Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **31** (2011) n. 1, 117–127. doi:10.1285/i15900932v31n1p117

Possible a priori motivations for the use of Hilbert spaces in Quantum Mechanics

Carlo Franchetti
Dipartimento di Sistemi e Informatica,
Facoltà di Ingegneria,
via S. Marta 3, 50139 Firenze (Italy)
carlo.franchetti@unifi.it

Received: 30/06/2010; accepted: 29/10/2010.

Abstract. In this note we try to show that some a priori justifications can be given for the use of Hilbert spaces in Quantum Mechanics.

Keywords: Quantum Mechanics, Axioms, Rearrangement invariant spaces, L_p-spaces, Hilbert spaces, Projections.

MSC 2000 classification: primary 81P16, secondary 46C15

Dedicated to the memory of V.B. Moscatelli

Introduction

The theory and the results of Quantum Mechanics are firmly established from more then half a century and no man of science would have the least doubt that QM "works", however what is usually called the "interpretation" of the rules of QM is still controversial. An exhaustive discussion of this problem is contained in the books by R.B. Griffiths (2002) and by R. Omnès (1999) (references [6], [8]). An attempt to state the principles of QM in an axiomatic form is presented also in introductory books, for an example of clear and useful description see [9], ch 4. Omnès states in [8] as fundamental for QM four "Principles", we quote the first one (ch 8, page 91)
One can associate with every isolated physical system a definite Hilbert space. Every physical concept entering in the description or analysis of the system should be expressible in a mathematical language into which only notions involving this Hilbert space can enter.
Our goal is to discuss the nature of this principle from a mathematical point of view.

1 The superposition principle

If the first statement in an exposition of QM is (as it often happens) "the state of a quantum system is represented by a vector in a Hilbert space..", then there is no point of speaking of a superposition principle since by definition in a vector space we sum vectors: this is trivial! Of course it is intended that the big novelty will be in the physical interpretation of the superposed states: an a posteriori motivation.

1.1 Quotations from Dirac

We shall follow Dirac's view: the superposition principle should be regarded as fundamental. All the following quotations from Dirac are found in reference [4] (at the beginning of the book).
For this purpose [to provide any quantitative basis for the building up of QM] *a new set of accurate laws of nature is required. One of the most fundamental and most drastic of these is the Principle of Superposition of States.*
After having explained what should be meant by "state" of a quantum system, Dirac says:
When a state is formed by the superposition of two other states, it will have properties that are in some vague way intermediate between those of either of them according to the greater or less "weight" attached to this state in the superposition process. The new state is completely defined by the two original states when their relative weights in the superposition process are known, together with a certain phase difference, the exact meaning of weights and phases being provided in the general case by the mathematical theory.
About the role of probability we quote
The non-classical nature of the superposition process is brought out clearly if we consider the superposition of two states, A and B, such that there exists an observation which, when made on the system in the state A, is certain to lead to one particular result, a say, and when made on the system in the state B, is certain to lead to some different result, b say. What will be the result of the observation when made on the system in the superposed state? The answer is that the result will be sometimes a and sometimes b, according to a probability law depending on the relative weights of A and B in the superposition process. It will never be different from both a and b.
Here is Dirac's conclusion of the discussion (which contains the above quotations):
It is important to remember, however, that the superposition that occurs in QM is of an essentially different nature from any occuring in the classical theory, as is shown by the fact that the quantum superposition principle demands indeterminacy in the results of observations in order to be capable of a sensible physical interpretation. The analogies are thus liable to be misleading.

1.2 The space of states as a free vector space

We see that the "states", whatever they can be, should be intended from the mathematical point of view as abstract objects of unspecified nature. We take these objects and put them randomly in the emptiness of some "space", none of them may have a favoured position.
Now comes the superposition principle and tell us that these objects (pure states, eigenstates of some operator or whatever they are) can be associated in a group of two or more: anticipating the language of vector spaces we create all possible formal linear combinations of these objects (the surprising fact that we allow complex coefficients we do not discuss here). We learn also that the original objects must be "independent" otherwise it could happen that one of them is a superposition of other companions (this will just mean that the set of all the primitive objects is something like a basis for the space). This procedure leads to the construction of what the mathematicians call "free vector space".

1.3 Assigning a norm in the space

Since the results of measurements must be numbers we need a metric structure on the vector space and this is most easily done if we require to have a norm on the space. Let us call E the space, we consider the set $S = \{A_i\}$ of primitive objects as a basis for E. It is natural to assume that all the A_i have the same norm, 1 say; also for symmetry reasons it can be assumed that the mutual distance among them is constant, that is $i \neq j \Rightarrow \|A_i - A_j\|_E = k$ (a positive constant). We shall see later that S could be a symmetric basis or a symmetric Schauder basis.

1.4 Norm is conditioned by probability

Let $A \neq B$ be objects in S and consider a vector v in a superposition state: $v = \lambda A + \mu B$ where λ and μ are nonzero complex numbers: how do we know the values of the coefficients in the linear combination? After measurement the state of the system will be A with probability p or B with probability $(1 - p)$ (for an exhaustive discussion on measurements in QM see [6, ch. 17 and 18]). The value of p may be determined empirically after many measurements. The way of assigning (a posteriori) a norm on the space and the values of the coefficients of the linear combination may be considered to some extent conventional. It is however natural to assume a priori that if the states A and B will result with the same probability, then then the values taken by some function on these coefficients must be equal, if it is known a priori that in the superposition state the coefficients are equal (for example because of some symmetry) then the probabilities must be equal; moreover if the resulting of the state A is certain then the coefficient of B is zero and that of A has any nonzero value (and vice-versa). Incidentally these remarks suggest why the physical meaning of a state is unchanged when multiplied by a nonzero (complex) scalar.

Consider a general superposition state $v = \sum_j \lambda_j A_j$: we assume that a probability law for the transition from v to A_j (which implies the relation between the coefficient λ_j and the probability p_j) must be the same for every j; we also assume that for a given state A_j coefficients of the same modulus depend on the same probability. In order to specify this relation between λ_j and p_j it is necessary to have some normalization: this will be done using the norm of the state $v = \sum_k \lambda_k A_k$. Let us set

$$p_j = \frac{h(|\lambda_j|)}{h(\|v\|)} \tag{1}$$

where h is a positive increasing function with $h(0) = 0, h(1) = 1$ and the norm is such that $|\lambda_j| = \|\lambda_j A_j\| \leq \|v\|$. Note that since $\sum_k p_k = 1$ and since a state multiplied by a nonzero scalar does not change his physical meaning and consequently the underlying probabilities, from (1) we get for any $t \neq 0$

$$\sum_k h(|\lambda_k|) = h(\|v\|) \quad \text{and} \quad \frac{h(|\lambda_j|)}{h(\|v\|)} = \frac{h(|t||\lambda_j|)}{h(|t|\|v\|)} \tag{2}$$

taking $\|v\| = 1$ we obtain $h(|\lambda_j|)h(|t|) = h(|t||\lambda_j|)$, thus h is a homogeneous function, that is, $h(x) = x^s$ for some $s > 0$. Thus we have

$$p_j = \frac{|\lambda_j|^s}{\|v\|^s} \quad \text{and} \quad |\lambda_j| = \|v\|(p_j)^{1/s} \tag{3}$$

So if we want to recover the wave function from the probabilities empirically determined, all we can say about the coefficients λ_j is that

$$\lambda_j = \mathrm{e}^{i\theta_j}\|v\|(p_j)^{1/s}$$

where θ_j is any real number. The factor $\|v\|$ is of no importance but the phases (dependent on j) $\mathrm{e}^{i\theta_j}$ cause an enormous indetermination!

We can give a "negative" motivation for the use of complex scalars in QM: the wave function is such a mysterious thing that it must be impossible to recover it! A more serious (a posteriori)

reason given by S. Caser is discussed in [8] (ch.5 page 42).
We will show that the probability law determines or severely restricts the possible choices of a norm in E.

2 Localization on [0,1]

We assume that the state of a particle is represented by a vector in a (complex) normed space E of (Lebesgue measurable) functions defined on the interval $[0,1]$. Let A be any (measurable) subset of $[0,1]$, χ_A the characteristic function of A and $\mu = \mu(A)$ the (Lebesgue) measure of $A\,(0 \le \mu \le 1)$. We assume that

$$\|\chi_{[0,1]}\|_E = 1 \tag{4}$$

Let $f \in E$ be the wave function of a particle α_f: we will describe its relation with probability in two different ways.

2.1 1st method - Rearrangement invariant spaces

If we define $P_A f = f\chi_A$ then P_A is a projection onto the subspace of E of functions vanishing in $[0,1]\setminus A = A^c$; also $(I-P_A)f = P_{A^c}f = f\chi_{A^c}$ (where I is the identity projector). The meaning of the state $P_A f$ is "particle α_f is localized on A", that of $P_{A^c}f$ is "particle α_f is localized on A^c ". We assume that for a particle α_f the probability that $\{\alpha_f$ is localized in $A\}$ is "true" (we will write for simplicity $\{\alpha_f \in A\}$) is a function ϕ of the quantity $\frac{\|P_A f\|_E}{\|f\|_E}$, thus $0 \le \phi(\frac{\|P_A f\|_E}{\|f\|_E}) \le 1$.
Note that the condition that for any (complex) $\lambda \neq 0$ the functions f and λf represent the same state for a particle α is fulfilled by this definition. If $\{\alpha_f \in A\}$ is "true" then $\{\alpha_f \in A^c\}$ is "false" and vice-versa, hence

$$\phi(\frac{\|P_A f\|_E}{\|f\|_E}) + \phi(\frac{\|(I-P_A)f\|_E}{\|f\|_E}) = 1 \tag{5}$$

If $A \subset B \subset [0,1]$ we must have $p\,\{\alpha_f \in A\} \le p\,\{\alpha_f \in B\}$ that is
$\phi(\frac{\|f\chi_A\|_E}{\|f\|_E}) \le \phi(\frac{\|f\chi_B\|_E}{\|f\|_E})$ and for $f = \chi_{[0,1]}$, using (4), we obtain
$\phi(\|\chi_A\|_E) \le \phi(\|\chi_B\|_E)$. On the other hand by (5) we also have
$\phi(\|\chi_A\|_E) + \phi(\|\chi_{A_c}\|_E) = 1$ thus $0 \le \phi(t) \le 1$ and ϕ is (strictly) increasing.
Note that if the particle α_f is in the state $f = \chi = \chi_{[0,1]}$ we have no information on its localization, therefore we have

$$p\,\{\alpha_\chi \in A\} = \mu(A) = \phi(\|\chi_A\|_E)$$

so that $\|\chi_A\|_E = \phi^{-1}(\mu(A))$ depends only on the measure of A. This is a condition that the norm on E must satisfy.
Two functions f, g in E are called equimeasurable if they have the same distribution function, the distribution function D_h of a function $h \in E$ being defined by

$$D_h(y) = \mu\{t : |h(t)| > y\} \quad (y \ge 0).$$

Recall also the following

Definition. A Banach space E of measurable functions on [0, 1] is called rearrangement invariant (or symmetric) if:
i) from $|x(t)| \le |y(t)|, y \in E$, it follows that $x \in E$ and $\|x\|_E \le \|y\|_E$;
ii) from the equimeasurability of the functions x, y and from $y \in E$ it follows that $x \in E$ and $\|x\|_E = \|y\|_E$.

Standard examples of such spaces are L_p , Orlicz, Lorentz and Marcinkievicz spaces. We can now prove

Theorem 1. *The quantum space E must be a rearrangement invariant (symmetric) space.*

PROOF. It will be enough to prove that the following two properties hold for every f and g in E

i) $|f| \leq |g| \Rightarrow \|f\|_E \leq \|g\|_E$

ii) f and g equimeasurable $\Rightarrow \|f\|_E = \|g\|_E$

It will be enough to consider simple functions.

i) Assume that

$$f = \sum_i c_i \chi_{A_i} \; ; \; g = \sum_i d_i \chi_{A_i}$$

where $\{A_i\}$ is a partition of $[0,1]$ and $|c_i| \leq |d_i|$. Using formula (5) when generalized to partitions we obtain

$$1 = \sum_i \phi(\frac{\|c_i \chi_{A_i}\|_E}{\|f\|_E}) = \sum_i \phi(\frac{|c_i| \|\chi_{A_i}\|_E}{\|f\|_E}) = \sum_i \phi(\frac{|d_i| \|\chi_{A_i}\|_E}{\|g\|_E})$$

since $|c_i| \leq |d_i|$ and ϕ is increasing we have $1 \leq \sum_i \phi(\frac{|d_i| \|\chi_{A_i}\|_E}{\|f\|_E})$; if it were $\|g\|_E < \|f\|_E$ we would have $1 < \sum_i \phi(\frac{|d_i| \|\chi_{A_i}\|_E}{\|g\|_E}) = 1$: a contradiction.

ii) Assume that

$$f = \sum_{i=1}^{n} c_i \chi_{A_i} \; ; \; g = \sum_{i=1}^{n} d_i \chi_{B_i}$$

where $\{A_i\}$ and $\{B_i\}$are partitions of $[0,1]$. The functions f and g will be equimeasurable if $|c_i| = |d_i|$ and $\mu(A_i) = \mu(B_i)$.

As above we have

$$1 = \sum_i \phi(\frac{|c_i| \|\chi_{A_i}\|_E}{\|f\|_E}) = \sum_i \phi(\frac{|d_i| \|\chi_{B_i}\|_E}{\|g\|_E})$$

Since we know that $\|\chi_{A_i}\|_E = \|\chi_{B_i}\|_E$ the last argument in i) shows here that $\|f\|_E = \|g\|_E$. QED

2.2 2nd method -L^p spaces

We assume that the probability that the particle α_f is found in A is given by

$$p\{\alpha_f \in A\} = \frac{\phi(\|f\chi_A\|_E)}{\phi(\|f\|_E)} \tag{6}$$

where ϕ is a positive real function. For $f = \chi = \chi_{[0,1]}$ we get

$$p\{\alpha_\chi \in A\} = \frac{\phi(\|\chi_A\|_E)}{\phi(1)} \tag{7}$$

and it is natural to assume in this case that $p\{\alpha_\chi \in A\} = \mu(A)$. Thus we have that

$$\phi(\|\chi_A\|_E) = \phi(1)\mu(A) \tag{8}$$

Since for any (complex) $\lambda \neq 0$ the functions f and λf represent the same state for the particle, using (7) for $f = \chi = \chi_{[0,1]}$ we get

$$p\{\alpha_\chi \in A\} = \frac{\phi(\|\chi_A\|_E)}{\phi(1)} = \frac{\phi(|\lambda| \|\chi_A\|_E)}{\phi(|\lambda|)}$$

that is in general one has $\phi(ab)\phi(1) = \phi(a)\phi(b)$. Assuming that ϕ is differentiable we see that ϕ must be of the form $\phi(x) = kx^p$ where p is positive and $k\,(= \phi(1)) \geq 1$. We therefore conclude that

$$\phi(x) = kx^p\,;\ \|\chi_A\|_E = \mu^{1/p} \quad (\mu = \mu(A))$$

Note that there is no restriction in assuming that $k = 1 = \phi(1)$.

Lemma. Assume that two elements f and g of E have disjoint supports, then

$$\phi(\|f+g\|_E) = \phi(\|f\|_E) + \phi(\|g\|_E) \tag{9}$$

PROOF. With no restriction we assume that $\text{supp}(f) = A$, $\text{supp}(g) = A^c$. We have that $f + g = f\chi_A + g\chi_{A^c}$. For the particle α_{f+g} we must have $p\,\{\alpha_{f+g} \in A\} + p\,\{\alpha_{f+g} \in A^c\} = 1$ so that by (6) we get

$$\frac{\phi(\|f\|_E)}{\phi(\|f+g\|_E)} + \frac{\phi(\|g\|_E)}{\phi(\|f+g\|_E)} = 1$$

and the assertion follows immediately. QED

For $x \in [0,1]$ and $f \in E$ let us define

$$Q_f(x) = Q(x) = \phi(\|f\chi_{[0,x]}\|_E) = \phi(\|f\|)\,p\,\{\alpha_f \in [0,x]\}.$$

Note that since $f\chi_{[0,x+h]} = f\chi_{[0,x]} + f\chi_{[x,x+h]}$ (here $h > 0$) we have by (9)

$$\phi(\|f\chi_{[0,x+h]}\|_E) = \phi(\|f\chi_{[0,x]}\|_E) + \phi(\|f\chi_{[x,x+h]}\|_E)$$

and consequently $Q(x+h) - Q(x) = \phi(\|f\chi_{[x,x+h]}\|_E)$, this last quantity, assuming some regularity for f and recalling (8), is approximated by

$$\phi(\||f(x)|\chi_{[x,x+h]}\|_E) = \phi(|f(x)|\phi^{-1}(h)) = h|f(x)|^p$$

Theorem 2. *We have $p \geq 1$ and for $f \in E$*

$$\|f\|_E = \|f\|_p = (\int_0^1 |f(t)|^p dt)^{1/p}$$

and consequently $E = L^p[0,1]$.

PROOF. From the above calculation we get immediately that $Q'(x) = |f(x)|^p$, consequently

$$Q(x) = \int_0^x |f(t)|^p dt = (\|f\chi_{[0,x]}\|_E)^p \quad Q(1) = \|f\|_E^p = \int_0^1 |f(t)|^p dt$$

$$\|f\|_E = (\int_0^1 |f(t)|^p dt)^{1/p}$$

and the last formula defines a norm in E only if $p \geq 1$. QED

2.3 Which norm on E ?

The conditions imposed in the first method force E to be a rearrangement invariant space, imposing the more restrictive conditions of the second method results in E being a L_p space. Up to this point we have seen that a space E for a quantum system, also taking the more restrictive point of view, can have any L_p or l_p norm. Apparently there is no a priori reason that p should be equal to 2, all the p-norms are likely to give a correct description (see 6.2 in the Appendix). In order to find a possible a priori motivation for using the euclidean norm we will perform some further steps.

2.4 What is a property ?

Once it is established that the state of a quantum particle is somehow represented by a vector in a (possibly infinite dimensional) complex normed vector space E, one might want to define what it is meant by a specific "property" of the particle.
Recall that an operator $P : E \to A$ where A is a (closed) subspace of E is a projection if it is linear continuous and acts as the identity on A.
J. von Neumann first suggested to identify a "property" with such a projection P or, equivalently, with the subspace A. This point of view has been exhaustively described in the books [6, ch 4] and [8, ch 9].
Let P be a (physically meaningful) projection from E onto a (closed) subspace, (we have already considered such an example in 2.1 and 2.2). We assume that for a particle α_f the probability that the "property" P is "true" depends only on the two vectors f and Pf. If property P is true its negation must also be a property and must be false: the negation of P must be the projection $(I - P)$. For particle α_f the union of the two "events" $P, (I - P)$ must give the event certain: indeed we have $P + (I - P) = I$ (see [6] for a complete discussion); thus passing to the probabilities we must have

$$p\{Pf\} + p\{(I - P)f\} = 1. \tag{10}$$

From (10) it follows the

Result 1. If P is a physically meaningful property (projection) then

$$\|P\|_E = \|I - P\|_E = 1$$

Proof. If (for example) $\|I-P\|_E > 1$ there exists an f with $\|f\|_E = 1$ and $\|(I-P)f\|_E > 1$ but by (5) this is impossible since ϕ is positive. $\boxed{QED}$

Calling now the properties "subspaces" one has to keep in mind that the two subspaces $P(E)$ and $(I-P)(E)$ do not exhaust E: there are vectors of E which belong to neither subspace: the property for a vector to belong to a different subspace is meaningless and unrelated with our P (see again [6, 4.4 page 56]). One has also to consider the following: is it true that any (closed) subspace or any (norm one) projection can be considered as a physically meaningful property? This can be questionable.

3 Characterizations of Hilbert spaces

Since we are interested in possible a priori motivations for the use of Hilbert spaces in QM, it is natural to ask which are the characteristic properties of Hilbert spaces that are meaningful for this purpose. Many Hilbert space characterizations are known, see for example [1] for references up to 1985.

3.1 Kakutani's characterization of Hilbert spaces

The Jordan-von Neumann parallelogram equality (1935) is perhaps the best known characterization:

Theorem 3. *A normed space E is an inner product space (ips) if and only if for every $x, y \in E$ one has*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{11}$$

This is essentially a 2-dimensional characterization: E is ips if and only if every plane in E is ips and a plane is ips if and only if (11) holds. From (11) many 2-dimensional characterizations are deduced, but we must look to a different type of characterizations.
In a ips space E every (closed) subspace is the range of a norm-1 projection, namely the orthogonal projection (note that "norm-1 projection" is a meaningful concept in every normed space, not so the concept of "orthogonal projection"), the converse is also true but much more can be said. Here we state a case of the celebrated Kakutani's theorem

Theorem 4. *A normed space E of dimension at least 3 is a Hilbert space if and only if every closed hyperplane V of E is range of a norm-1 projection P of E .*

In this characterization a dimension greater than 2 is needed for the following reason: if E is 2-dimensional the hyperplanes are just the 1-dimensional subspaces but, as it is easily seen, a 1-dimensional subspace is 1-complemented in any normed space. So this is essentially a 3-dimensional characterization. From now on 2 and 1-dimensional Hilbert spaces will be ruled out from our considerations. Incidentally there is another good reason for this: in fact it has been proved that no universal value functional defined on the collection of all observable or Hermitian operators satisfying certain two natural conditions can exist in a QM space of dimension at least 3, thus the "paradoxical" feature of QM can be detected only in spaces of dimension at least 3, spaces of lesser dimension having no QM interest (see [6] 22.2 page 299 and 22.3 page 301).

Kakutani's theorem above and Result 1 imply immediately the following

Result 2. If all the hyperplanes of a quantum space E are physically meaningful properties (projections) then E must be a Hilbert space.

Indeed one can do much better: if the space E has some symmetry properties one can prove that it is Hilbert even assuming that just one of its hyperplanes is 1-complemented. Let E be a rearrangement invariant space, then

$$Pf = \left(\int_0^1 f(t)\, dt\right) \chi_{[0,1]} \tag{12}$$

defines a projection P of E onto the 1-dimensional subspace of the constant functions. In every E we have $\|P\|_E = 1$; $(I-P)$ is a projection onto the hyperplane of the null average functions, $(I-P)$ is minimal and $1 \le \|I-P\|_E \le 2$. The following characterization is known:

Theorem 5. *(see [5],[7]) Let E be a rearrangement invariant space of measurable functions on [0, 1] such that $\|I-P\|_E = 1$. Then $E = L_2[0,1]$.*

As a consequence of Result 1 and of the above theorem we have

Result 3. If the projection P (or the projection $I-P$) defined in (12) is a physically meaningful property then the quantum space E must be Hilbert.

Unfortunately we do not know whether P (or $I-P$) is related with a physically meaningful "property" for a particle.

4 The finite dimensional case

We assume that the state of a particle is represented by a vector x in a n-dimensional (complex) normed space E with canonical basis $\{e_i\}$ so that $x = \sum x_i e_i$.
The results of section 3 apply with the obvious modifications. Regard E as a space of functions defined on $X = \{1, 2, .., n\}$ with the counting measure and the normalization $\|\chi_{\{i\}}\| = 1$.
The conditions imposed in the first method force E to be a symmetric space with respect the

canonical basis: explicitly this means

$$\|\sum_{i=1}^{n} |a_i| e_i\| = \|\sum_{i=1}^{n} a_{\pi(i)} e_i\|$$

for every scalar a_i and every permutation π of $\{1, 2, .., n\}$.

Second method restricts E to be a $l_p(n)$ space: write $f = \sum_{k=1}^{n} f\chi_k$, by (9) we get $\phi(\|f\|) = \sum_{k=1}^{n} \phi(\|f\chi_k\|), \|f\|^p = \sum_{k=1}^{n} |f(k)|^p$. The projection defined in (12) takes now the form

$$Px = (\frac{1}{n}\sum_{i=1}^{n} x_i)u \tag{13}$$

where u is the constant vector (1, 1,..,1), $(I - P)$ is a projection onto the hyperplane of the null average vectors. The symmetry of the basis implies that in any such space E we have $\|P\|_E = 1$, we also have $1 \leq \|I - P\|_E \leq 2$. However the condition $\|I - P\|_E = 1$ does not imply that E is Hilbert since there exist non Hilbert symmetric spaces E with $\|I - P\|_E = 1$ (this is proved in [3, Theorem 5]). We see here a difference between the infinite and the finite dimensional case. If however we assume by hypothesis that E has a p-norm (as it happens via the second method), then

Theorem 6. *If $E = l_p(n)$ for some $p \geq 1$ then the condition $\|I - P\|_E = 1$ implies that E is Hilbert.*

Using again Result 1 and the above theorem (which we prove in 6.1 in the Appendix) we obtain

Result 4. If the projection P (or the projection $I - P$) defined in (13) is a physically meaningful property then the quantum space E must be Hilbert.

4.1 A special case

In order to investigate the relevance of the projector onto the null average functions we consider a special case.

Let us first recall that a square matrix $H = \{h_{ij}\}$ is a Hadamard matrix of order n if $|h_{ij}| = 1$ and $HH^T = nI$. Beside the trivial cases $n = 1, 2$ a Hadamard matrix must have order $4n$, existence is known for infinitely many n, the existence for any n is still an open conjecture. Let the integer n be such that there exists a Hadamard matrix H of order $4n$, it is known that H can be normalized in such a way that in the first row and in the first column all the entries are 1; let $h_i = (\sigma_{i1}, \sigma_{i2}, .., \sigma_{i4n})$ be the rows of H $(|\sigma_{ij}| = 1, h_1 = (1, 1, .., 1))$, for $i > 1$ $(2n)$ entries of h_i are equal to 1 and $(2n)$ to -1, any two rows have exactly $2n$ matches (they are mutually orthogonal).

Consider the space $E = R^{4n}$ with a norm symmetric with respect to the standard basis and let H be a Hadamard matrix as above. Note that $\{h_i\}$ is a basis for E. Let P_i be the projection onto the span of h_i defined by

$$P_i x = (\frac{1}{n}\sum_{k=1}^{4n} \sigma_{ik} x_k)h_i$$

The symmetry of the space implies that $\|P_i\| = 1$; note that $(I - P_1)$ is the projector onto the hyperplane of the null average functions, $\{h_2, h_3, .., h_{4n}\}$ being a basis for this hyperplane. Assume that E has a l_p-norm, then Result 4 implies

Result 5. If $(I - P_1)$ or at least one of the projections P_i is a physically meaningful property then the quantum space E must be Hilbert.

5 Final remarks

Two basic facts of QM, the superposition principle and the probabilistic nature of phenomena, seem to imply that the quantum space E has to be a rearrangement invariant space or more specifically a L_p space. E will become a Hilbert space as soon as one can show that some special projections define "properties" which possibly have "physical meaning" (see for example 4.1 above).
In conclusion there is some evidence that the axiom "E is a Hilbert space" could be removed from the list of axioms of QM and replaced with a weaker one or even deduced from the others (here the term "axiom" should not be intended as mathematicians and/or logicians do but in a somewhat looser way).

6 Appendix

6.1 Proof of Theorem 3

Indeed the result is practically contained in [2, section 3] where it is proved that, setting

$$h_p(t) = [(t^{1/p})^q + [(1-t)^{1/p}]^q]^{1/q}[(t^{1/q})^p + [(1-t)^{1/q}]^p]^{1/p} \quad (1/p + 1/q = 1)$$

one has

$$\|I - P\|_p = \max\{h_p(k/n), 0 \le k \le n\}\,.$$

Clearly $\|I - P\|_p \ge 1$, however it is not shown explicitly when inequality is strict. In fact we prove that for $n > 2$ and $p \ne 2$ we have $\|I - P\|_p > 1$.
Using obvious notations we can write

$$h_p(t) = \|(t^{1/p}, (1-t)^{1/p})\|_q \|(t^{1/q}, (1-t)^{1/q})\|_p$$

by the Hölder inequality we obtain

$$h_p(t) \ge \|(t^{1/p+1/q}, (1-t)^{1/p+1/q})\|_1 = \|(t, 1-t)\|_1 = 1$$

It is easily seen that we have equality in the Hölder inequality for any t for $p = 2$; for $p \ne 2$ equality holds only in the points $t = 0, 1/2, 1$ but if $n > 2$ the $\max\{h_p(k/n), 0 \le k \le n\}$ cannot be attained in such points.

6.2 A toy model

Consider the toy Mach-Zehnder interferometer discussed in [6] (13.2, pages 178-179), we try to show that the model could be worked also with a p-norm in the space making the natural modifications. We start defining in a parallel way an operator S which will be the time "unitary" transformation in the p-generalized sense; keeping the same notations as in [6] we set:
$S\,|mz\rangle = |(m+1)z\rangle$ with the exceptions

$$S\,|0a\rangle = 2^{-(1/p)}(|1c\rangle + |1d\rangle)\,,\; S\,|0b\rangle = 2^{-(1/p)}(-\,|1c\rangle + |1d\rangle)\,,$$
$$S\,|2c\rangle = \mathrm{e}^{i\phi}\,|3c\rangle\;,\; S\,|2d\rangle = \mathrm{e}^{i\psi}\,|3d\rangle\;,$$

$$S\ |3c\rangle = 2^{-(1/p)}(|4e\rangle + \mathrm{e}^{i\alpha}\ |4f\rangle)\,,\ S\ |3d\rangle = 2^{-(1/p)}(-\,|4e\rangle + \mathrm{e}^{i\beta}\ |4f\rangle)\,.$$

A particle (photon) which enters the a channel undergoes a unitary time evolution of the form (we just quote from [6] every time it is possible)

$$\begin{aligned}|0a\rangle\ &\mapsto 2^{-(1/p)}(|1c\rangle + |1d\rangle) \mapsto 2^{-(1/p)}(|2c\rangle + |2d\rangle) \mapsto 2^{-(1/p)}(\mathrm{e}^{i\phi}\ |3c\rangle + \mathrm{e}^{i\psi}\ |3d\rangle)\\ &\mapsto 2^{-(2/p)}\left[(\mathrm{e}^{i\phi} - \mathrm{e}^{i\psi})\,|4e\rangle + (\mathrm{e}^{i(\phi+\alpha)} + \mathrm{e}^{i(\psi+\beta)})\,|4f\rangle\right] \qquad (14)\end{aligned}$$

Set $\tau = (\phi - \psi)/2\,,\ \theta = (\alpha - \beta)/2$ and note that

$$\left|\mathrm{e}^{i\phi} - \mathrm{e}^{i\psi}\right| = 2|\sin(\tau)|\ ,\ \left|\mathrm{e}^{i(\phi+\alpha)} + \mathrm{e}^{i(\psi+\beta)}\right| = 2|\cos(\tau+\theta)|$$

S will be p-unitary if the p-norm of the term in (14) is 1, that is if

$$|\sin(\tau)|^p + |\cos(\tau+\theta)|^p = 2^{(2-p)}\,. \qquad (15)$$

For every $p \geq 1$ there are solutions of (15) in τ and θ.

References

[1] D. Amir: Characterizations of inner product spaces, Birkhäuser, Basel 1986.

[2] M. Baronti, C. Franchetti: *Minimal and polar projections onto hyperplanes in the spaces l_p and l_∞*, Riv.Mat.Univ.Parma, no. 16 **4** (1990), 331–342.

[3] M. Baronti, C. Franchetti: *Norm-one complemented hyperplanes in spaces with a Schauder basis*, Analysis Mathematica, **25** (1999), 171–178.

[4] P. Dirac: The Principles of Quantum Mechanics, London 1930.

[5] C. Franchetti, E. M. Semenov: *A Hilbert space characterization among function spaces*, Analysis Mathematica, **21** (1995), 85–93.

[6] R. B. Griffiths: Consistent Quantum Theory, Cambridge 2002.

[7] N. Kalton, B. Randrianantoanina: *Isometries on rearrangement-invariant spaces*, C.R. Acad.Sci. Paris Sér.I Math., **316** (1993), 351–355.

[8] R. Omnès: Understanding Quantum Mechanics, Princeton 1999.

[9] R. Shankar: Principles of Quantum Mechanics, Kluwer 1994.