Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **31** (2011) n. 1, 129–138. doi:10.1285/i15900932v31n1p129

Weak* closures and derived sets in dual Banach spaces

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Received: 26/03/2010; accepted: 29/09/2010.

Abstract. The main results of the paper: **(1)** The dual Banach space X^* contains a linear subspace $A \subset X^*$ such that the set $A^{(1)}$ of all limits of weak* convergent bounded nets in A is a proper norm-dense subset of X^* if and only if X is a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual. **(2)** Let X be a non-reflexive Banach space. Then there exists a convex subset $A \subset X^*$ such that $A^{(1)} \neq \overline{A}^*$ (the latter denotes the weak* closure of A). **(3)** Let X be a quasi-reflexive Banach space and $A \subset X^*$ be an absolutely convex subset. Then $A^{(1)} = \overline{A}^*$.

Keywords: norming subspace, quasi-reflexive Banach space, total subspace, weak* closure, weak* derived set, weak* sequential closure

MSC 2000 classification: primary 46B10, secondary 46B15, 46B20

Dedicated to the memory of V.B. Moscatelli

Let A be a subset of a dual Banach space X^*, we denote the weak* closure of A by $\overline{A}^*$. The *weak* derived set* of A is defined as

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{X^*}}^{\,*},$$

where B_{X^*} is the unit ball of X^*, that is, $A^{(1)}$ is the set of all limits of weak* convergent bounded nets in A. If X is separable, $A^{(1)}$ coincides with the set of all limits of weak* convergent sequences from A, called the *weak* sequential closure*. The strong closure of a set A in a Banach space is denoted $\overline{A}$. A subset $A \subset X^*$ is called *total* if for every $0 \neq x \in X$ there exists $f \in A$ such that $f(x) \neq 0$. A subset $A \subset X^*$ is called *norming* if there is $c > 0$ such that for every $0 \neq x \in X$ there exists $f \in A$ satisfying $||f|| = 1$ and $f(x) \geq c||x||$.

The study of weak* derived sets was initiated by Banach and continued by many authors, see [2, 9, 17, 18, 21, 23, 34], and references therein. Weak* derived sets and their relations with weak* closures found applications in many areas: the structure theory of Fréchet spaces (see [1, 3, 5, 19, 20, 22, 24]), Borel and Baire classification of linear operators, including the theory of ill-posed problems ([28, 31–33]), Harmonic Analysis ([12, 16, 18, 29]), theory of biorthogonal systems ([10,30]; I have to mention that the historical information on weak* sequential closures in [10] is inaccurate). The survey [25] contains a historical account and an up-to-date-in-2000 information on weak* sequential closures.

Recently derived sets were used in the study of extension problems for holomorphic functions on dual Banach spaces [8].

The main purpose of the present paper is to answer the following two questions asked in [8]:

1. [8, Question 6.3(a)]. Let X be a quasi-reflexive Banach space. Is $\overline{A}^* = A^{(1)}$ for each (absolutely) convex set $A \subset X^*$?

2. [8, Question 6.5]. For which Banach spaces X there is a linear subspace $A \subset X^*$ such that $A^{(1)}$ is a proper norm-dense subset of X^*? Is it true whenever X is not quasi-reflexive?

The main results of the paper:

Theorem 1. *The dual Banach space X^* contains a linear subspace $A \subset X^*$ such that $A^{(1)}$ is a proper norm-dense subset of X^* if and only if X is a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual.*

Theorem 2. *Let X be a non-reflexive Banach space. Then there exists a convex subset $A \subset X^*$ such that $A^{(1)} \neq \overline{A}^*$.*

Theorem 3. *Let X be a quasi-reflexive Banach space and $A \subset X^*$ be an absolutely convex subset. Then $A^{(1)} = \overline{A}^*$.*

Some parts of Theorems 1 and 2 are proved for separable spaces with basic sequences of special kinds first, and then are extended to the general case.

To describe the way in which results are extended from subspaces we need some more notation. Let Z be a subspace in a Banach space X and $E : Z \to X$ be the natural isometric embedding. Then $E^* : X^* \to Z^*$ is a quotient mapping which maps each functional in X^* onto its restriction to Z. Let A be a subset of Z^*. It is clear that $D = (E^*)^{-1}(A)$ is the set of all extensions of all functionals in A to the space X.

Lemma 1.
$$D^{(1)} = (E^*)^{-1}(A^{(1)}), \tag{1}$$
where the derived set $D^{(1)}$ is taken in X^* and the derived set $A^{(1)}$ - in Z^*.

PROOF. The inclusion $D^{(1)} \subset (E^*)^{-1}(A^{(1)})$ follows immediately from the inclusion $E^*(D^{(1)}) \subset A^{(1)}$, which in turn follows from the weak* continuity of E^*.

To prove the inverse inclusion it suffices to show that for every bounded net $\{f_\nu\} \subset Z^*$ with $w^*-\lim_\nu f_\nu = f$ and every $g \in (E^*)^{-1}(\{f\})$ there exist $g_\nu \in (E^*)^{-1}(\{f_\nu\})$ such that some subnet of $\{g_\nu\}$ is bounded and weak* convergent to g. Let h_ν be such that $h_\nu \in (E^*)^{-1}(\{f_\nu\})$ and $||h_\nu|| = ||f_\nu||$ (Hahn-Banach extensions). Then $\{h_\nu\}_\nu$ is a bounded net in X^*. Hence it has a weak* convergent subnet, let h be its limit. Then $g - h \in (E^*)^{-1}(\{0\})$, therefore $g_\nu = h_\nu + g - h$ is a desired net. QED

PROOF OF THEOREM 1. First we suppose that X is such that X^* contains a subspace A for which $A^{(1)}$ is a proper norm-dense subset in X^*.

The space X cannot be quasi-reflexive because the norm-density of $A^{(1)}$ in X^* implies that A is total, and the condition $A^{(1)} \neq X^*$ implies that A is not norming [6], and total non-norming subspaces do not exist in duals of quasi-reflexive spaces ([27], [37]).

To show that X contains an infinite-dimensional subspace with separable dual, assume the contrary, that is, all infinite-dimensional subspaces of X have non-separable duals.

Define the Banach space X_A as the completion of X with respect to the norm $||x||_A = \sup\{|f(x)| : f \in A, ||f|| = 1\}$. Since the subspace A is non-norming, the natural mapping $N : X \to X_A$ is not an isomorphism. Since the subspace A is total, the mapping N is injective.

Using the standard argument [15, Proposition 2.c.4] we find a separable infinite-dimensional subspace $Z \subset X$ such that the restriction $N|_Z$ is a compact operator. By duality [7, VI.5.2], this implies that $R = (N|_Z)^*(B_{X_A^*})$ is a norm-compact subset of Z^*. Observe that $A \cap B_{X^*}$ is embedded in a natural way into $B_{X_A^*}$. Therefore $E^*(A \cap B_{X^*}) \subset R$. Therefore E^* maps each weak* convergent net in $A \cap B_{X^*}$ onto a strongly convergent net in Z^*, therefore $E^*(A^{(1)})$ is contained in the linear span of R, which is a separable subspace of Z^*. Since by our assumption Z^* is non-separable, the subspace $E^*(A^{(1)})$ is not dense in Z^*. Hence $A^{(1)}$ is not dense in X^*, this contradiction completes the first part of the proof.

Now we prove the converse. Assume that X is a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual. We use terminology of [15]. The following result is proved using the techniques of [4]. We use the notation $n_k = \frac{k(k+1)}{2}$ for $k = 0, 1, 2, \dots$.

Lemma 2. *Let X be a non-quasi-reflexive Banach space containing an infinite-dimensional subspace with separable dual. Then there exists a minimal system*

$$\{u_i\}_{i=0}^{\infty} \cup \{x_i\}_{i=0}^{\infty}$$

in X satisfying the conditions:

(1) The system $\{u_i\}_{i=0}^{\infty} \cup \{x_i\}_{i=0}^{\infty}$ and the system of its biorthogonal functionals $\{u_i^*\}_{i=0}^{\infty} \cup \{x_i^*\}_{i=0}^{\infty}$ are uniformly bounded.

(2) The sequence $\{u_i\}_{i=0}^{\infty}$ spans a subspace U with separable dual U^*, and the restrictions of the biorthogonal sequence $\{u_i^*\}_{i=0}^{\infty}$ to U span U^*.

(3) The set $\left\{\sum_{p=j}^{k} x_{n_p+j} :\ 0 \le j \le k \le \infty\right\}$ is bounded.

Proof. This proof is a modification of the proof of Proposition 1 in [4, pp. 360–362]. For this reason we mostly follow the terminology and notation of [4] and the reader is expected to consult [4] if more details are needed (making this proof readable independently from [4] would lead to too much copying from [4]).

For the same reason as in [4, Proposition 1] we may assume that X is a separable non-quasi-reflexive Banach space containing an infinite-dimensional subspace Y with separable dual. Let $\{s_i\}_{i=0}^{\infty} \subset X^*$ be a sequence such that its restrictions to Y^* are dense in Y^*. By [4, Theorem 1], there is a weak* null sequence $\{y_n\} \subset X^*$, a bounded sequence $\{f_n\}_{n=0}^{\infty} \subset X^{**}$, and a partition I_n of the integers into pairwise disjoint infinite subsets such that

$$f_k(y_n) = \begin{cases} 1 & \text{if } n \in I_k \\ 0 & \text{if } n \notin I_k. \end{cases}$$

Let $\lambda > \|f_n\|$ for all n and choose $0 < \varepsilon_i < 1$ for all i with $\prod_i (1 + \varepsilon_i) < \infty$.

We are going to use induction to show that for each p we can find $\{x_k |\ 0 \le k \le n_p + p\} \subset X$, $\{f_i' |\ 0 \le i \le p\} \subset X^{**}$, $\{u_i |\ 0 \le i \le p\} \subset Y$, and finite-dimensional subspaces $G_0 \subset G_1 \subset \dots \subset G_p$ of X^* such that the following conditions are satisfied:

(1) G_i is $(1 + \varepsilon_i)^2$ norming over the linear span of $\{x_k |\ 0 \le k \le n_i + i\} \cup \{u_j : 0 \le j \le i\}$ for $0 \le i \le p$.

(2) $s_i \in G_i$ for each $i = 0, \dots, p$.

(3) $u_i \in Y \cap (G_{i-1})_\perp$ for $1 \le i \le p$, $||u_i|| = 1$.

(4) $\{x_{n_i+j} |\ 0 \le j \le i\} \subset (G_{i-1})_\perp$ for $1 \le i \le p$.

(5) $\left(\left\|\sum_{i=j}^{k} x_{n_i+j}\right\| \ \middle|\, j \le k \le p\right)$ is bounded by 6λ, for $0 \le j \le p$.

(6) $g\left(\sum_{i=j}^{p} x_{n_i+j}\right) = f_j'(g)$ for $g \in G_p$, $0 \le j \le p$.

(7) There is a constant C depending only on $\sup_n ||y_n||$ such that for each j, $0 \le j \le p$ there are functionals $\{\varphi_{n_j+i}|\ 0 \le i \le j\} \subset X^*$ of norm $\le C$ such that the system $\{x_{n_j+i}, \varphi_{n_j+i}|\ 0 \le i \le j\}$ is biorthogonal.

(8) For each j, $0 \le j \le p$ there is a functional $v_j^* \in X^*$ such that $||v_j^*|| = 1$, $v_j^*(u_j) = 1$, and $v_j^*(x_{n_j+i}) = 0$ for $0 \le i \le j$.

(9) $||f_i'|| \le 3\lambda$ for $0 \le i \le p$.

(10) There exist infinite sets $I_k' \subset I_k$, $k = 0, 1, \dots$, so that for each $i, 0 \le i \le p$, f_i' agrees with f_i on $[y_n | n \in \cup I_k']$. The sets I_k' depend on i, although we do not reflect this dependence in our notation.

For the first step, let U_0 be a 2-dimensional subspace of Y and G_0 be a finite-dimensional subspace of X^* which $(1+\varepsilon_0)$-norms $[\{f_0\} \cup U_0]$. By local reflexivity [11, 14], we pick x_0 in X with $||x_0|| \le \min\{\lambda, (1+\varepsilon_0)||f_0||\}$ such that $g(x_0) = f_0(g)$ for g in G_0. For convenience of notation later, we rename f_0 by f_0'. By the well-known result of [13] (see [15, Lemma 2.c.8]) there is $u_0 \in U_0$, $||u_0|| = 1$ such that for some $v_0^* \in X^*$ we have $||v_0^*|| = 1$, $v_0(u_0) = 1$ and $v_0(x_0) = 0$.

Let $(1'), \dots, (10')$ be the statements above for $p+1$. By [4, Lemma 1], pick infinite sets $I_k'' \subset I_k'$ for all k so that the natural projection onto G_p from $G_p \oplus [y_n |\ n \in \cup I_k'']$ has norm ≤ 2. Hence, there exists f_{p+1}' in X^{**} with $||f_{p+1}'|| < 3\lambda$ so that $f_{p+1}'(g) = 0$ for $g \in G_p$, and such that f_{p+1}' agrees with f_{p+1} on $[y_n |\ n \in \cup I_k'']$. This satisfies $(9')$ and $(10')$.

Since $y_n \overset{w^*}{\to} 0$, and each I_k'' is infinite, there exist, for $0 \le i \le p+1$, $q_i \in I_i''$ so that $\sum_{k=0}^{n_p+p} |y_{q_i}(x_k)| < 1/4p$. Now we select a $(p+2)$-dimensional subspace $U_{p+1} \subset Y \cap (G_p)_\perp$ and a finite-dimensional subspace $G_{p+1} \subset X^*$ containing $G_p \cup \{s_{p+1}\} \cup \{y_{q_i} |\ 0 \le i \le p+1\}$ and such that G_{p+1} is $(1+\varepsilon_{p+1})$-norming over the linear span H of $\{x_k\}_{k=0}^{n_p+p} \cup \{f_i'\}_{i=0}^{p+1} \cup \{u_i\}_{i=0}^{p} \cup U_{p+1}$ in X^{**}. This definition of G_{p+1} implies that $(2')$ is satisfied. By the principle of local reflexivity [11,14], there is an operator $T : H \to X$ such that T is the identity on $\{x_k\}_{k=0}^{n_p+p} \cup \{u_i\}_{i=0}^{p} \cup U_{p+1}$, T is an $(1+\varepsilon_{p+1})$-isometry and $g(Tf) = f(g)$ for $f \in H$, $g \in G_{p+1}$. Define $x_{n_{p+1}}, \dots, x_{n_{p+1}+p+1}$ by $x_{n_{p+1}+j} = Tf_j' - \sum_{i=j}^{p} x_{n_i+j}$ for $0 \le j \le p$ and $x_{n_{p+1}+p+1} = Tf_{p+1}'$.

Thus $Tf_j' = \sum_{i=j}^{p+1} x_{n_i+j}$ for $0 \le j \le p+1$, so that $(5')$ and $(6')$ hold. Since $G_p \subset G_{p+1}$ and $f_{p+1}' \in G_p^\perp$, using (6) we get $(4')$. Now, for $0 \le i \le p+1$, $0 \le j \le p+1$, one again has from local reflexivity that $y_{q_i}(x_{n_{p+1}+j}) = f_j'(y_{q_i}) - \sum_{k=j}^{p} y_{q_i}(x_{n_k+j})$, so that $y_{q_i}(x_{n_{p+1}+i}) \ge 1 - 1/4p \ge 3/4$ and $|y_{q_i}(x_{n_{p+1}+j})| < 1/4p$ when $i \ne j$. It is easy to derive from these inequalities that the Hahn-Banach extensions of the functionals defined by $\varphi_{n_p+i}(x_{n_p+j}) = \delta_{ij}$, $i, j = 0, 1, \dots, p+1$, satisfy $||\varphi_{n_p+i}|| \le C$ where C depends on $\sup_n ||y_n||$ only, so $(7')$ is satisfied.

Now we use [15, Lemma 2.c.8] and pick $u_{p+1} \in U_{p+1}$ such that $||u_{p+1}|| = 1$ such that for some $v_{p+1}^* \in X^*$ we have $||v_{p+1}^*|| = 1$, $v_{p+1}(u_{p+1}) = 1$ and $v_{p+1}(x_{n_{p+1}+i}) = 0$ for $i = 1, \dots, p+1$. It is clear that $(3')$ and $(8')$ are satisfied.

Since G_{p+1} is $(1+\varepsilon_{p+1})$-norming over H, local reflexivity guarantees that G_{p+1} is $(1+\varepsilon_{p+1})^2$-norming over the linear span of $\{x_k |\ 0 \le k \le n_{p+1}+p+1\} \cup \{u_i : 0 \le i \le p+1\}$ so that $(1')$ holds. This completes the construction of $\{x_n\}$ and $\{u_n\}$.

The conditions (1), (3), and (4) and the choice of $\{\varepsilon_i\}$ imply that the sequence

$$[x_0, u_0], [x_1, x_2, u_1], \dots, [x_{n_p}, \dots, x_{n_p+p}, u_p], \dots$$

of subspaces forms a finite-dimensional decomposition of the closed linear span of

$$\{u_i\}_{i=0}^{\infty} \cup \{x_i\}_{i=0}^{\infty}$$

Now we check that conditions **(1–3)** of Lemma 2 are satisfied.

The sequences $\{x_i\}$ and $\{u_i\}$ are bounded by construction, for $\{x_i\}$ we use also (5). The biorthogonal functionals of the system $\{u_i\}_{i=0}^{\infty} \cup \{x_i\}_{i=0}^{\infty}$ are bounded because of the finite decomposition property and the conditions (7) and (8). It remains to check the condition **(2)**.

Since $\{u_i\}_{i=0}^{\infty}$ is a basis in its closed linear span, it suffices to show that this basis is shrinking, that is, that for each $u^* \in U^*$ and $\delta > 0$ there is $n \in \mathbb{N}$ such that $||u^*|_{[u_i]_{i=n}^{\infty}}|| < \delta$ (see [15, Proposition 1.b.1]). Let $\tilde{u}$ be a norm-preserving extension of u^* to Y. By density there exists $n \in \mathbb{N}$ such that $||s_{n-1}|_Y - \tilde{u}|| < \delta$. The conditions (2) and (3) above imply that $s_{n-1}|_{[u_i]_{i=n}^{\infty}} = 0$. Hence $||u^*|_{[u_i]_{i=n}^{\infty}}|| < \delta$. QED

We consider the subspace W spanned by the system $\{u_i\}_{i=0}^{\infty} \cup \{x_i\}_{i=0}^{\infty}$ constructed in Lemma 2. Denote by h_j, $j \in \mathbb{N} \cup \{0\}$, a weak*-cluster point of the sequence

$$\left\{\sum_{i=j}^{k} x_{n_i+j}\right\}_{k=j}^{\infty}$$

in W^{**}, and by $\{u_i^*\}_{i=0}^{\infty} \cup \{x_i^*\}_{i=0}^{\infty} \subset W^*$ the biorthogonal functionals of $\{u_i\}_{i=0}^{\infty} \cup \{x_i\}_{i=0}^{\infty}$.

It is easy to see that Lemma 1 and the Hahn-Banach theorem imply that it suffices to find a subspace $A \subset W^*$ such that $A^{(1)} \neq W^*$ and $A^{(1)}$ is norm-dense in W^*.

To construct such A we pick a sequence $\{a_i\}_{i=1}^{\infty}$ of real numbers satisfying $a_i > 0$ and $\sum_{i=1}^{\infty} a_i < \infty$, and let $K = \{u_i + a_i h_i :\ i \in \mathbb{N} \cup \{0\}\} \subset W^{**}$ and $A = K_\perp \subset W^*$.

We claim that

(A) $u_i^* \in A^{(1)}$ for all i.

In fact, u_i^* is a weak* limit of $u_i^* - \frac{1}{a_i} x_{n_j+i}^*$ as $j \to \infty$ and $u_i^* - \frac{1}{a_i} x_{n_j+i}^* \in K_\perp$ for $j \geq i$.

(B) If $y^* \in U^\perp \subset W^*$, then

$$y^* - \sum_i a_i h_i(y^*) u_i^* \in A.$$

This immediately follows from the condition $h_i(u_j^*) = 0$. The series is norm-convergent because $\{h_j\}$ and $\{u_i^*\}$ are bounded sequences and $\sum_{i=1}^{\infty} a_i < \infty$. Therefore $y^* \in \overline{A^{(1)}}$.

We show that conditions (A) and (B) imply that $\overline{A^{(1)}} = W^*$. In fact, let $z^* \in W^*$, $\varepsilon > 0$. By Lemma 2**(2)**, the restriction of z^* to U can be ε-approximated by a finite linear combination of restrictions of u_i^* to U. Therefore there exists a vector s in U^* such that $||s|| < \varepsilon$ and $z^*|_U - s$ is a finite linear combination of $\{u_i^*|_U\}$. Let s^* be a Hahn-Banach extension of s to W, so $||s^*|| < \varepsilon$ and only finitely many of the numbers $\{(z^* - s^*)(u_i)\}_{i=1}^{\infty}$ are non-zero. Subtracting from $z^* - s^*$ the corresponding finite linear combination of $\{u_i^*\}$ we get a vector from $U^\perp$. Thus every vector $z^* \in W^*$ can be arbitrarily well approximated by vectors of $\mathrm{lin}(\{u_i^*\} \cup U^\perp)$, hence $W^* = \overline{A^{(1)}}$.

It remains to prove that $A^{(1)} \neq W^*$. Let $z^* \in A \cap B_{W^*}$. Then $|z^*(u_i)| = |-a_i h_i(z^*)| \leq a_i C$, where $C = \sup_i ||h_i||$. It is easy to see that the conditions on $\{u_i\}$ and $\{a_i\}$ imply that the set $T = \{u^* \in U^* : |u^*(u_i)| \leq a_i C\ \forall i \in \mathbb{N} \cup \{0\}\}$ is norm-compact. The inequality $|z^*(u_i)| \leq a_i C$ implies that $E^*(A \cap B_{W^*}) \subset T$, where E is the natural isometric embedding of U into W. Since T is norm-compact, the set $E^*\left(\overline{A \cap B_{W^*}}^{\,*}\right)$ is also contained in T. Therefore $E^*(A^{(1)}) \subset \mathrm{lin}(T) \neq U^*$ and $A^{(1)} \neq W^*$. QED

Proof of Theorem 2. We are going to use the following result proved in [26, 36]: If a Banach space X is non-reflexive, then it contains a normalized basic sequence $\{z_i\}_{i=0}^{\infty}$ such that the sequence $\left\{\sum_{i=1}^{k} z_i\right\}_{k=1}^{\infty}$ is bounded. Let Z be the closed linear span of the sequence $\{z_i\}_{i=0}^{\infty}$ and z^{**} be a weak*-cluster point of the sequence $\left\{\sum_{i=1}^{k} z_i\right\}_{k=1}^{\infty}$ in Z^{**}. (We added an extra element z_0 to the sequence, because it is needed for our construction, of course, it does not affect the validity of the result of [26,36].) By Lemma 1, it suffices to find a convex subset $A \subset Z^*$ such that $A^{(1)} \neq \overline{A}^*$. In fact, if we have such A, we let $D = (E^*)^{-1}(A)$. We have, by Lemma 1, $D^{(1)} = (E^*)^{-1}(A^{(1)})$. Also, by the bipolar theorem $\overline{A}^* = A^{\circ\circ}$, where the first polar is in Z and the second in Z^*. It is easy to see that the polar D° of D in X coincides with A°. Therefore $\overline{D}^* = A^{\circ\circ}$, where the first polar is in Z, and the second in X^*. Hence $\overline{D}^* \supset (E^*)^{-1}(\overline{A}^*)$ and $D^{(1)} \neq \overline{D}^*$.

Let $\{z_i^*\}$ be the biorthogonal functionals of $\{z_i\}$, $\{\alpha_i\}$ and $\{\beta_i\}$ be strictly increasing sequences of positive real numbers satisfying $\lim_{i\to\infty} \alpha_i = 1$ and $\lim_{i\to\infty} \beta_i = \infty$. We split $\mathbb{N}$ into infinitely many infinite subsequences $\mathbb{N}_j$. Let $A \subset Z^*$ be the convex hull of all vectors of the form $\alpha_j z_0^* + \beta_j z_k^*$, where $k \in \mathbb{N}_j$.

It is enough to show that the set $A^{(1)}$ is not strongly closed. First we observe that $\alpha_j z_0^* \in A^{(1)}$. In fact, $\alpha_j z_0^*$ is the weak* limit of the sequence $\{\alpha_j z_0^* + \beta_j z_k^*\}_{k\in\mathbb{N}_j}$.

It remains to show that $z_0^* \notin A^{(1)}$. Assume the contrary. Since Z is separable, there is a bounded sequence $\{y_r^*\}_{r=1}^\infty$ of vectors in A such that z_0^* is a weak* limit of $\{y_r^*\}_{r=1}^\infty$. By the definition of A, vectors y_r^* are finite convex combinations of the form $y_r^* = \sum_{j,k} a_{j,k}(r)(\alpha_j z_0^* + \beta_j z_k^*)$. Since z_0 is a weak* continuous functional on Z^*, we get

$$\lim_{r\to\infty} \sum_{j=1}^\infty a_{j,k}(r)\alpha_j = 1.$$

It is clear that this implies

$$\lim_{r\to\infty} \sum_{\substack{j \\ \alpha_j < 1-\varepsilon}} a_{j,k}(r) = 0.$$

Since $\lim_{j\to\infty} \beta_j = \infty$, this implies that for each $M < \infty$

$$\lim_{r\to\infty} \sum_{\substack{j \\ \beta_j > M}} a_{j,k}(r) = 1.$$

Therefore

$$\begin{aligned}\limsup_{r\to\infty} z^{**}(y_r) &= \limsup_{r\to\infty} \sum_{j=1}^\infty a_{j,k}(r)\beta_j \\ &\geq M \limsup_{r\to\infty} \sum_{\substack{j \\ \beta_j > M}} a_{j,k}(r) = M.\end{aligned}$$

Since M is arbitrary, this implies that the sequence y_r^* is unbounded, and we get a contradiction. QED

Proof of Theorem 3. Assume the contrary, let A be an absolutely convex subset of the dual of a quasi-reflexive Banach space X such that $A^{(1)} \neq \overline{A}^*$.

By [7, Theorem V.5.7] this implies that $\left(A^{(1)}\right)^{(1)} \neq A^{(1)}$, so there exists a bounded weak* convergent net $\{x_\alpha\}$ in $A^{(1)}$ such that any nets $\{x_{\alpha,\beta}\} \subset A$ satisfying

$$\sup_\beta ||x_{\alpha,\beta}|| < \infty \quad \text{and} \quad w^* - \lim_\beta x_{\alpha,\beta} = x_\alpha \tag{2}$$

are not uniformly bounded, that is,

$$\sup_\alpha \sup_\beta ||x_{\alpha,\beta}|| = \infty.$$

Since X is quasi-reflexive, we have $X^{**} = X \oplus F$ where F is a finite-dimensional subspace. We pick nets $\{x_{\alpha,\beta}\}_\beta \subset X^*$ satisfying the condition (2). We may assume that β in all of them runs through the same ordered set (it can be chosen to be a subnet of the naturally ordered set of weak* neighborhoods of 0 in X^*) and that the natural images of these nets in F^* converge strongly. Denote the corresponding limits in F^* by v_α. First we show that $\limsup_\alpha ||v_\alpha|| = \infty$.

Assume the contrary, that is, $\limsup_\alpha ||v_\alpha|| < \infty$. Using local reflexivity [11, 14] we find, for sufficiently large α, uniformly bounded nets $\{\ell_{\alpha,\delta}\}_\delta \in X^*$ such that $\ell_{\alpha,\delta}|_F = v_\alpha - (x_\alpha|_F)$ and $\lim_\delta \ell_{\alpha,\delta}(x) = 0$ for all $x \in X$.

Then the combined nets $\{x_{\alpha,\beta} - x_\alpha - \ell_{\alpha,\delta}\}_{(\beta,\delta)}$ (where the order is defined by: $(\beta_1, \delta_1) \succ (\beta_2, \delta_2)$ if and only if both $\beta_1 \succ \beta_2$ and $\delta_1 \succ \delta_2$) are weakly null. In fact, if $x \in X$ then $\lim_\beta x_{\alpha,\beta}(x) = x_\alpha(x)$ and $\lim_\delta \ell_{\alpha,\delta}(x) = 0$. If $f \in F$ then $\lim_\beta x_{\alpha,\beta}(f) = v_\alpha(f)$ and $\lim_\delta \ell_{\alpha,\delta}(f) = v_\alpha(f) - x_\alpha(f)$. Therefore, by [7, Theorem V.3.13], for each $\varepsilon > 0$ and β_0 there is a convex combination of $\{x_{\alpha,\beta} - x_\alpha - \ell_{\alpha,\delta}\}_{\beta\succ\beta_0}$ satisfying

$$\left\|\sum a_{\beta,\alpha,\delta}(\beta_0,\varepsilon)(x_{\alpha,\beta} - x_\alpha - \ell_{\alpha,\delta})\right\| < \varepsilon.$$

But then the nets

$$\left\{\sum a_{\beta,\alpha,\delta}(\beta_0,1)x_{\alpha,\beta}\right\}_{\beta_0}$$

are contained in A, are uniformly bounded, and

$$w^* - \lim_{\beta_0} \sum a_{\beta,\alpha,\delta}(\beta_0,1)x_{\alpha,\beta} = x_\alpha.$$

We get a contradiction with the assumption made at the beginning of the proof.

We consider the set of all vectors $\{v_\alpha - x_\alpha|_F\}$. It is clear that it is an unbounded set. We need the following observation from Convex Geometry.

Lemma 3. *Let $\{m(\alpha)\}_{\alpha\in\Omega} \subset \mathbb{R}^n$, where Ω is a partially ordered set, be such that $\limsup_\alpha \|m(\alpha)\| = \infty$. Then there exist $0 < C < \infty$ and $\alpha' \in \Omega$ such that for each $\alpha_0 \succ \alpha'$ and each $\varepsilon > 0$ there is a finitely non-zero collection $a(\alpha)$ of real numbers supported on $\alpha \succ \alpha_0$ and satisfying $\sum_\alpha |a(\alpha)| = 1$, $a(\alpha_0) = 1 - \varepsilon$, and $\left\|\sum_\alpha a(\alpha)m(\alpha)\right\| \leq C$.*

We do not specify the norm on $\mathbb{R}^n$ because the lemma holds for any norm, only the constant C changes.

Proof of Lemma 3. For each $\alpha_0 \in \Omega$ consider the closed absolutely convex hull M_α of $\{m(\alpha)\}_{\alpha\succ\alpha_0}$. By [35, Lemma 1.4.2], each M_α is a (Minkowski) sum of a compact set K_α and a linear subspace L_α.

Since $\limsup_\alpha ||m(\alpha)|| = \infty$, the subspaces L_α are non-trivial. Also it is clear that $L_{\alpha_1} \subset L_{\alpha_2}$ for $\alpha_1 \succ \alpha_2$. Since all of these subspaces are finite-dimensional, they stabilize in the sense that there exists α' such that $L_\alpha = L_{\alpha'}$ for any $\alpha \succeq \alpha'$. Let $L = L_{\alpha'} (= \cap_\alpha L_\alpha)$. Then $M_\alpha = K_\alpha + L$ for each $\alpha \succeq \alpha'$ and we may assume that $K_\alpha \subset K_{\alpha'}$ (we may assume that all K_α are in the same complement of the subspace L, see [35]). Set $C = \max\{||x|| : x \in K_{\alpha'}\}$.

We have $m(\alpha_0) = k(\alpha_0) + \ell(\alpha_0)$, where $k(\alpha_0) \in K_{\alpha_0} \subset K_{\alpha'}$, $\ell(\alpha_0) \in L$. Since the vector $-\frac{1-\varepsilon}{\varepsilon}\ell(\alpha_0)$ is in L, it can be arbitrarily well approximated by absolutely convex combinations of $\{m(\alpha)\}_{\alpha\succ\alpha_0}$. Therefore there is a finitely nonzero collection $\{b(\alpha)\}_{\alpha\succ\alpha_0}$ such that $\sum_\alpha |b(\alpha)| = 1$ and

$$\left\|\sum_{\alpha\succ\alpha_0} b(\alpha)m(\alpha) + \frac{1-\varepsilon}{\varepsilon}\ell(\alpha_0)\right\| < C.$$

We introduce $a(\alpha)$ by $a(\alpha) = \varepsilon b(\alpha)$ for $\alpha \succ \alpha_0$, $a(\alpha_0) = 1 - \varepsilon$ and $a(\alpha) = 0$ for all other α. We have

$$\sum_\alpha a(\alpha)m(\alpha) = (1-\varepsilon)k(\alpha_0) + (1-\varepsilon)\ell(\alpha_0) + \varepsilon \sum_{\alpha\succ\alpha_0} b(\alpha)m(\alpha),$$

where $||(1-\varepsilon)k(\alpha_0)|| \le (1-\varepsilon)C$ and $||(1-\varepsilon)\ell(\alpha_0)+\varepsilon\sum_{\alpha\succ\alpha_0} b(\alpha)m(\alpha)|| < C\varepsilon$. The conclusion follows. QED

We apply Lemma 3 to the set $\{v_\alpha - x_\alpha|_F\}_\alpha$ and find that there is C (independent of α) such that for large enough α and an arbitrary $\varepsilon > 0$ there is a finite combination

$$(1-\varepsilon)(v_\alpha - x_\alpha|_F) + \sum_{\delta\succ\alpha} a(\delta)(v_\delta - x_\delta|_F) \tag{3}$$

having norm $\le C$ and such that $\sum_\delta |a(\delta)| = \varepsilon$. Using local reflexivity [11, 14] we can find a net $\{p_\gamma\} \subset X^*$ whose weak* limit is 0 and whose restrictions to F converge to the vector (3), and $\sup_\gamma ||p_\gamma|| \le C_1$, where C_1 does not depend on α.

Then the (β,γ)-net

$$(1-\varepsilon)(x_{\alpha,\beta} - x_\alpha) + \sum_{\delta\succ\alpha} a(\delta)(x_{\delta,\beta} - x_\delta) - p_\gamma \tag{4}$$

is weakly null, where the ordering on pairs (β,γ) is defined as above. Therefore, by [7, Theorem V.3.13], for each β_0 and $\omega > 0$ there is a convex combination satisfying

$$\left\|\sum_{\beta\succ\beta_0,\gamma} d_{\alpha,\beta,\gamma,\delta}(\beta_0,\omega)\left((1-\varepsilon)(x_{\alpha,\beta} - x_\alpha) + \sum_{\delta\succ\alpha} a(\delta)(x_{\delta,\beta} - x_\delta) - p_\gamma\right)\right\| < \omega. \tag{5}$$

Consider the net

$$\left\{\sum_{\beta\succ\beta_0,\gamma} d_{\alpha,\beta,\gamma,\delta}(\beta_0,1)\left((1-\varepsilon)x_{\alpha,\beta} + \sum_{\delta\succ\alpha} a(\delta)x_{\delta,\beta}\right)\right\}_{\beta_0}.$$

It is clear that this net is weak* convergent to $(1-\varepsilon)x_\alpha + \sum_{\delta\succ\alpha} a(\delta)x_\delta$. Since A is absolutely convex each element of this net is in A. By (5), the elements of this net are norm-bounded independently of α.

Now we consider the net

$$\left\{\sum_{\beta\succ\beta_0,\gamma} d_{\alpha,\beta,\gamma,\delta}(\beta_0,1)\left((1-\varepsilon)x_{\alpha,\beta} + \sum_{\delta\succ\alpha} a(\delta)x_{\delta,\beta}\right)\right\}_{\beta_0,\varepsilon},$$

where $(\beta_1,\varepsilon_1) \succ (\beta_2,\varepsilon_2)$ if and only if $\beta_1 \succ \beta_2$ and $\varepsilon_1 < \varepsilon_2$. It is clear that this net is weak* convergent to x_α and its elements are bounded independently of α. This contradicts the assumption made at the beginning of the proof. QED

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