reveals the optimality of modulation spaces even for the compactness property. These topics shall be detailed in Section 5.

Another application of Time-frequency analysis refers to Fourier Integral Operators. They are a mathematical tool to study a variety of problems arising in partial differential equations. Originally introduced by Lax [38] for the construction of parametrices in the Cauchy problem for hyperbolic equations, they have been widely employed to represent solutions to Cauchy problems, in the framework of both pure and applied mathematics (see, e.g., the papers [10, 20, 21, 36], the books [37, 46, 50] and references therein). In particular, they were employed by Helffer and Robert [33, 34] to study the spectral property of a class of globally elliptic operators, generalizing the harmonic oscillator of the Quantum Mechanics.

The Fourier Integral operators we shall present, possess a phase function similar to those of [33, 34]. A simple example is the resolvent of the Cauchy problem for the Schrödinger equation with a quadratic Hamiltonian.

Precisely, consider the Cauchy problem

$$\begin{cases} i\dfrac{\partial u}{\partial t} + Hu = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{9}$$

where H is the Weyl quantization of a quadratic form on $\mathbb{R}^d \times \mathbb{R}^d$ (see, e.g., [29]). Simple examples are $H = -\frac{1}{4\pi}\Delta + \pi|x|^2$, or $H = -\frac{1}{4\pi}\Delta - \pi|x|^2$ (see [5]). The solution to (9) is a one-parameter family of FIOs:

$$u(t,x) = e^{itH}u_0,$$

with symbol $\sigma \equiv 1$ and a phase given by a quadratic form $\Phi(x,\omega)$. We address to Section 6 for a debited study of such operators.

Let us just define these operators and give a flavour of the results. For a given function f on $\mathbb{R}^d$ the *Fourier Integral Operator* (FIO) T with symbol σ and phase Φ on $\mathbb{R}^{2d}$ can be formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i\Phi(x,\omega)}\sigma(x,\omega)\hat{f}(\omega)d\omega. \tag{10}$$

The phase function $\Phi(x,\eta)$ is smooth on $\mathbb{R}^{2d}$, fulfills the estimates

$$|\partial_z^\alpha \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2, \quad z \in \mathbb{R}^{2d}, \tag{11}$$

and the nondegeneracy condition

$$|\det \partial^2_{x,\eta}\Phi(x,\omega)| \geq \delta > 0, \qquad (x,\omega) \in \mathbb{R}^{2d}. \tag{12}$$

The symbol σ on $\mathbb{R}^{2d}$ satisfies

$$|\partial_z^\alpha \sigma(z)| \leq C_\alpha, \quad |\alpha| \leq 2N, \quad \text{a.e. } z \in \mathbb{R}^{2d}, \tag{13}$$

for a fixed $N > 0$.

Time-frequency analysis is used to rephrase the operator T as an infinite matrix. Similar ideas go back at least to [18] and were recently employed to study PDEs with not smooth coefficients, see e.g. [45, 47].

The key role is played by the so-called Gabor frames. For $\alpha, \beta > 0$, $g \in L^2(\mathbb{R}^d)$, the set of time-frequency shifts $\mathcal{G}(g, \alpha, \beta) = \{g_{m,n} := M_n T_m g\}$, with $(m, n) \in \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, is a *Gabor frame* if there exist positive constants $A, B > 0$, such that

$$A\|f\|_{L^2} \leq \sum_{m,n} |\langle f, T_m M_n g\rangle|^2 \leq B\|f\|_{L^2}, \quad \forall f \in L^2(\mathbb{R}^d). \tag{14}$$

We show (see Section 6) that the matrix representation of a FIO T with respect to a Gabor frame with $g \in \mathcal{S}(\mathbb{R}^d)$ is well-organized, provided that the symbol σ satisfies the decay estimate for every $N > 0$ (see Theorem 11):

Theorem 1.1. *For each $N > 0$, there exists a constant $C_N > 0$ such that*

$$|\langle T g_{m,n}, g_{m',n'}\rangle| \leq C_N \langle \chi(m, n) - (m', n')\rangle^{-2N}, \tag{15}$$

where χ is the canonical transformation generated by Φ.

In the special case of pseudodifferential operators such an almost diagonalization was already obtained in [40]. Indeed, pseudodifferential operators correspond to the phase $\Phi(x, \eta) = x\eta$ and canonical transformation $\chi(y, \eta) = (y, \eta)$.

One should mention that the use of almost diagonal estimates in proving continuity results goes back to the pioneering work [30], where the Calderón-Zygmund class of operators was studied via such a technique, and this was achieved by working with wavelets. Also, observe that simple cases of a FIOs are the Fourier multipliers, and the usefulness of Gabor frames was first exhibited for these operators in [3].

Notation. We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi it\omega}dt$, the involution g^* is $g^*(t) = \overline{g(-t)}$.

The singular values $\{s_k(L)\}_{k=1}^{\infty}$ of a compact operator $L \in B(L^2(\mathbb{R}^d))$ are the eigenvalues of the positive self-adjoint operator $\sqrt{L^*L}$. For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in l^p. For consistency, we define $S_\infty := B(L^2(\mathbb{R}^d))$ to be the space of bounded operators on $L^2(\mathbb{R}^d)$. In particular, S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ if $A \leq cB$ and $B \leq kA$, for suitable $c, k > 0$.

2 Time-Frequency Methods

First we summarize some concepts and tools of time-frequency analysis , for an extended exposition we refer to the textbooks [29, 31].

The time-frequency representations required for localization operators and the Weyl calculus are the *short-time Fourier transform* and the *Wigner distribution* .

The short-time Fourier transform (STFT) is defined in (2). The *cross-Wigner distribution* $W(f, g)$ of $f, g \in L^2(\mathbb{R}^d)$ is given by

$$W(f, g)(x, \omega) = \int f(x + \frac{t}{2})\overline{g(x - \frac{t}{2})}e^{-2\pi i\omega t}\, dt. \tag{16}$$

The quadratic expression $Wf = W(f, f)$ is usually called the Wigner distribution of f.

Both the STFT $V_g f$ and the Wigner distribution $W(f, g)$ are defined for f, g in many possible pairs of Banach spaces. For instance, they both map $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$. Furthermore, they can be extended to a map from $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^{2d})$.

For a non-zero $g \in L^2(\mathbb{R}^d)$, we write V_g^* for the adjoint of V_g, given by

$$\langle V_g^* F, f\rangle = \langle F, V_g f\rangle, \quad f \in L^2(\mathbb{R}^d),\ F \in L^2(\mathbb{R}^{2d}).$$

In particular, for $F \in \mathcal{S}(\mathbb{R}^{2d})$, $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$V_g^* F(t) = \int_{\mathbb{R}^{2d}} F(x, \omega) M_\omega T_x g(t)\, dx\, d\omega \in \mathcal{S}(\mathbb{R}^d). \tag{17}$$

Take $f \in \mathcal{S}(\mathbb{R}^d)$ and set $F = V_g f$, then

$$f(t) = \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x g(t)\, dx\, d\omega = \frac{1}{\|g\|_{L^2}^2} V_g^* V_g f(t) \in \mathcal{S}(\mathbb{R}^d). \tag{18}$$

We refer to [31, Proposition 11.3.2] for a detailed treatment of the adjoint operator.

Representation of localization operators as Weyl/Kohn-Nirenberg operators. Let $W(g, f)$ be the cross-Wigner distribution as defined in (16). Then the Weyl operator L_σ of symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is defined by

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g, f)\rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{19}$$

Every linear continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a Weyl operator, and a calculation in [8, 29, 43] reveals that

$$A_a^{\varphi_1, \varphi_2} = L_{a * W(\varphi_2, \varphi_1)}, \tag{20}$$

so the (Weyl) symbol of $A_a^{\varphi_1, \varphi_2}$ is given by

$$\sigma = a * W(\varphi_2, \varphi_1)\,. \tag{21}$$

To get boundedness results for a localization operator, it is sometimes convenient to write it in a different pseudodifferential form. Consider the *Kohn-Nirenberg* form of a pseudodifferential operator, given by

$$T_\tau f(x) = \int_{\mathbb{R}^d} \tau(x, \omega) \hat{f}(\omega) e^{2\pi i x\omega}\, d\omega, \quad f \in \mathcal{S}(\mathbb{R}^d), \tag{22}$$

where τ is a measurable function, or even a tempered distribution on $\mathbb{R}^{2d}$.

If we define the *rotation* operator $\mathcal{U}$ acting on a function F on $\mathbb{R}^{2d}$ by

$$\mathcal{U}F(x, \omega) = F(\omega, -x), \quad \forall\, (x, \omega) \in \mathbb{R}^{2d}, \tag{23}$$

then, the identity of operators below holds [16]:

$$A_a^{\varphi_1, \varphi_2} = T_\tau, \tag{24}$$

with the Kohn-Nirenberg symbol τ given by

$$\tau = a * \mathcal{U}\mathcal{F}(V_{\varphi_1} \varphi_2). \tag{25}$$

The expression $\mathcal{U}\mathcal{F}(V_{\varphi_1} \varphi_2)$ is usually called the Rihaczek distribution.

3 Function Spaces

Weight Functions. In the sequel v will always be a continuous, positive, even, submultiplicative function (submultiplicative weight), i.e., $v(0) = 1$, $v(z) = v(-z)$, and $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z, z_1, z_2 \in \mathbb{R}^{2d}$. Moreover, v is assumed to be even in each group of coordinates, that is, $v(\pm x, \pm\omega) = v(x, \omega)$, for all $(x, \omega) \in \mathbb{R}^{2d}$ and all choices of signs. Submultiplicativity implies that $v(z)$ is *dominated* by an exponential function, i.e.

$$\exists\, C, k > 0 \quad \text{such that} \quad v(z) \leq Ce^{k|z|}, \quad z \in \mathbb{R}^{2d}. \tag{26}$$

Associated to every submultiplicative weight we consider the class of so-called *v-moderate* weights $\mathcal{M}_v$. A positive, even weight function m on $\mathbb{R}^{2d}$ belongs to $\mathcal{M}_v$ if it satisfies the condition

$$m(z_1 + z_2) \leq Cv(z_1)m(z_2), \quad \forall z_1, z_2 \in \mathbb{R}^{2d}.$$

We note that this definition implies that $\frac{1}{v} \lesssim m \lesssim v$, $m \neq 0$ everywhere, and that $1/m \in \mathcal{M}_v$.

For the investigation of localization operators the weights mostly used are defined by

$$\begin{aligned} v_s(z) &= v_s(x,\omega) = \langle z\rangle^s = (1 + x^2 + \omega^2)^{s/2}, \quad z = (x,\omega) \in \mathbb{R}^{2d} && (27)\\ w_s(z) &= w_s(x,\omega) = e^{s|(x,\omega)|}, \quad z = (x,\omega) \in \mathbb{R}^{2d}, && (28)\\ \tau_s(z) &= \tau_s(x,\omega) = \langle\omega\rangle^s && (29)\\ \mu_s(z) &= \mu_s(x,\omega) = e^{s|\omega|}. && (30) \end{aligned}$$

Definition 2. Let m be a weight in $\mathcal{M}_v$, and g a non-zero *window* function in $\mathcal{S}$. For $1 \leq p, q \leq \infty$ and $f \in \mathcal{S}$ we define the modulation space norm (on $\mathcal{S}$) by

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p\, dx\right)^{q/p} d\omega\right)^{1/q},$$

(with obvious changes if either $p = \infty$ or $q = \infty$). If $p, q < \infty$, the modulation space $M^{p,q}_m$ is the norm completion of $\mathcal{S}$ in the $M^{p,q}_m$-norm. If $p = \infty$ or $q = \infty$, then $M^{p,q}_m$ is the completion of $\mathcal{S}$ in the weak* topology. If $p = q$, $M^p_m := M^{p,p}_m$, and, if $m \equiv 1$, then $M^{p,q}$ and M^p stand for $M^{p,q}_m$ and $M^{p,p}_m$, respectively.

The class of modulation spaces contains the following well-known function spaces:
Weighted L^2-spaces: $M^2_{\langle x\rangle^s}(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) = \{f : f(x)\langle x\rangle^s \in L^2(\mathbb{R}^d)\}, s \in \mathbb{R}$.
Sobolev spaces: $M^2_{\langle\omega\rangle^s}(\mathbb{R}^d) = H^s(\mathbb{R}^d) = \{f : \hat{f}(\omega)\langle\omega\rangle^s \in L^2(\mathbb{R}^d)\}, s \in \mathbb{R}$.
Shubin-Sobolev spaces [8, 43]: $M^2_{\langle(x,\omega)\rangle^s}(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H^s(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$.
Feichtinger's algebra: $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$.

The characterization of the Schwartz class of tempered distributions: $\mathcal{S}(\mathbb{R}^d) = \bigcap_{s\geq 0} M^1_{\langle\cdot\rangle^s}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d) = \bigcup_{s\geq 0} M^\infty_{1/\langle\cdot\rangle^s}(\mathbb{R}^d)$.

Potential spaces. For $s \in \mathbb{R}$ the Bessel kernel is

$$G_s = \mathcal{F}^{-1}\{(1 + |\cdot|^2)^{-s/2}\}, \tag{31}$$

and the *potential space* [6] is defined by

$$W^p_s = G_s * L^p(\mathbb{R}^d) = \{f \in \mathcal{S}',\, f = G_s * g,\, g \in L^p\}$$

with norm $\|f\|_{W^p_s} = \|g\|\,|_{L^p}$.

For comparison we list the following embeddings between potential and modulation spaces [11].

Lemma 1. *We have*
(i) If $p_1 \leq p_2$ *and* $q_1 \leq q_2$*, then* $M_m^{p_1,q_1} \hookrightarrow M_m^{p_2,q_2}$.
(ii) For $1 \leq p \leq \infty$ *and* $s \in \mathbb{R}$

$$W_s^p(\mathbb{R}^d) \hookrightarrow M_{\tau_s}^{p,\infty}(\mathbb{R}^d).$$

Consequently, $L^p \subseteq M^{p,\infty}$, and in particular, $L^\infty \subseteq M^\infty$. But M^∞ contains all bounded measures on $\mathbb{R}^d$ and other tempered distributions. For instance, the point measure δ belongs to M^∞, because for $g \in \mathcal{S}$ we have

$$|V_g\delta(x,\omega)| = |\langle \delta, M_\omega T_x g\rangle| = |\bar{g}(-x)| \leq \|g\|_{L^\infty}, \quad \forall (x,\omega) \in \mathbb{R}^{2d}.$$

Convolution Relations. In view of the relation between the multiplier a and the Weyl symbol (21), we need to understand the convolution relations between modulation spaces and some properties of the Wigner distribution.

We first state a *convolution relation for modulation spaces* proven in [11], in the style of Young's theorem. Let v be an arbitrary submultiplicative weight on $\mathbb{R}^{2d}$ and m a v-moderate weight. We write $m_1(x) = m(x,0)$ and $m_2(\omega) = m(0,\omega)$ for the restrictions to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^d$, and likewise for v.

Proposition 1. *Let* $\nu(\omega) > 0$ *be an arbitrary weight function on* $\mathbb{R}^d$ *and* $1 \leq p,q,r,s,t \leq \infty$*. If*

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}, \qquad \text{and} \qquad \frac{1}{t} + \frac{1}{t'} = 1,$$

then

$$M_{m_1\otimes\nu}^{p,st}(\mathbb{R}^d) * M_{v_1\otimes v_2\nu^{-1}}^{q,st'}(\mathbb{R}^d) \hookrightarrow M_m^{r,s}(\mathbb{R}^d) \tag{32}$$

with norm inequality $\|f*h\|_{M_m^{r,s}} \lesssim \|f\|_{M_{m_1\otimes\nu}^{p,st}} \|h\|_{M_{v_1\otimes v_2\nu^{-1}}^{q,st'}}$.

Remark 1. 1. Despite the large number of indices, the statement of this proposition has some intuitive meaning: a function $f \in M^{p,q}$ behaves like $f \in L^p$ and $\hat{f} \in L^q$; so the parameters related to the x-variable behave like those in Young's theorem for convolution, whereas the parameters related to ω behave like Hölder's inequality for pointwise multiplication.

2. A special case of Proposition 1 with a different proof is contained in [48].

4 Gabor frames

Fix a function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, for $\alpha, \beta > 0$. For $(m,n) \in \Lambda$, define $g_{m,n} := M_n T_m g$. The set of time-frequency shifts $\mathcal{G}(g,\alpha,\beta) = \{g_{m,n}, (m,n) \in \Lambda\}$ is called Gabor system. Associated to $\mathcal{G}(g,\alpha,\beta)$ we define the coefficient operator C_g, which maps functions to sequences as follows:

$$(C_g f)_{m,n} = (C_g^{\alpha,\beta} f)_{m,n} := \langle f, g_{m,n}\rangle, \quad (m,n) \in \Lambda, \tag{33}$$

the synthesis operator

$$D_g c = D_g^{\alpha,\beta} c = \sum_{(m,n)\in\Lambda} c_{m,n} T_m M_n g, \quad c = \{c_{m,n}\}_{(m,n)\in\Lambda}$$

and the Gabor frame operator

$$S_g f = S_g^{\alpha,\beta} f := D_g S_g f = \sum_{(m,n)\in\Lambda} \langle f, g_{m,n}\rangle g_{m,n}. \tag{34}$$

The set $\mathcal{G}(g,\alpha,\beta)$ is called a Gabor frame for the Hilbert space $L^2(\mathbb{R}^d)$ if S_g is a bounded and invertible operator on $L^2(\mathbb{R}^d)$. Equivalently, C_g is bounded from $L^2(\mathbb{R}^d)$ to $l^2(\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d)$ with closed range, i.e., $\|f\|_{L^2}\asymp\|C_gf\|_{l^2}$. If $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, then the so-called *dual window* $\gamma=S_g^{-1}g$ is well-defined and the set $\mathcal{G}(\gamma,\alpha,\beta)$ is a frame (the so-called canonical dual frame of $\mathcal{G}(g,\alpha,\beta)$). Every $f\in L^2(\mathbb{R}^d)$ possesses the frame expansion

$$f=\sum_{(m,n)\in\Lambda}\langle f,g_{m,n}\rangle\gamma_{m,n}=\sum_{(m,n)\in\Lambda}\langle f,\gamma_{m,n}\rangle g_{m,n} \tag{35}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$, and norm equivalence:

$$\|f\|_{L^2}\asymp\|C_gf\|_{l^2}\asymp\|C_\gamma f\|_{l^2}.$$

This result is contained in [31, Proposition 5.2.1]. In particular, if $\gamma=g$ and $\|g\|_{L^2}=1$ the frame is called *normalized tight* Gabor frame and the expansion (35) reduces to

$$f=\sum_{(m,n)\in\Lambda}\langle f,g_{m,n}\rangle g_{m,n}. \tag{36}$$

If we ask for more regularity on the window g, then the previous result can be extended to suitable Banach spaces, as shown below [25].

Theorem 3. *Let $\mu\in\mathcal{M}_v$, $\mathcal{G}(g,\alpha,\beta)$ be a normalized tight Gabor frame for $L^2(\mathbb{R}^d)$, with lattice $\Lambda=\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$, and $g\in M_v^1$. Define $\tilde\mu=\mu_{|\Lambda}$.*
(i) For every $1\le p,q\le\infty$, $C_g: M_\mu^{p,q}\to l_{\tilde\mu}^{p,q}$ and $D_g: l_{\tilde\mu}^{p,q}\to M_\mu^{p,q}$ continuously and, if $f\in M_\mu^{p,q}$, then the Gabor expansions (36) *converge unconditionally in $M_\mu^{p,q}$ for $1\le p,q<\infty$ and all weight μ, and weak*-M_μ^∞ unconditionally if $p=\infty$ or $q=\infty$.*
(ii) The following norms are equivalent on $M_\mu^{p,q}$:

$$\|f\|_{M_\mu^{p,q}}\asymp\|C_gf\|_{l_{\tilde\mu}^{p,q}}. \tag{37}$$

We also establish the following properties. Denote by $\tilde M_\mu^{p,q}$ the closure of the Schwartz class in $M_\mu^{p,q}$. Hence, $\tilde M_\mu^{p,q}=M_\mu^{p,q}$ if $p<\infty$ and $q<\infty$. Also, denote by $\tilde l_{\tilde\mu}^{p,q}$ the closure of the space of eventually zero sequences in $l_{\tilde\mu}^{p,q}$. Hence $\tilde l_{\tilde\mu}^{p,q}=l_{\tilde\mu}^{p,q}$ if $p<\infty$ and $q<\infty$. We recall the following result [17]

Theorem 4. *Under the assumptions of Theorem 3, for every $1\le p,q\le\infty$ the operator C_g is continuous from $\tilde M_\mu^{p,q}$ into $\tilde l_{\tilde\mu}^{p,q}$, whereas the operator D_g is continuous from $\tilde l_{\tilde\mu}^{p,q}$ into $\tilde M_\mu^{p,q}$.*

5 Regularity Results for localization operators

In this section, we first give general sufficient conditions for boundedness and Schatten classes of localization operators. Then, we show the results are optimal.

5.1 Sufficient Conditions for Boundedness and Schatten Class

Using the tools of time-frequency analysis in Section 3, we can now obtain the properties of localization operators with symbols in modulation spaces, by reducing the problem to the corresponding one for the Weyl calculus.

First, we recall a boundedness and trace class result for the Weyl operators in terms of modulation spaces.

Theorem 5. *(i) If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then L_σ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \le p,q \le \infty$, with a uniform estimate $\|L_\sigma\|_{S_\infty} \lesssim \|\sigma\|_{M^{\infty,1}}$ for the operator norm. In particular, L_σ is bounded on $L^2(\mathbb{R}^d)$.*

(ii) If $\sigma \in M^1(\mathbb{R}^{2d})$, then $L_\sigma \in S_1$ and $\|L_\sigma\|_{S^1} \lesssim \|\sigma\|_{M^1}$.

(iii) If $1 \le p \le 2$ and $\sigma \in M^p(\mathbb{R}^{2d})$, then $L_\sigma \in S_p$ and $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^p}$.

(iv) If $2 \le p \le \infty$ and $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$, then $L_\sigma \in S_p$ and $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^{p,p'}}$,

where p' is the conjugate exponent of p.

One of many proofs of (i) can be found in [31, Thm. 14.5.2], the L^2-boundedness was first discovered by Sjöstrand [44]. The trace class property (ii) is proved in [32], whereas (iii) and (iv) follow by interpolation from the first two statements, since $[M^1, M^2]_\theta = M^p$ for $1 \le p \le 2$, and $[M^{\infty,1}, M^{2,2}]_\theta = M^{p,p'}$ for $2 \le p \le \infty$.

Based on the Thm. 5 and Prop. 1, we present the most general boundedness results for localization operators obtained so far.

Theorem 6. *Let $s \ge 0$, $a \in M^\infty_{1/\tau_s}(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$. Then $A_a^{\varphi_1,\varphi_2}$ is bounded on $M^{p,q}(\mathbb{R}^d)$ for all $1 \le p,q \le \infty$, and the operator norm satisfies the uniform estimate*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \lesssim \|a\|_{M^\infty_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^1_{v_s}}.$$

Remark 2. To compare Theorem 6 to existing results, we recall that the standard condition for $A_a^{\varphi_1,\varphi_2}$ to be bounded is $a \in L^\infty(\mathbb{R}^{2d})$, see [51]. A more subtle result of Feichtinger and Nowak [27] shows that the condition a in the Wiener amalgam space $W(M, L^\infty)$ is sufficient for boundedness. Since we have the proper embeddings $L^\infty \subset W(M, L^\infty) \subset M^\infty \subset M^\infty_{1/\tau_s}$ for $s \ge 0$, Theorem 6 appears as a significant improvement. A special case of Theorem 6 follows also from Toft's work [49].

The Schatten class properties of localization operators with symbols in modulation spaces are achieved accordingly. Combining Proposition 1 with Theorem 5, almost optimal conditions for $A_a^{\varphi_1,\varphi_2} \in S_p$ are derived in [11, 14]. Precisely, the following result is [11, Theorem 3.4]:

Theorem 7. *(i) If $1 \le p \le 2$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{1/\tau_s}(\mathbb{R}^{2d}) \times M^1_{v_s}(\mathbb{R}^d) \times M^p_{v_s}(\mathbb{R}^d)$ into S_p, in other words,*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^p_{v_s}}.$$

(ii) If $2 \le p \le \infty$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{1/\tau_s} \times M^1_{v_s} \times M^{p'}_{v_s}$ into S_p, and

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^{p'}_{v_s}}.$$

Using the embeddings $W^p_{-s} \hookrightarrow M^{p,\infty}_{1/\tau_s}$ (Lemma 1) and $M^1_{v_s} \hookrightarrow M^p_{v_s}$, one obtains a slightly weaker statement for symbols in potential spaces. This result was already derived in [8, Thm. 4.7].

Corollary 1. *Let $a \in W^p_{-s}(\mathbb{R}^{2d})$ for some $s \ge 0$, $1 \le p \le \infty$, and $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$. Then*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{W^p_{-s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^1_{v_s}}.$$

5.2 Compactness of Localization Operators

Localization operators with symbols and windows in the Schwartz class are compact [8]. If we define by M^0 the closed subspace of M^∞, consisting of all $f \in \mathcal{S}'$ such that its STFT $V_g f$ (with respect to a non-zero Schwartz window g) vanishes at infinity, it is easy to show that localization operators with symbols in M^0 and Schwartz windows are compact. Namely, let $a \in M^0(\mathbb{R}^{2d})$ and $g \in \mathcal{S}(\mathbb{R}^{2d})$, for simplicity normalized to be $\|g\|_{L^2} = 1$; consider then $V_g a$. The Schwartz class is dense in M^0, hence there exists a sequence F_n of Schwartz functions on $\mathbb{R}^{4d}$ that converge to $V_g a$ in the L^∞-norm. Define the sequence $a_n := V_g^* F_n$, $n \in \mathbb{N}$, where V_g^* is the adjoint operator defined in (17). Then $a_n \in \mathcal{S}(\mathbb{R}^{2d})$ and $a_n \to a$ in the M^∞-norm, since by (18)

$$\|a - a_n\|_{M^\infty} = \|V_g a - V_g V_g^* F_n\|_{L^\infty} = \|V_g a - F_n\|_{L^\infty} \to 0,$$

for $n \to \infty$. From Theorem 6 we have

$$\|A_{a_n}^{\varphi_1,\varphi_2} - A_a^{\varphi_1,\varphi_2}\|_{B(L^2)} = \|A_{(a_n-a)}^{\varphi_1,\varphi_2}\|_{B(L^2)} \le C\|a - a_n\|_{M^\infty}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1} \to 0.$$

Since compact operators are a closed subspace of the space of all bounded operators $B(L^2)$, then the localization operator $A_a^{\varphi_1,\varphi_2}$ is compact.

The symbol class $M^0(\mathbb{R}^{2d})$ is not optimal as the next simple example shows. Consider $a = \delta \notin M^0(\mathbb{R}^{2d})$. Since $V_g\delta(z,\zeta) = \bar{g}(z)$, it does not tend to zero when $z \in \mathbb{R}^{2d}$ is fixed and $|\zeta|$ goes to infinity. Hence $\delta \notin M^0(\mathbb{R}^{2d})$. However $A_\delta^{\varphi_1,\varphi_2}$ is a trace class operator for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, in fact, a rank-one operator, and therefore it is compact.

The example just mentioned has been the inspiration for the following compactness result [28, Proposition 3.6.]:

Proposition 2. *Let $g \in \mathcal{S}(\mathbb{R}^{2d})$ be given and $a \in M^\infty(\mathbb{R}^{2d})$. If $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and*

$$\lim_{|z|\to\infty} \sup_{|\zeta|\le R} |V_g a(z,\zeta)| = 0, \qquad \forall\, R > 0, \tag{38}$$

then $A_a^{\varphi_1,\varphi_2}$ is a compact operator.

5.3 Necessary Conditions

In this subsection we show that the sufficient conditions obtained so far are essentially optimal. This investigation requires different techniques and we limit ourselves to state the main results. A first attempt is done in Theorems 4.3, 4.4 of [11], where a converse for bounded and Hilbert-Schmidt operators is obtained for modulation spaces with polynomial weights:

Theorem 8. *i) Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and fix $s \ge 0$. If there exists a constant $C = C(a) > 0$ depending only on a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \le C\, \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^1_{v_s}}$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $a \in M^\infty_{1/\tau_s}$.

ii) Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$. If there exists a constant $C = C(a) > 0$ depending only on a such that

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_2} \le C\, \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $a \in M^{2,\infty}$.

Necessary conditions for localization operators belonging to the Schatten class S_p have been obtained for unweighted modulation spaces in [12]:

Theorem 9. *Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $1 \le p \le \infty$. Assume that $A_a^{\varphi_1,\varphi_2} \in S_p$ for* all *windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and that there exists a constant $B > 0$ depending only on the symbol a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \le B\, \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}, \quad \forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d),$$

then $a \in M^{p,\infty}$.

The techniques employed for the converse results are thoroughly different from the techniques for the sufficient conditions. Gabor frames and equivalent norms for modulation spaces are some of the crucial ingredients in the proofs. For the sake of completeness, we shall sketch the main features. First, by using the Gabor frame of the form

$$\{M_{\beta n} T_{\alpha k} \Phi\}_{k,n\in\mathbb{Z}^{2d}}, \quad 0<\alpha,\beta<1,$$

with the Gaussian window $\Phi(x,\omega)=2^{-d}\, e^{-\pi(x^2+\omega^2)}$, the $M^{p,\infty}(\mathbb{R}^{2d})$-norm of a can be expressed by the equivalent norms

$$\|a\|_{M^{p,\infty}(\mathbb{R}^{2d})} \asymp \|\langle a, M_{\beta n} T_{\alpha k}\Phi\rangle_{n,k\in\mathbb{Z}^{2d}}\|_{\ell^{p,\infty}(\mathbb{Z}^{4d})}. \tag{39}$$

Then one relates the action of the localization operator on certain time-frequency shift of the Gaussian φ to the Gabor coefficients, and for a diligent choice of (x,ξ) and (u,η) one obtains that $\langle A_a^{\varphi_1,\varphi_2} M_\xi T_x\varphi, M_\eta T_u \varphi\rangle = \langle a, M_{\beta n}T_{\alpha k}\Phi\rangle$. The result is then obtained by using (39).

We end up with the compactness necessary result of [28, Theorem 3.15].

Theorem 10. *Let $a\in M^\infty(\mathbb{R}^{2d})$ and $g\in\mathcal{S}(\mathbb{R}^{2d})$ be given. Then, the following conditions are equivalent:*

(a) The localization operator $A_a^{\varphi_1,\varphi_2} : L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is compact for every pair $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^d)$.

(b) The symbol a satisfies condition (38).

6 Fourier Integral Operators

We now come to a class of operators which include, as special examples, pseudo-differential operators.

Namely, we study the FIO T defined in (10), having phase satisfying (11) and (12) and symbol enjoying (13).

In this section we present an almost diagonalization result for FIOs as above, with respect to Gabor frames. For simplicity, we consider a normalized tight frame $\mathcal{G}(g,\alpha,\beta)$, with $g\in\mathcal{S}(\mathbb{R}^d)$.

Using the action of the Fourier transform over the time-frequency shifts: $(T_x f)^\wedge = M_{-x}\hat f$ and $(M_\omega f)^\wedge = T_\omega \hat f$, the smoothness of Φ (so that we expand $\Phi(x,\omega)$ into a Taylor series around (m',n)), the integration by parts and the Leibniz' formula, one can prove (see [17, Thm. 3.3]):

Theorem 11. *Consider a phase function Φ satisfying* (11) *and* (12) *and a symbol satisfying* (13). *Let $g\in\mathcal{S}(\mathbb{R}^d)$. There exists a constant $C_N>0$ such that*

$$|\langle T g_{m,n}, g_{m',n'}\rangle| \le C_N \langle \chi(m,n)-(m',n')\rangle^{-2N}, \tag{40}$$

where χ is the canonical transformation generated by Φ.

This result shows that the matrix representation of a FIO with respect a Gabor frame is well-organized. More precisely, if $\sigma\in S^0_{0,0}$, namely if (13) is satisfied for every $N\in\mathbb{N}$, then the Gabor matrix of T is highly concentrated along the graph of χ.

The previous result, together with the following lemma, are the key to prove the continuity of these FIOs on the modulation spaces M^p_μ, with a weight function $\mu\in\mathcal{M}_{v_s}$, $s\ge 0$. Observe that the first part of the lemma is the classical Schur's test (see e.g. [31, Lemma 6.2.1]), the second part is proved in [17, Lemma 4.1]:

Lemma 2. *Consider a lattice* Λ *and an operator* K *defined on sequences as*

$$(Kc)_\lambda = \sum_{\nu\in\Lambda} K_{\lambda,\nu} c_\nu,$$

where

$$\sup_{\nu\in\Lambda} \sum_{\lambda\in\Lambda} |K_{\lambda,\nu}| < \infty, \quad \sup_{\lambda\in\Lambda} \sum_{\nu\in\Lambda} |K_{\lambda,\nu}| < \infty.$$

Then K *is continuous on* $l^p(\Lambda)$ *for every* $1 \le p \le \infty$ *and moreover maps the space* $c_0(\Lambda)$ *of sequences vanishing at infinity into itself.*

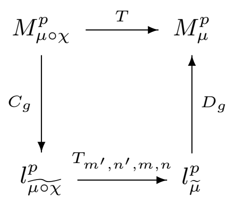
We can now state and give the proof of the following result, that allows the reader to understand the way of using Gabor frame theory to reach the goal.

Theorem 12. *Consider a phase function satisfying* (11) *and* (12) *and a symbol satisfying* (13). *Let* $0 \le s < 2N - 2d$, *and* $\mu \in \mathcal{M}_{v_s}$. *For every* $1 \le p < \infty$, T *extends to a continuous operator from* $M^p_{\mu\circ\chi}$ *into* M^p_μ, *and for* $p = \infty$ *it extends to a continuous operator from* $\tilde{M}^\infty_{\mu\circ\chi}$ *into* $\tilde{M}^\infty_\mu$.

Recall that $\tilde{M}^\infty_\mu$ is the closure of $\mathcal{S}(\mathbb{R}^d)$ in M^∞_μ. Moreover, observe that $\mu\circ\chi \in \mathcal{M}_{v_s}$. Indeed, $v_s \circ \chi \asymp v_s$, due to the bilipschitz property of χ.

Proof. We first prove the theorem in the case $p < \infty$.

For $T = C_g \circ T_{m',n',m,n} \circ D_g$, the following diagram is commutative:



where T is viewed as an operator with dense domain $\mathcal{S}(\mathbb{R}^d)$. Whence, it is enough to prove the continuity of the infinite matrix $T_{m',n',m,n}$ from $l^p_{\widetilde{\mu\circ\chi}}$ into $l^p_{\tilde{\mu}}$.

This follows from Schur's test (Lemma 2) if we prove that, upon setting

$$K_{m',n',m,n} = T_{m',n',m,n} \frac{\mu(m',n')}{\mu(\chi(m,n))},$$

we have

$$K_{m',n',m,n} \in l^\infty_{m,n} l^1_{m',n'}, \tag{41}$$

and

$$K_{m',n',m,n} \in l^\infty_{m',n'} l^1_{m,n}. \tag{42}$$

In view of (40) we have

$$|K_{m',n',m,n}| \lesssim \langle \chi(m,n) - (m',n') \rangle^{-2N+s} \frac{\mu(m',n')}{\langle \chi(m,n) - (m',n') \rangle^s \mu(\chi(m,n))}. \tag{43}$$

Now, the last quotient in (43) is bounded because μ is v_s-moderate, so we deduce (41).

Finally, since χ is a bilipschitz function we have

$$|\chi(m,n) - (m',n')| \asymp |(m,n) - \chi^{-1}(m',n')| \tag{44}$$

so that (42) follows as well.

The case $p = \infty$ follows analogously by using the last part of the statement of Lemma 2. QED

Remark 3. Theorem 12 with $v \equiv 1$ gives, in particular, continuity on the unweighted modulation spaces M^p.

Also, Theorem 12 applies to $\mu = v_t$, with $|t| \leq s$. In that case we obtain continuity on M_{v_t}, because $v_t \circ \chi \asymp v_t$.

6.1 The case of quadratic phases: metaplectic operators

We briefly discuss the particular case of quadratic phases, namely phases of the type

$$\Phi(x,\eta) = \frac{1}{2}Ax\cdot x + Bx\cdot\eta + \frac{1}{2}C\eta\cdot\eta + \eta_0\cdot x - x_0\cdot\eta, \tag{45}$$

where $x_0, \eta_0 \in \mathbb{R}^d$, A, C are real symmetric $d\times d$ matrices and B is a real $d\times d$ nondegenerate matrix.

It is easy to see that, if we take the symbol $\sigma \equiv 1$ and the phase (45), the corresponding FIO T is (up to a constant factor) a metaplectic operator. This can be seen by means of the easily verified factorization

$$T = M_{\eta_0} U_A D_B \mathcal{F}^{-1} U_C \mathcal{F} T_{x_0}, \tag{46}$$

where U_A and U_C are the multiplication operators by $e^{\pi i Ax\cdot x}$ and $e^{\pi i C\eta\cdot\eta}$ respectively, and D_B is the dilation operator $f \mapsto f(B\cdot)$. Each of the factors is (up to a constant factor) a metaplectic operator (see e.g. the proof of [37, Theorem 18.5.9]), so T is.

For the benefit of the reader, some important special cases are detailed in the table below.

operator	phase $\Phi(x,\eta)$	canonical transformation
T_{x_0}	$(x-x_0)\cdot\eta$	$\chi(y,\eta) = (y+x_0,\eta)$
M_{η_0}	$(\eta+\eta_0)\cdot x$	$\chi(y,\eta) = (y,\eta+\eta_0)$
D_B	$Bx\cdot\eta$	$\chi(y,\eta) = (B^{-1}y, {}^t B\eta)$
U_A	$x\cdot\eta + \frac{1}{2}Ax\cdot x$	$\chi(y,\eta) = (y,\eta+Ay)$

We end up by observing that there are metaplectic operators, as the Fourier transform, which cannot be expressed as FIOs of this type.

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