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Behavior of bivariate interpolation operators at points of discontinuity of the first kind

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Abstract. We introduce an index of convergence for double sequences of real numbers. This index is used to describe the behaviour of some bivariate interpolation sequences at points of discontinuity of the first kind. We consider in particular the case of bivariate Lagrange and Shepard operators.

Keywords: Index of Convergence, Interpolation, Bivariate Lagrange Operators, Bivariate Shepard Operators

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Dedicated to the memory of V.B. Moscatelli

1 An index of convergence for double sequences

In this paper we consider a general index of convergence for multiple sequences of real numbers. This index turns out to be useful in the description of non converging sequences and in some cases it can give complete information of the behavior of these sequences. This is the case for example of some phenomena in interpolation theory where we have at our disposal some results on the failure of the convergence at points of discontinuity of the function but a complete behavior has not yet obtained. In particular we shall concentrate ourselves on double sequences of real numbers and on the Lagrange and Shepard operators in the bivariate case, where we shall be able to furnish complete description on their behavior at points of discontinuity of the first kind of a function in terms of the index of convergence.

We start with the definition of the index of convergence for multi-indexed sequences of real numbers, which generalizes in a natural way that of index of convergence for a sequence of real numbers given in [2].

In general, if $K \subset \mathbb{N}^m$, $m \geq 1$, the lower density and, respectively, the upper density of K can be defined by

$$\delta_-(K) := \liminf_{n\to+\infty} \frac{|K \cap \{1,\dots,n\}^m|}{n^m}, \quad \delta_+(K) := \limsup_{n\to+\infty} \frac{|K \cap \{1,\dots,n\}^m|}{n^m}.$$

In the case where $\delta_-(K) = \delta_+(K)$ the density of K is defined as follows

$$\delta(K) := \delta_-(K) = \delta_+(K).$$

The equalities $\delta_-(K) = 1 - \delta_+(K^c)$ and $\delta_+(K) = 1 - \delta_-(K^c)$ remain true and can be shown as in [2].

It follows the definition of index of convergence.

Definition 1. Let $(x_{n_1,\dots,n_m})_{n_1,\dots,n_m\geq 1}$ be a multi-indexed sequence of real numbers. If $L \in \mathbb{R}$, the *index of convergence of the sequence* $(x_{n_1,\dots,n_m})_{n_1,\dots,n_m\geq 1}$ *to* L is defined by

$$i\,(x_{n_1,\dots,n_m}; L) := 1 - \sup_{\varepsilon>0} \delta_+(\{\,(n_1,\dots,n_m) \in \mathbb{N}^m \mid x_{n_1,\dots,n_m} \notin]L-\varepsilon, L+\varepsilon[\,\}).$$

Moreover, we also set

$$i\,(x_{n_1,\dots,n_m}; +\infty) := 1 - \sup_{M\in\mathbb{R}} \delta_+\left(\{\,(n_1,\dots,n_m) \in \mathbb{N}^m \mid x_{n_1,\dots,n_m} \notin]M,+\infty[\,\}\right),$$
$$i\,(x_{n_1,\dots,n_m}; -\infty) := 1 - \sup_{M\in\mathbb{R}} \delta_+\left(\{\,(n_1,\dots,n_m) \in \mathbb{N}^m \mid x_{n_1,\dots,n_m} \notin]-\infty,M[\,\}\right).$$

Finally, we can also define the *index of convergence of* $(x_{n_1,\dots,n_m})_{n_1,\dots,n_m\geq 1}$ *to* a subset A of $\mathbb{R}$ as follows

$$i(x_{n_1,\dots,n_m}, A) := 1 - \sup_{\varepsilon>0} \delta_+(\{\,(n_1,\dots,n_m) \in \mathbb{N}^m \mid x_{n_1,\dots,n_m} \notin A + B_\varepsilon\,\}),$$

where $B_\varepsilon :=]-\varepsilon,\varepsilon[^m$.

In the case $m = 1$ we obtain exactly the index of convergence considered in [2]. Since all results can be easily extended from double sequences to multi-indexed sequences, for the sake of simplicity in the sequel we shall consider only the case $m = 2$ of double sequences of real numbers.

Remark 1. If necessary, we shall use the following explicit expressions of the index of convergence of a double sequence $(x_{n,m})_{n\geq 1}$

$$\begin{aligned} i(x_{n,m}; L) &= 1 - \sup_{\varepsilon>0} \delta_+\left(\{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \notin]L-\varepsilon, L+\varepsilon[\,\}\right) \\ &= 1 + \inf_{\varepsilon>0}\left(-\delta_+\left(\{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \notin]L-\varepsilon, L+\varepsilon[\,\}\right)\right) \\ &= \inf_{\varepsilon>0}\left(1 - \delta_+\left(\{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \notin]L-\varepsilon, L+\varepsilon[\,\}\right)\right) \\ &= \inf_{\varepsilon>0} \delta_-\left(\{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \in]L-\varepsilon, L+\varepsilon[\,\}\right), \end{aligned}$$

and if $A \subset \mathbb{R}$

$$i(x_{n,m}, A) = \inf_{\varepsilon>0} \delta_-(\{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \in A + B_\varepsilon\,\}).$$

Example 1. As a simple example, we can take $x_{n,m} := \cos n\pi/2 \cos m\pi/2$. It is easy to recognize that

$$i(x_{n,m}; 0) = \frac{3}{4}, \quad i(x_{n,m}; 1) = \frac{1}{8}, \quad i(x_{n,m}; -1) = \frac{1}{8}.$$

In the next proposition we point out some relations between the index of convergence and the density of a suitable converging subsequences.

Proposition 1. *Let $(x_{n,m})_{n,m\geq 1}$ be a double sequence of real numbers and $\sigma \in]0,1]$. Then $i(x_{n,m}, L) \geq \sigma$ if and only if there exists a subsequence $\left(x_{k(n,m)}\right)_{n,m\geq 1}$ converging to L such that*

$$\delta_-\left(\{\, k(n,m) \mid n,m \in \mathbb{N} \,\}\right) \geq \sigma.$$

Proof. $\Rightarrow$) For every $k \geq 1$, we consider the set $M_{1/k} := \{\, (n,m) \in \mathbb{N}^2 \mid |x_{n,m} - L| < 1/k \,\}$. From Remark 1, for every $k \in \mathbb{N}$ there exists $\tilde{\nu}_k$ such that

$$\frac{\left|M_{1/k} \cap \{\, 1,2,\ldots,j \,\}^2\right|}{j^2} \geq \sigma - \frac{1}{k}$$

whenever $j > \tilde{\nu}_k$. At this point we define recursively the sequence $(\nu_k)_{k\geq 1}$ by setting $\nu_1 = \tilde{\nu}_1$ and $\nu_k = \max\{\, \tilde{\nu}_k, \nu_{k-1} + 1 \,\}$. We have

$$\frac{\left|M_{1/k} \cap \{\, 1,2,\ldots,j \,\}^2\right|}{j^2} \geq \sigma - \frac{1}{k} \text{ for all } j > \nu_k. \tag{1}$$

Consider the set of integers

$$K = \bigcup_{k\geq 1} \left(M_{1/k} \cap \{\, 1,2,\ldots,\nu_{k+1} \,\}^2\right)$$

and the subsequence $\{\, x_{n,m} \mid (n,m) \in K \,\}$.

For every $\varepsilon > 0$, let $\ell \in \mathbb{N}$ such that $1/\ell \leq \varepsilon$. Then for every $(n,m) \in K$ satisfying $n,m > \nu_\ell$ we have $(n,m) \in \bigcup_{k\geq \ell} \left(M_{1/k} \cap \{\, 1,2,\ldots,\nu_{k+1} \,\}^2\right)$ and hence $|x_{n,m} - L| < \frac{1}{\ell} \leq \varepsilon$. This shows that the subsequence $\{\, x_{n,m} \mid (n,m) \in K \,\}$ converges to L.

On the other hand, for every $j > \nu_\ell$, there exists $\tilde{\ell} \geq \ell$ such that $\nu_{\tilde{\ell}} < j \leq \nu_{\tilde{\ell}+1}$ and thanks to (1) we have

$$\begin{aligned} \frac{|K \cap \{\, 1,2,\ldots,j \,\}^2|}{j^2} &\geq \frac{\left|M_{1/\tilde{\ell}} \cap \{\, 1,2,\ldots,\nu_{\tilde{\ell}+1} \,\}^2 \cap \{\, 1,2,\ldots,j \,\}^2\right|}{j^2} \\ &= \frac{\left|M_{1/\tilde{\ell}} \cap \{\, 1,2,\ldots,j \,\}^2\right|}{j^2} \geq \sigma - \frac{1}{\tilde{\ell}} \geq \sigma - \frac{1}{\ell} \geq \sigma - \varepsilon \end{aligned}$$

that is

$$\liminf_{n\to\infty} \frac{|K \cap \{\, 1,2,\ldots,j \,\}^2|}{j^2} \geq \sigma.$$

$\Leftarrow$) We suppose that there exists a subsequence $(x_{k(n,m)})_{n\geq 1}$ converging to L such that $\delta_-\left(\{\, k(n,m) \mid n,m \in \mathbb{N} \,\}\right) \geq \sigma$. For every $\varepsilon > 0$ there exists $\nu_\varepsilon \in \mathbb{N}$ such that $|x_{k(n,m)} - L| < \varepsilon$ whenever $n,m \geq \nu_\varepsilon$. Hence

$$\begin{aligned} \delta_-(\{\, (n,m) \in \mathbb{N}^2 \mid |x_{n,m} - L| < \varepsilon \,\}) &\geq \delta_-(\{\, (n,m) \in \mathbb{N}^2 \mid |x_{k(n,m)} - L| < \varepsilon \,\}) \\ &= \delta_-(\{\, k(n,m) \mid n,m \geq \nu_\varepsilon \,\}) \\ &= \delta_-(\{\, k(n,m)) \mid n,m \in \mathbb{N} \,\}) \geq \sigma \end{aligned}$$

and therefore, from Remark 1, we obtain $i(x_{n,m}, L) \geq \sigma$. $\boxed{QED}$

Proposition 2. *Let $(x_{n,m})_{n,m\geq 1}$ be a double sequence of real numbers and $(A_j)_{j\geq 1}$ a sequence of subsets of $\mathbb{R}$ such that $\overline{A_k}\cap\overline{A_j}=\emptyset$ for all $k\neq j$. Then*

$$0\leq\sum_{k=1}^{+\infty} i(x_{n,m},A_k)\leq 1.$$

In particular, if $(L_k)_{k\geq 1}$ is a sequence of distinct elements of $[-\infty,\infty]$ such that, for every $m\geq 1$

$$i(x_{n,m};L_k)=\alpha_k,$$

for some $\alpha_k\geq 0$, then

$$0\leq\sum_{k=1}^{+\infty}\alpha_k\leq 1.$$

Proof. Let $N\geq 1$; since $\overline{A_k}\cap\overline{A_j}=\emptyset$ whenever $k\neq j$, we can choose ε such that

$$(A_k+B_\varepsilon)\cap(A_j+B_\varepsilon)=\emptyset$$

for all $k,j=1,\dots,N$, $k\neq j$.

Now consider the set

$$M_\varepsilon^{(k)}:=\{\,(n,m)\in\mathbb{N}^2\mid x_{n,m}\in A_k+B_\varepsilon\,\}$$

and observe that $M_\varepsilon^{(k)}\cap M_\varepsilon^{(j)}=\emptyset$ whenever $k,j=1,\dots,N$, $k\neq j$. Then we can conclude that

$$\begin{aligned}0&\leq\sum_{k=1}^{N} i(x_{n,m},A_k)\leq\sum_{k=1}^{N}\delta_-(\{\,(n,m)\in\mathbb{N}^2\mid x_{n,m}\in A_k+B_\varepsilon\,\})\\&=\sum_{k=1}^{N}\liminf_{n\to\infty}\frac{|M_\varepsilon^{(k)}\cap\{\,1,\dots,n\,\}^2|}{n^2}\leq\liminf_{n\to\infty}\left(\sum_{k=1}^{N}\frac{|M_\varepsilon^{(k)}\cap\{\,1,\dots,n\,\}^2|}{n^2}\right)\\&=\liminf_{n\to\infty}\frac{\left|\bigcup_{k=1}^{N}M_\varepsilon^{(k)}\cap\{\,1,\dots,n\,\}^2\right|}{n^2}=\delta_-\left(\bigcup_{k=1}^{N}M_\varepsilon^{(k)}\right)\leq 1\end{aligned}$$

$\boxed{QED}$

Remark 2. Observe that if in the preceding proposition we have $\sum_{k=1}^{+\infty}\alpha_k=1$, then every subsequence $\big(x_{k(n,m)}\big)_{n,m\geq 1}$ of $(x_{n,m})_{n,m\geq 1}$ which converges to a limit L different from each L_k, $k\geq 1$, necessarily satisfies $\delta_-(\{\,k(n,m)\mid n,m\in\mathbb{N}\,\})=0$ and therefore $i(x_{n,m};L)=0$.

Indeed, if a subsequence $\big(x_{k(n,m)}\big)_{n,m\geq 1}$ of $(x_{n,m})_{n,m\geq 1}$ exists such that $\delta_-(\{\,k(n,m)\mid n,m\in\mathbb{N}\,\})=\alpha>0$, then by Proposition 1 we get $i(x_{n,m},L)\geq\alpha$ and therefore

$$i(x_{n,m},L)+\sum_{k=1}^{\infty} i(x_{n,m},L_k)\geq\alpha+\sum_{k=1}^{\infty}\alpha_k>1$$

which contradicts Proposition 2.

Proposition 3. *Let $(x_{n,m})_{n,m\geq 1}$ be a double sequence of real numbers and $(y_n)_{n\geq 1}$ a sequence of real numbers. If there exists a subsequence $(k(m))_{m\geq 1}$ such that $\lim_{m\to\infty}x_{n,k(m)}=y_n$ uniformly with respect to n and if $\delta\{\,k(m)\mid m\in\mathbb{N}\,\}=\alpha$ then*

$$i(x_{n,m};A)\geq\alpha\, i(y_n;A)$$

for every A subset of $\mathbb{R}$.

Proof. Let us consider $K_1, K_2 \subset \mathbb{N}$; since

$$\begin{aligned}
&\delta_-(\{\,(i,j) \in \mathbb{N}^2 \mid (i,j) \in K_1 \times K_2\,\}) = \delta_-(\{\,(i,j) \in \mathbb{N}^2 \mid i \in K_1, j \in K_2\,\})\\
&\quad= \liminf_{n\to\infty} \frac{|\{\,(i,j) \in \mathbb{N}^2 | i \in K_1, j \in K_2\,\} \cap \{\,1,\dots,n\,\}^2|}{n^2}\\
&\quad= \liminf_{n\to\infty} \frac{|\{\,i \in \mathbb{N} \mid i \in K_1\,\} \cap \{\,1,\dots,n\,\}|}{n} \frac{|\{\,j \in \mathbb{N} \mid j \in K_2\,\} \cap \{\,1,\dots,n\,\}|}{n}\\
&\quad\geq \liminf_{n\to\infty} \frac{|\{\,i \in \mathbb{N} \mid i \in K_1\,\} \cap \{\,1,\dots,n\,\}|}{n} \liminf_{n\to\infty} \frac{|\{\,j \in \mathbb{N} \mid j \in K_2\,\} \cap \{\,1,\dots,n\,\}|}{n}\\
&\quad= \delta_-(\{\,i \in \mathbb{N} \mid i \in K_1\,\})\delta_-(\{\,j \in \mathbb{N} \mid j \in K_2\,\}),
\end{aligned}$$

we have

$$\delta_-(K_1 \times K_2) \geq \delta_-(K_1)\delta_-(K_2). \tag{2}$$

For every $\varepsilon > 0$ there exists $\eta \in \mathbb{N}$ such that $|x_{n,k(m)} - y_n| < \varepsilon$ whenever $m \geq \eta$ and $n \in \mathbb{N}$. Then

$$\{\,(n,k(m)) \in \mathbb{N}^2 \mid m \geq \eta, y_n \in A + B_\varepsilon\,\} \subset \{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \in A + B_{2\varepsilon}\,\}$$

and consequently

$$\begin{aligned}
&\delta_-\left(\{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \in A + B_{2\varepsilon}\,\}\right) \geq \delta_-\left(\{\,(n,k(m)) \in \mathbb{N}^2 \mid m \geq \eta, y_n \in A + B_\varepsilon\,\}\right)\\
&\quad\geq \delta_-(\{\,n \in \mathbb{N} \mid y_n \in A + B_\varepsilon\,\})\delta_-(\{\,k(m) \mid m \geq \eta\,\})\\
&\quad= \alpha\delta_-(\{\,n \in \mathbb{N} \mid y_n \in A + B_\varepsilon\,\}) \geq \alpha i(y_n; A).
\end{aligned}$$

Taking the infimum with respect to ε we obtain the desired result. $\boxed{QED}$

Proposition 4. *Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ sequences of real numbers, let $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be injective differentiable functions with differentiable inverses. If $i(x_n; A) = |f^{-1}(A)|$ and $i(y_n; A) = |g^{-1}(A)|$ for every Peano-Jordan measurable set $A \subset [0,1]$, then*

$$i(x_n y_m, A) = |G^{-1}(A)|,$$

where $G(x,y) = g(x)f(y)$.

Proof. Firstly we prove that $i(x_n; I) = \delta(\{\,n \in \mathbb{N} \mid x_n \in I\,\})$ for every interval $I \subset \mathbb{R}$. Notice that for every interval $[a,b] \subset \mathbb{R}$ and for all $\varepsilon > 0$

$$\delta_-(\{\,n \in \mathbb{N} | \, x_n \in]a-\varepsilon, b+\varepsilon[\,\}) \geq \delta_-(\{\,n \in \mathbb{N} | \, x_n \in [a,b]\,\})$$

and then, taking the infimum over $\varepsilon > 0$,

$$i(x_n; [a,b]) \geq \delta_-(\{\,n \in \mathbb{N} | \, x_n \in [a,b]\,\}). \tag{3}$$

On the other hand, for every $\delta > 0$ we have that

$$\begin{aligned}
|f^{-1}([a+\delta, b-\delta])| &= \inf_{\varepsilon>0} \delta_-(\{\,n \in \mathbb{N} \mid x_n \in]a+\delta-\varepsilon, b-\delta+\varepsilon[\,\})\\
&\leq \delta_-(\{\,n \in \mathbb{N} \mid x_n \in [a,b]\,\}).
\end{aligned}$$

Since f^{-1} is continuous, the function $\delta \mapsto |f^{-1}([a+\delta, b-\delta])|$ is continuous at 0 and taking the limit as $\delta \to 0$ we have

$$i(x_n; [a,b]) = |f^{-1}([a,b])| \leq \delta_-(\{\,n \in \mathbb{N} | \, x_n \in [a,b]\,\}),$$

which jointly with (3) yields

$$i(x_n; I) = \delta_-(\{\,n \in \mathbb{N} \mid x_n \in I\,\})$$

for every interval I. Finally we have

$$\begin{aligned}&\delta_+(\{\, n \in \mathbb{N} \mid x_n \in I \,\}) = 1 - \delta_-(\{\, n \in \mathbb{N} \mid x_n \in I^c \,\}) = 1 - |f^{-1}(I^c)| \\ &\quad = |f^{-1}(I)| = \delta_-(\{\, n \in \mathbb{N} \mid x_n \in I \,\}),\end{aligned}$$

so

$$i(x_n; I) = |f^{-1}(I)| = \delta(\{\, n \in \mathbb{N} \mid x_n \in I \,\}).$$

Let I, J be real intervals, we have

$$\delta(\{\,(n,m) \in \mathbb{N} \mid (x_n, y_m) \in I \times J \,\}) = \delta(\{\, n \in \mathbb{N} \mid x_n \in I \,\})\delta(\{\, m \in \mathbb{N} \mid y_m \in J \,\}),$$

indeed

$$\begin{aligned}&\delta(\{\,(i,j) \in \mathbb{N}^2 \mid (x_i, y_j) \in I \times J \,\}) = \delta(\{\,(i,j) \in \mathbb{N}^2 \mid x_i \in I, y_j \in J \,\}) \\ &\quad = \lim_{n\to\infty} \frac{|\{\,(i,j) \in \mathbb{N}^2 | x_i \in I, y_j \in J \,\} \cap \{\,1,\dots,n\,\}^2|}{n^2} \\ &\quad = \lim_{n\to\infty} \frac{|\{\, i \in \mathbb{N} \mid x_i \in I \,\} \cap \{\,1,\dots,n\,\}|}{n} \frac{|\{\, j \in \mathbb{N} \mid y_j \in J \,\} \cap \{\,1,\dots,n\,\}|}{n} \\ &\quad = \lim_{n\to\infty} \frac{|\{\, i \in \mathbb{N} \mid x_i \in I \,\} \cap \{\,1,\dots,n\,\}|}{n} \lim_{n\to\infty} \frac{|\{\, j \in \mathbb{N} \mid y_j \in J \,\} \cap \{\,1,\dots,n\,\}|}{n} \\ &\quad = \delta(\{\, i \in \mathbb{N} \mid x_i \in I \,\})\delta(\{\, j \in \mathbb{N} \mid y_j \in J \,\}).\end{aligned}$$

Then we have

$$\delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in I \times J \,\}\right) = |f^{-1}(I)||g^{-1}(J)| = |(f,g)^{-1}(I \times J)|;$$

moreover, thank to the linearity of the limit, if Q is a pluri-interval of $\mathbb{R}^2$ we have

$$\delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in Q \,\}\right) =\mid (f,g)^{-1}(Q) \mid .$$

Now let $\tilde{A}$ be a Peano-Jordan measurable subset of $[0,1]^2$ and fix $\varepsilon > 0$. Since $F := (f,g)^{-1}$ is a diffeomorphism from $[0,1]^2$ into $[0,1]^2$, the subset $F(\tilde{A})$ is measurable and therefore there exist pluri-intervals Y_1, Y_2 such that $Y_1 \subset F(\tilde{A}) \subset Y_2$ and $|Y_2| - |Y_1| \le \varepsilon$. Let $Q_1 = F^{-1}(Y_1)$ and $Y_2 = F^{-1}(Q_2)$; these set are pluri-intervals and $Q_1 \subset \tilde{A} \subset Q_2$. We have

$$\begin{aligned}&\delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in Q_1 \,\}\right) \le \delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in \tilde{A} \,\}\right) \\ &\quad \le \delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in Q_2 \,\}\right),\end{aligned}$$

from which

$$|Y_1| = |F(Q_1)| \le \delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in \tilde{A} \,\}\right) \le |F(Q_2)| = |Y_2|.$$

On the other hand

$$|Y_1| \le |F(\tilde{A})| \le |Y_2|$$

and therefore

$$\left|\delta\left(\{(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in \tilde{A}\}\right) - \mid F(\tilde{A}) \mid\right| \le\mid Y_2 \mid - \mid Y_1 \mid\le \varepsilon$$

and we get

$$\delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in \tilde{A} \,\}\right) = |F(\tilde{A})|.$$

Finally we consider a Peano-Jordan measurable set $A \subset [0,1]$ and the function $P : \mathbb{R}^2 \to \mathbb{R}$, $P(x,y) = xy$. We have

$$\begin{aligned}\{\,(n,m) \in \mathbb{N}^2 \mid x_n y_m \in A\,\} &= \{\,(n,m) \in \mathbb{N}^2 \mid P(x_n, y_m) \in A\,\}\\ &= \{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in P^{-1}(A)\,\}\end{aligned}$$

and hence

$$\begin{aligned}\delta(\{\,(n,m) \in \mathbb{N}^2 \mid x_n y_m \in A\,\}) &= \delta\left(\{\,(n,m) \in \mathbb{N}^2 \mid (x_n, y_m) \in P^{-1}(A)\,\}\right)\\ &= |(f,g)^{-1}P^{-1}(A)| = |(f\cdot g)^{-1}(A)|\end{aligned}$$

where $(f\cdot g)(x,y) = G(x,y) = f(x)g(y)$. Therefore for every Peano-Jordan measurable set $A \subset [0,1]$ we have

$$\begin{aligned}i(x_n y_m; A) &= \inf_{\varepsilon>0} \delta(\{\,(n,m) \in \mathbb{N}^2 | \, x_n y_m \in A + B_\varepsilon\,\}) = \inf_{\varepsilon>0} |G^{-1}(A + B_\varepsilon)|\\ &\geq |G^{-1}(A)|. \end{aligned} \tag{4}$$

To prove the converse inequality, we argue by contradiction and suppose that $\underline{i(x_n y_m; A) > |G^{-1}(A)|}$. So there exists $\delta > 0$ such that $i(x_n y_m; A) = |G^{-1}(A + B_\delta)|$. Notice that $\overline{A} \cap (A + B_{\delta/2})^c = \emptyset$ and, by Proposition 2, we get

$$i(x_n y_m; A) + i(x_n y_m; (A + B_{\delta/2})^c) \leq 1.$$

Since $G^{-1}([0,1]^2) \subset [0,1]^2$, by (4) we have

$$|G^{-1}(A + B_\delta)| \leq 1 - |G^{-1}((A + B_{\delta/2})^c)| = |G^{-1}(A + B_{\delta/2})|.$$

This leads to a contradiction since the map $\delta > 0 \to |G^{-1}(A + B_\delta)|$ is monotone increasing. Then our claim is achieved. $\boxed{QED}$

Example 2. As a further example, let $\alpha, \gamma \in [0,1)$ be irrational, $\beta, \delta \in [0,1)$ and consider

$$x_{n,m} := (n\alpha + \beta - [n\alpha + \beta])(m\gamma + \delta - [m\gamma + \delta])$$

where $[x]$ denotes the integer part of x.

We already know that (see [2, Example 1.4)-ii)])

$$i\,(n\alpha + \beta - [n\alpha + \beta]; A) = |A|$$

for every Peano-Jordan measurable set $A \subset [0,1[$, where $|\cdot|$ denotes the Peano-Jordan measure.

Now, applying Proposition 4 with the identity function in place of f and g, we get

$$i(x_{n,m}; A) = \delta\left(\left\{\,(n,m) \in \mathbb{N}^2 \mid x_{n,m} \in A\,\right\}\right) = |G^{-1}(A)|$$

for every Peano-Jordan measurable set $A \subset [0,1[$, where $G(x,y) = x\,y$.

2 Bivariate Lagrange operators on discontinuous functions

We begin by considering the univariate Lagrange operators $(L_n)_{n\geq 1}$ at the Chebyshev nodes of second type, which are defined by

$$L_n f(x) = \sum_{k=1}^{n} \ell_{n,k}(x) f(x_{n,k}),$$

where f is a suitable function from $[-1,1]$ to $\mathbb{R}$,

$$x_{n,k} = \cos\theta_{n,k}, \qquad \theta_{n,k} = \frac{k-1}{n-1}\pi \qquad k = 1, \dots, n,$$

are the Chebyshev nodes of second type and

$$\ell_{n,k}(x) = \prod_{i \neq k} \frac{x - x_{n,i}}{x_{n,k} - x_{n,i}}$$

are the corresponding fundamental polynomials.

Setting $x = \cos\theta$, with $\theta \in [0,\pi]$, the polynomials $\ell_{n,k}$ can be rewritten as follows

$$\ell_{n,k}(\cos\theta) = \frac{(-1)^k}{(n-1)(1+\delta_{k,1}+\delta_{k,n})} \frac{\sin((n-1)\theta)\sin\theta}{\cos\theta - \cos\theta_{n,k}},$$

where $\delta_{i,j}$ denotes the Kronecker symbol, that is

$$\delta_{i,j} := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Our first aim is to study the behavior of the sequence of Lagrange operators for a particular class of functions having a finite number of points of discontinuity of the first kind. This will simplify the subsequent discussion on the bivariate case.

We consider the function $h_{x_0,d} : [-1,1] \to \mathbb{R}$ defined by

$$h_{x_0,d}(x) := \begin{cases} 0, & x < x_0, \\ d, & x = x_0, \\ 1, & x > x_0, \end{cases} \qquad x \in [-1,1], \tag{5}$$

where d is a fixed real number.

We also need to define the function $g :]0,1[\mapsto \mathbb{R}$ by setting

$$g(x) := \frac{\sin(\pi x)}{\pi} J(1,x), \qquad \text{if } x \in]0,1[, \tag{6}$$

where $J(s,a)$ denotes the Lerch zeta function

$$J(s,a) := \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+a)^s}, \qquad a \in]0,1], \quad \Re[s] > 0.$$

The following result describes the behavior of Lagrange operators at the point x_0 in terms of the index of convergence defined in [2] and corresponding to the case $m = 1$ in Definition 1.

Theorem 1. *Let $x_0 = \cos\theta_0 \in]-1,1[$ and consider the functions $h := h_{x_0,d}$ defined by* (5)*. Then, the sequence of functions $(L_n h)_{n\geq 1}$ converges uniformly to h on every compact subsets of $[-1,1] \setminus \{x_0\}$.*

As regards the behaviour of the sequence $(L_n h(x_0))_{n\geq 1}$ we have the following cases.

i) If $\frac{\theta_0}{\pi} = \frac{p}{q}$ with $p,q \in \mathbb{N}$, $q \neq 0$ and $GCD(p,q) = 1$, then

$$i\,(L_n h(x_0); d) = \frac{1}{q}, \qquad i\left(L_n h(x_0); g\left(\frac{m}{q}\right)\right) = \frac{1}{q}, \quad m = 1, \dots, q-1.$$

ii) If $\frac{\theta_0}{\pi}$ is irrational and if $A \subset \mathbb{R}$ is a Peano-Jordan measurable set, then

$$i\,(L_n h(x_0); A) = |g^{-1}(A)|,$$

where $|\cdot|$ denotes the Peano-Jordan measure.

Proof. Let $a = \cos\theta_1 \in [-1, x_0[$ and $x = \cos\theta \in [-1, a]$; for sufficiently large $n \geq 1$ there exists k_0 such that $0 \leq \theta_{n,k_0} \leq \theta_0 < \theta_{n,k_0+1} < \theta_1 \leq \theta \leq \pi$ and therefore

$$0 < \cos\theta_0 - \cos\theta_1 \leq \cos\theta_{n,k_0} - \cos\theta.$$

We have $L_n h(\cos\theta) = \sum_{k=1}^{k_0-1} \ell_{n,k}(\cos\theta) + d\ell_{n,k_0}(\cos\theta)$ if $\theta_{n,k_0} = \theta_0$, and $L_n h(\cos\theta) = \sum_{k=1}^{k_0} \ell_{n,k}(\cos\theta)$ if $\theta_{n,k_0} < \theta_0$; hence

$$\begin{aligned} &L_n h(\cos\theta) \\ &= \sum_{k=1}^{k_0} \frac{(-1)^k}{(n-1)(1+\delta_{k,1})} \frac{\sin((n-1)\theta)\sin\theta}{\cos\theta - \cos\theta_{n,k}} + (d-1)\chi_{\{\theta_{n,k_0}\}}(\theta_0)\ell_{n,k_0}(\cos\theta) \\ &= \frac{\sin((n-1)\theta)\sin\theta}{2(n-1)(\cos\theta - 1)} + \sum_{k=1}^{k_0} \frac{(-1)^{k-1}}{n-1} \frac{\sin((n-1)\theta)\sin\theta}{\cos\theta_{n,k} - \cos\theta} \\ &\quad + (d-1)\chi_{\{\theta_{n,k_0}\}}(\theta_0)\ell_{n,k_0}(\cos\theta). \end{aligned}$$

The function $t \to \frac{1}{\cos t - \cos\theta}$ is positive and monotone increasing on the interval $[0, \theta[$; since $0 < \theta_{n,k} < \theta_{n,k+1} < \theta$ for every $1 \leq k \leq k_0$, we have

$$\begin{aligned} |L_n h(x)| &= |L_n(h)(\cos\theta)| \\ &\leq \frac{1}{2(n-1)(1-\cos\theta)} + \left| \frac{\sin((n-1)\theta)\sin\theta}{n-1} \frac{1}{\cos\theta_{n,k_0} - \cos\theta} \right| \\ &\quad + |d-1| \left| \frac{\sin((n-1)\theta)\sin\theta}{n-1} \frac{1}{\cos\theta_{n,k_0} - \cos\theta} \right| \\ &\leq \frac{1}{2(n-1)(1-\cos\theta_1)} + \frac{1+|d-1|}{n-1} \frac{1}{\cos\theta_0 - \cos\theta_1}. \end{aligned}$$

It follows that $(L_n h)_{n\geq 1}$ converges uniformly to h in $[-1, a]$.

Now let $b = \cos\theta_2 \in]x_0, 1[$ and $x = \cos\theta \in [b, 1]$. For sufficiently large $n \geq 1$ there exists k_0 such that $0 \leq \theta \leq \theta_2 < \theta_{n,k_0} \leq \theta_0 < \theta_{n,k_0+1} \leq 2\pi$ and consequently

$$0 < \cos\theta_2 - \cos\theta_0 \leq \cos\theta - \cos\theta_{n,k_0+1}.$$

Then

$$\begin{aligned} |1 - L_n h(x)| &= |1 - L_n h(\cos\theta)| = \left| \sum_{k=1}^{n} \ell_{n,k}(\cos\theta) - \sum_{k=1}^{k_0} \ell_{n,k}(\cos\theta) h(\cos\theta_{n,k}) \right| \\ &= \left| \frac{(-1)^{n+1}}{2(n-1)} \frac{\sin((n-1)\theta)\sin\theta}{\cos\theta + 1} \right. \\ &\quad \left. + \sum_{k=k_0+1}^{n} \frac{(-1)^k}{n-1} \frac{\sin((n-1)\theta)\sin\theta}{\cos\theta - \cos\theta_{n,k}} - (d-1)\chi_{\{\theta_{n,k_0}\}}(\theta_0)\ell_{n,k_0}(\cos\theta) \right| \\ &\leq \frac{1}{n-1} \left[\frac{1}{2(\cos\theta + 1)} + \left| \frac{1}{\cos\theta - \cos\theta_{n,k_0+1}} \right| + |d-1| \left| \frac{1}{\cos\theta - \cos\theta_{n,k_0}} \right| \right] \\ &\leq \frac{1}{2(n-1)(\cos\theta_2 + 1)} + \frac{1+|d-1|}{n-1} \frac{1}{\cos\theta_2 - \cos\theta_0}, \end{aligned}$$

since the function $t \to \frac{1}{\cos\theta - \cos t}$ is positive and monotone decreasing in $]\theta, \pi]$ and $\theta < \theta_{n,k-1} < \theta_{n,k} < \pi$ for every $k_0 + 1 \leq k \leq n$. So $(L_n h)_{n\geq 1}$ converges uniformly to h also in $[b, 1]$.

Now, we study the behavior of $(L_n h(x_0))_{n\geq 1}$.

We identify $x_0 = \cos\theta_0$, $\theta_0 \in [0,\pi]$. For sufficiently large $n \geq 1$ there exists k_0 such that $\theta_{n,k_0} \leq \theta_0 < \theta_{n,k_0+1}$. Let us denote $\sigma_n = \dfrac{n-1}{\pi}(\theta_0 - \theta_{n,k_0})$. From $\frac{k_0-1}{n-1}\pi \leq \theta_0 < \frac{k_0}{n-1}\pi$ we have that $0 \leq \sigma_n < 1$; then

$$n - 1 = \frac{\pi}{\theta_0}(\sigma_n + k_0 - 1)$$

and moreover

$$k_0 - 1 \leq \frac{n-1}{\pi}\theta_0 < k_0,$$

that is $k_0 - 1 = \left[\frac{n-1}{\pi}\theta_0\right]$ and

$$\sigma_n = \frac{n-1}{\pi}\theta_0 - \left[\frac{n-1}{\pi}\theta_0\right]. \tag{7}$$

If x_0 is a Chebyshev node, that is $\theta_0 = \theta_{n,k_0}$ and $\sigma_n = 0$, then

$$L_n h(\cos\theta_0) = d. \tag{8}$$

If x_0 is not a Chebyshev node we have $\theta_{n,k_0} < \theta_0$, $0 < \sigma_n < 1$ and

$$L_n h(\cos\theta_0) = \sum_{k=1}^{k_0} \ell_{n,k}(\cos\theta_0). \tag{9}$$

Let us consider the case where x_0 is not a Chebyshev node and observe that

$$\begin{aligned}
\sin((n-1)\theta_0) &= (-1)^{k_0-1}\sin((n-1)\theta_0 - (k_0-1)\pi)\\
&= (-1)^{k_0-1}\sin\left(\pi\left(\frac{n-1}{\pi}\theta_0 - (k_0-1)\right)\right)\\
&= (-1)^{k_0-1}\sin(\pi\sigma_n).
\end{aligned}$$

Then we can rewrite $L_n h(x_0)$ in the following way

$$\begin{aligned}
L_n h(x_0) &= \sum_{k=1}^{k_0} \frac{(-1)^{-k}}{(n-1)(1+\delta_{k,1})}\,\frac{\sin((n-1)\theta_0)\sin\theta_0}{\cos\theta_0 - \cos\theta_{n,k}}\\
&= \frac{\sin((n-1)\theta_0)\sin\theta_0}{2(n-1)(\cos\theta_0 - 1)} + \sum_{k=1}^{k_0}\frac{(-1)^{-k}}{(n-1)}\,\frac{\sin((n-1)\theta_0)\sin\theta_0}{\cos\theta_0 - \cos\theta_{n,k}}\\
&= \frac{\sin((n-1)\theta_0)\sin\theta_0}{2(n-1)(\cos\theta_0 - 1)} + \frac{\sin(\pi\sigma_n)}{n-1}\sum_{m=0}^{k_0-1}(-1)^m\frac{\sin\theta_0}{\cos\theta_{n,k_0-m} - \cos\theta_0}\\
&= \frac{\sin((n-1)\theta_0)\sin\theta_0}{2(n-1)(\cos\theta_0 - 1)} + \frac{\sin(\pi\sigma_n)}{\pi}\sum_{m=0}^{k_0-1}\frac{(-1)^m}{\sigma_n + m}\\
&\quad + \frac{\sin(\pi\sigma_n)}{n-1}\sum_{m=0}^{k_0-1}(-1)^m\left[\frac{\sin\theta_0}{\cos\theta_{n,k_0-m} - \cos\theta_0} - \frac{n-1}{\pi(\sigma_n+m)}\right]\\
&= \frac{\sin((n-1)\theta_0)\sin\theta_0}{2(n-1)(\cos\theta_0 - 1)} + \frac{\sin(\pi\sigma_n)}{\pi}\sum_{m=0}^{k_0-1}\frac{(-1)^m}{\sigma_n + m}\\
&\quad + \frac{\sin(\pi\sigma_n)}{n-1}\sum_{m=0}^{k_0-1}(-1)^m\left[\frac{\sin\theta_0}{\cos\theta_{n,k_0-m} - \cos\theta_0} - \frac{1}{\theta_0 - \theta_{n,k_0-m}}\right],
\end{aligned}$$

where

$$\theta_0 - \theta_{n,k_0-m} = \theta_0 - \frac{k_0-m-1}{n-1}\pi = \theta_0 - \theta_{n,k_0} + \frac{m}{n-1}\pi = \frac{\pi}{n-1}(\sigma_n + m).$$

If we consider the function

$$g_{\theta_0}(x) := \frac{\sin\theta_0}{\cos x - \cos\theta_0} - \frac{1}{\theta_0 - x}, \quad x \in [0, \theta_0[,$$

we can write

$$\begin{aligned} L_n h(x_0) =& \frac{\sin((n-1)\theta_0)\sin\theta_0}{2(n-1)(\cos\theta_0 - 1)} + \frac{\sin(\pi\sigma_n)}{\pi}\sum_{m=0}^{k_0-1}\frac{(-1)^m}{\sigma_n+m} \\ &+ \frac{\sin(\pi\sigma_n)}{n-1}\sum_{m=0}^{k_0-1}(-1)^m g_{\theta_0}(\theta_{n,k_0-m}). \end{aligned} \tag{10}$$

The function g_{θ_0} is monotone decreasing and bounded since $g_{\theta_0}(0) = \frac{\sin\theta_0}{1-\cos\theta_0} - \frac{1}{\theta_0}$ and

$$\lim_{x\to\theta_0^-} g_{\theta_0}(x) = \frac{1}{2}\cot(\theta_0).$$

For all $n \geq 1$ and $\sigma \in [0,1[$, consider the function $f_n : [0,1[\to \mathbb{R}$ defined by setting

$$f_n(\sigma) := \begin{cases} \dfrac{\sin((n-1)\theta_0)\sin\theta_0}{2(n-1)(\cos\theta_0 - 1)} + \dfrac{\sin(\pi\sigma)}{\pi}\displaystyle\sum_{m=0}^{k_0-1}\frac{(-1)^m}{\sigma+m} & \\ \qquad + \dfrac{\sin(\pi\sigma)}{n-1}\displaystyle\sum_{m=0}^{k_0-1}(-1)^m g_{\theta_0}(\theta_{n,k_0-m}), & \text{if } \sigma \in]0,1[, \\ d, & \text{if } \sigma = 0; \end{cases}$$

taking into account (8), (9) and (10) we have $L_n h(\cos\theta_0) = f_n(\sigma_n)$.

For all $\sigma \in]0,1[$

$$\begin{aligned} &|f_n(\sigma) - g(\sigma)| \\ &\leq \left|\frac{\sin(\pi\sigma)}{\pi}\sum_{m=k_0}^{\infty}\frac{(-1)^m}{\sigma+m}\right| + \frac{1}{2(n-1)(1-\cos\theta_0)} + \frac{\sin(\pi\sigma)}{n-1}\left(\frac{1}{\theta_0} + |g_{\theta_0}(\theta_{n,k_0})|\right) \\ &\leq \frac{\sin(\pi\sigma)}{\pi}\left|\frac{(-1)^{k_0}}{\sigma+k_0}\right| + \frac{1}{2(n-1)(1-\cos\theta_0)} + \frac{\sin(\pi\sigma)}{n-1}\left(\frac{1}{\theta_0} + |g_{\theta_0}(\theta_{n,k_0})|\right) \\ &\leq \frac{1}{\pi k_0} + \frac{1}{n-1}\left(\frac{1}{\theta_0} + |g_{\theta_0}(\theta_{n,k_0})|\right); \end{aligned}$$

the right-hand side is independent of $\sigma \in]0,1[$ and it converges to 0 as $n \to \infty$ since

$$\lim_{n\to\infty} g_{\theta_0}(\theta_{n,k_0}) = \lim_{x\to\theta_0^-} g_{\theta_0}(x) = \frac{1}{2}\cot(\theta_0) < \infty.$$

Then we can conclude that the sequence $(f_n)_{n\geq 1}$ converges uniformly on $[0,1[$ to the function $\tilde{g} : [0,1[\to \mathbb{R}$ defined as follows

$$\tilde{g}(x) := \begin{cases} g(x), & \text{if } x \in]0,1[, \\ d, & \text{if } x = 0. \end{cases}$$

Now, we will construct q subsequences $\left(L_{k_m(n)}h(x_0)\right)_{n\geq 1}$, $m = 0,\dots,q-1$, of $(L_n h(x_0))_{n\geq 1}$ with density $\frac{1}{q}$ such that

$$\lim_{n\to\infty} L_{k_m(n)}h(x_0) = \tilde{g}\left(\frac{m}{q}\right) \quad \text{for all } m = 0,\dots,q-1.$$

Fix $m = 0,\dots,q-1$; since $GCD(p,q) = 1$ we can set $k_m(n) := l + nq + 1$, where $l \in \{\,0,\dots,q-1\,\}$ is such that $lp \equiv m \mod q$, that is there exists $s \in \mathbb{Z}$ such that $lp = sq+m$.

So, consider $\left(L_{k_m(n)}h(x_0)\right)_{n\geq 1}$ and observe that for all $m = 0,\dots,q-1$, we have $\delta(\{\,k_m(n) \mid n \in \mathbb{N}\,\}) = \frac{1}{q}$. It follows, for all $n \geq 1$

$$\begin{aligned}
\sigma_{k_m(n)} &= (k_m(n)-1)\frac{p}{q} - \left[(k_m(n)-1)\frac{p}{q}\right] \\
&= (l+nq)\frac{p}{q} - \left[(l+nq)\frac{p}{q}\right] = \frac{sq+m+nqp}{q} - \left[\frac{sq+m+nqp}{q}\right] \\
&= s + np + \frac{m}{q} - \left[s + np + \frac{m}{q}\right] = \frac{m}{q}
\end{aligned}$$

since $s, np \in \mathbb{Z}$, while $0 \leq \frac{m}{q} < 1$. Then

$$\lim_{n\to\infty} L_{k_m(n)}h(x_0) = \lim_{n\to\infty} f_n\left(\sigma_{k_m(n)}\right) = \lim_{n\to\infty} f_n\left(\frac{m}{q}\right) = \tilde{g}\left(\frac{m}{q}\right).$$

Therefore, by [2, Proposition 1.6] we have that for all $m = 0,\dots,q-1$

$$i\left(L_n h(x_0), \tilde{g}\left(\frac{m}{q}\right)\right) \geq \frac{1}{q}.$$

Now, we have q different statistical limits with index $\frac{1}{q}$, so by [2, Proposition 1.7] it necessarily follows

$$i\left(L_n h(x_0); \tilde{g}\left(\frac{m}{q}\right)\right) = \frac{1}{q}.$$

This completes the proof of part i).

The case where $\frac{\theta_0}{\pi}$ is irrational is similar to the analogous case considered in the proof of [2, Theorem 2.1 ii)] $\boxed{QED}$

At this point, we extend Theorem 1 to a larger classes of functions, namely on the space $\mathcal{C}+H$ where $\mathcal{C}$ denotes the space of all $f \in C([-1,1])$ such that f is either monotone on $[-1,1]$ or f satisfies the Dini-Lipschitz condition $\omega(f,\delta) = o(|\log\delta|^{-1})$, and H is the linear space generated by

$$\{\,h_{x_0,d} \mid x_0 \in]-1,1[,\ d \in \mathbb{R}\,\}.$$

Observe that if $f \in \mathcal{C}+H$ there exists at most a finite number of points $x_1,\dots,x_N$ of discontinuity with finite left and right limits $f(x_i-0)$ and $f(x_i+0)$, $i = 1,\dots,N$.

Then we can state the following theorem.

Theorem 2. *Let $f \in \mathcal{C}+H$ with a finite number N of points of discontinuity of the first kind at $x_1,\dots,x_N \in]-1,1[$. For every $i = 1,\dots,N$ consider $\theta_i \in]0,\pi[$ such that $x_i = \cos\theta_i$, $d_i := f(x_i)$ and define the function*

$$g_i(x) := f(x_i-0) + (f(x_i+0) - f(x_i-0))g(x).$$

Then, the sequence $(L_n f)_{n\geq 1}$ converges uniformly to f on every compact subset of $]-1,1[\setminus\{\,x_1,\dots,x_N\,\}$.

Moreover for all $i = 1,\dots,N$ the sequence $(L_n f(x_i))_{n\geq 1}$ has the following behavior *i) if $\frac{\theta_i}{\pi} = \frac{p}{q}$ with $p, q \in \mathbb{N}$, $q \neq 0$ and $GCD(p,q) = 1$, then*

$$i\left(L_n h(x_i); d_i\right) = \frac{1}{q}, \qquad i\left(L_n h(x_i); g_i\left(\frac{m}{q}\right)\right) = \frac{1}{q}, \quad m = 1, \ldots, q-1.$$

ii) if $\frac{\theta_i}{\pi}$ is irrational and if $A \subset \mathbb{R}$ is a Peano-Jordan measurable set, then

$$i\left(L_n h(x_i); A\right) = |g_i^{-1}(A)|,$$

where $|\cdot|$ denotes the Peano-Jordan measure.

PROOF. We assume $x_1 < \cdots < x_N$. We can write $f = F + \sum_{k=1}^N c_k h_k$, where $F \in \mathcal{C}$ and $h_i := h_{x_i, \tilde{d}_i}$ for every $i = 1, \ldots, N$.

Since F is continuous we have

$$f(x_i + 0) - \sum_{k=1}^{i-1} c_k - c_i = F(x_i + 0) = F(x_i - 0) = f(x_i - 0) - \sum_{k=1}^{i-1} c_k,$$

from which

$$c_i = f(x_i + 0) - f(x_i - 0)$$

and

$$F(x_i) = f(x_i - 0) - \sum_{k=1}^{i-1} c_k. \tag{11}$$

Moreover

$$\begin{aligned} d_i = f(x_i) &= F(x_i) + \sum_{k=1}^{i-1} c_k h_k(x_i) + c_i \tilde{d}_i \\ &= F(x_i) + \sum_{k=1}^{i-1} c_k + \left(f(x_i + 0) - f(x_i - 0)\right) \tilde{d}_i \\ &= f(x_i - 0) + \left(f(x_i + 0) - f(x_i - 0)\right) \tilde{d}_i. \end{aligned}$$

and hence

$$\tilde{d}_i = \frac{d_i - f(x_i - 0)}{f(x_i + 0) - f(x_i - 0)}.$$

The first part of our statement is a trivial consequence of the linearity of Lagrange interpolation operators. Indeed $F \in \mathcal{C}$ and therefore $L_n F \to F$ uniformly in compact subsets of $]-1,1[$ (see e.g. [13, Theorem 3.2, p. 24] and [10] in the case in which F is monotone, while we refer to [12, Theorem 14.4, p. 335] in the case F satisfies the Dini-Lipschitz condition); moreover for every $k = 1, \ldots, N$, by Theorem 1 $L_n h_k \to h_k$ converges uniformly to h_k on compact subsets of $[-1,1] \setminus \{x_k\}$. Then $L_n f = L_n F + \sum_{k=1}^N c_k L_n h_k$ converges uniformly to f on compact subsets of $]-1,1[\setminus\{x_1, \ldots, x_N\}$.

Now we establish property i). We fix a point x_i of discontinuity and following the same line of the proof of Theorem 1 we construct the subsequences $(k_m(n))_{n \geq 1}$, $m = 0, \ldots, q-1$. Since

$$L_{k_m(n)} f(x_i) = L_{k_m(n)} F(x_i) + \sum_{\substack{k=1 \\ k \neq i}}^{N} c_k L_{k_m(n)} h_k(x_i) + c_i L_{k_m(n)} h_i(x_i)$$

and taking into account (11) and that $F \in \mathcal{C}$, from Theorem 1 the right-hand side converges to

$$\begin{aligned} F(x_i) + \sum_{k=1}^{i-1} c_k h_k(x_i) + c_i g_i\left(\frac{m}{q}\right) \\ &= f(x_i - 0) + (f(x_i + 0) - f(x_i - 0))\, g_i\left(\frac{m}{q}\right) \\ &= g_i\left(\frac{m}{q}\right) \end{aligned}$$

for $m = 0, \dots, q-1$.

Finally, we prove property ii). For every $i = 1, \dots, N$ we have

$$L_n f(x_i) = L_n F(x_i) + \sum_{\substack{k=1 \\ k\neq i}}^{N} c_k L_n h_k(x_i) + c_i L_n h_i(x_i).$$

For the sake of simplicity let us denote

$$y_n := L_n f(x_i), \quad z_n := L_n F(x_i) + \sum_{\substack{k=1 \\ k\neq i}}^{N} c_k L_n h_k(x_i), \quad x_n := c_i L_n h_i(x_i)$$

(thus $y_n = z_n + x_n$) and

$$z := F(x_i) + \sum_{k=1}^{i-1} c_k h_k(x_i) = f(x_i - 0)$$

(see (11)).

Since (11) and that $F \in \mathcal{C}$ we can apply [13, Theorem 3.2, p. 24] (or [12, Theorem 14.4, p. 335]) and from Theorem 1 we obtain $z_n \to z$ and moreover $i(c_i^{-1}x_n; A) = |g^{-1}(A)|$ for every bounded Peano-Jordan measurable set $A \subset \mathbb{R}$. Hence $i(x_n; A) = |g^{-1}(c_i^{-1}A)|$, that is

$$|g^{-1}(c_i^{-1}A)| = \inf_{\varepsilon>0} \delta_-(\{\, n \in \mathbb{N} \mid x_n \in A + B_\varepsilon \,\}).$$

Fix $\varepsilon > 0$; if $x_n \in A + B_\varepsilon$, from the equality $x_n = y_n - z_n$ we get

$$y_n \in A + B_\varepsilon + z_n = A + B_\varepsilon + z + z_n - z.$$

Now, let $\nu \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ for all $n \geq \nu$, then for every $n \geq \nu$ we have $z_n - z \in B_\varepsilon$ and consequently $y_n \in A + B_{2\varepsilon} + z$. Therefore

$$\{\, n \geq \nu \mid x_n \in A + B_\varepsilon \,\} \subset \{\, n \geq \nu \mid y_n \in A + B_{2\varepsilon} + z \,\},$$

that is

$$\delta_-(\{\, n \in \mathbb{N} \mid x_n \in A + B_\varepsilon \,\}) \leq \delta_-(\{\, n \in \mathbb{N} \mid y_n \in A + B_{2\varepsilon} + z \,\}). \tag{12}$$

On the other hand, if $y_n \in A + B_{2\varepsilon} + z$, then $x_n = y_n - z_n \in A + B_{2\varepsilon} + z - z_n$. In this case for every $n \geq \nu$, we have $z - z_n \in B_\varepsilon$ and therefore $x_n \in A + B_{3\varepsilon}$; hence

$$\delta_-(\{\, n \in \mathbb{N} \mid x_n \in A + B_{3\varepsilon} \,\}) \geq \delta_-\{\, n \in \mathbb{N} \mid y_n \in A + B_{2\varepsilon} + z \,\}. \tag{13}$$

Taking the infimum over $\varepsilon > 0$ in (12) and (13) we can conclude that $i(x_n, A) \le i(y_n, A+z) \le i(x_n, A)$ which yields

$$i(y_n, A+z) = i(x_n, A) = |g^{-1}(c_i^{-1}A)|.$$

We conclude that $i(y_n, A) = |g^{-1}(c_i^{-1}(A-z))| = \left|g^{-1}\left(\frac{A-f(x_i-0)}{f(x_i+0)-f(x_i-0)}\right)\right| = |g_i^{-1}(A)|$ for every Peano-Jordan measurable set $A \subset \mathbb{R}$. $\boxed{QED}$

Now, we can consider the bivariate Lagrange interpolation polynomials $(L_{n,m})_{n,m\ge 1}$ on the Chebyshev nodes of second kind plus the endpoints ± 1 defined by

$$L_{n,m}(f)(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} \omega_{n,i}(x)\omega_{m,j}(j) f(x_{n,i}, y_{m,j}) \tag{14}$$

where f is a suitable function defined on $[-1,1]^2$ and

$$x_{n,i} = \cos\frac{i-1}{n-1}\pi, \quad i = 1,\dots,n.$$

Moreover, setting $x = \cos\theta$,

$$\omega_{n,i}(x) = \frac{(-1)^i}{(1+\delta_{i,1}+\delta_{i,n}(n-1)}\frac{\sin\theta\sin((n-1)\theta)}{x - x_{n,i}}.$$

Consider $z_0 = (x_0, y_0) \in]-1,1[\times]-1,1[$ and define the following function $h_{z_0}:\ [0,1]\times[0,1] \to \mathbb{R}$,

$$h_{z_0}(x,y) := \begin{cases} 1, & (x,y) \in [x_0,1]\times[y_0,1], \\ 0, & (x,y) \in [-1,1]\times[-1,1]\setminus[x_0,1]\times[y_0,1]. \end{cases} \tag{15}$$

In order to state the convergence properties of the sequence $(L_{n,m}h_{z_0})_{n\ge1}$, we consider the function $G:]0,1[\times]0,1[\to\mathbb{R}$ defined by

$$G(x,y) := g(x)g(y),$$

where $g:]0,1[\to\mathbb{R}$ is the function defined in (6).

In the following result we describe the behavior of the bivariate Lagrange polynomials evaluated at the function h_{z_0}, using the index of convergence for double sequences of real numbers.

Theorem 3. *Let $z_0 = (x_0,y_0) = (\cos\theta_0, \cos\gamma_0) \in]-1,1[\times]-1,1[$ and $h := h_{z_0}$ be defined by* (15)*. Then the sequence $(L_{n,m}h)_{n\ge1}$ converges uniformly to h on every compact subset of $[-1,1]\times[-1,1]\setminus Q$, where $Q := ([x_0,1]\times\{\,y_0\,\}) \cup (\{\,x_0\,\}\times[y_0,1])$.*

As regards the behaviour of the sequence $(L_{n,m}h(x,y))_{n,m\ge1}$ where $(x,y)\in Q$, we have:

1) if $x = x_0$ and $y \in]y_0,1]$, we have to consider the following cases:

i) if $\frac{\theta_0}{\pi} = \frac{p_1}{q_1}$ with $p_1, q_1 \in \mathbb{N}$, $q_1 \ne 0$, $GCD(p_1,q_1) = 1$, then

$$i\left(L_{n,m}h(x_0,y); g\left(\frac{m_1}{q_1}\right)\right) = \frac{1}{q_1}, \quad m_1 = 0,\dots,q_1-1;$$

ii) if $\frac{\theta_0}{\pi}$ is irrational, we have

$$i\left(L_{n,m}h(x_0,y); A\right) = |g^{-1}(A)|$$

for every Peano-Jordan measurable set $A \subset \mathbb{R}$.

2) If $x \in]x_0, 1]$ and $y = y_0$, we have to consider the following cases:

i) if $\frac{\gamma_0}{\pi} = \frac{p_2}{q_2}$ with $p_2, q_2 \in \mathbb{N}$, $q_2 \neq 0$, $GCD(p_2, q_2) = 1$, then

$$i\left(L_{n,m}h(x, y_0); g\left(\frac{m_2}{q_2}\right)\right) = \frac{1}{q_2}, \quad m_2 = 0, \dots, q_2 - 1;$$

ii) if $\frac{\gamma_0}{\pi}$ is irrational, we have

$$i\left(L_{n,m}h(x, y_0); A\right) = |g^{-1}(A)|$$

for every Peano-Jordan measurable set $A \subset \mathbb{R}$ if $s > 1$.

3) If $x = x_0$ and $y = y_0$, we have to consider the following cases:

i) if $\frac{\theta_0}{\pi} = \frac{p_1}{q_1}$ and $\frac{\gamma_0}{\pi} = \frac{p_2}{q_2}$ with $p_i, q_i \in \mathbb{N}$, $q_i \neq 0$, $GCD(p_i, q_i) = 1$, $i = 1, 2$, then

$$i\left(L_{n,m}h(x_0, y_0); g\left(\frac{m_1}{q_1}\right) g\left(\frac{m_2}{q_2}\right)\right) = \frac{1}{q_1 q_2}$$

for $m_1 = 0, \dots, q_1 - 1$ and $m_2 = 0, \dots, q_2 - 1$;

ii) if $\frac{\theta_0}{\pi} = \frac{p_1}{q_1}$ with $p_1, q_1 \in \mathbb{N}$, $q_1 \neq 0$, $GCD(p_1, q_1) = 1$ and $\frac{\gamma_0}{\pi}$ is irrational, then

$$i\left(L_{n,m}h(x_0, y_0); \left[0, g\left(\frac{j}{q_1}\right)\right]\right) \geq \frac{1}{q_1}, \qquad j = 0, \dots, q_1 - 1;$$

iii) if $\frac{\theta_0}{\pi}$ is irrational and $\frac{\gamma_0}{\pi} = \frac{p_2}{q_2}$ with $p_2, q_2 \in \mathbb{N}$, $q_2 \neq 0$, $GCD(p_2, q_2) = 1$, then

$$i\left(L_{n,m}h(x_0, y_0); \left[0, g\left(\frac{j}{q_2}\right)\right]\right) \geq \frac{1}{q_2}, \qquad j = 0, \dots, q_2 - 1;$$

iv) if $\frac{\theta_0}{\pi}$ and $\frac{\gamma_0}{\pi}$ are both irrational, then

$$i\left(L_{n,m}h(x_0, y_0); A\right) = |G^{-1}(A)|$$

for every Peano-Jordan measurable set $A \subset \mathbb{R}$.

Proof. We define the functions $h_1, h_2 : [-1, 1] \to \mathbb{R}$ as follows

$$h_1(t) := \begin{cases} 0 & \text{if } t < x_0, \\ 1 & \text{if } t \geq x_0, \end{cases} \qquad h_2(t) := \begin{cases} 0 & \text{if } t < y_0, \\ 1 & \text{if } t \geq y_0, \end{cases} \tag{16}$$

then we can write

$$h(x, y) = h_1(x) h_2(y),$$

and consequently we have

$$L_{n,m}h(x, y) = L_n h_1(x) L_m h_2(y). \tag{17}$$

Moreover, by Theorem 1, it follows that

$$\lim_{n\to\infty} L_n h_1 = h_1 \text{ uniformly on } [-1, 1] \setminus \{ x_0 \} \tag{18}$$

and

$$\lim_{m\to\infty} L_m h_2 = h_2 \text{ uniformly on } [-1, 1] \setminus \{ y_0 \}. \tag{19}$$

Let us consider a compact set $K \subset [-1,1]\times[-1,1]\setminus Q$; there exist $-1 \le a_1 < x_0 < a_2 \le 1$ and $-1 \le b_1 < y_0 < b_2 \le 1$ such that

$$K \subset [-1,1]\times[-1,b_1] \cup [-1,a_1]\times[-1,1] \cup [a_2,1]\times[b_2,1].$$

First let us consider $(x,y) \in [-1,1]\times[-1,b_1]$ and write $x = \cos\theta$ and $y = \cos\gamma$; then

$$|L_{n,m}h(x,y)| \le |L_n h_1(x)||L_m h_2(y)|. \tag{20}$$

For sufficiently large $n, m \ge 1$ there exist $k_0, \ell_0 \ge 1$ such that

$$x_{n,k_0+1} < x_0 \le x_{n,k_0}, \qquad y_{m,\ell_0+1} < y_0 \le y_{m,\ell_0},$$

then we can observe that

$$\begin{aligned}
|L_n h_1(x)| &\le \frac{|\sin(n-1)\theta\sin\theta|}{n-1}\left|\sum_{k=1}^{k_0}\frac{(-1)^k}{(1+\delta_{k,1})(\cos\theta-\cos\theta_{n,k})}\right| \\
&\le \frac{|\sin(n-1)\theta\sin\theta|}{2(n-1)|\cos\theta-\cos\theta_{n,1}|} + \frac{|\sin(n-1)\theta\sin\theta|}{(n-1)|\cos\theta-\cos\theta_{n,k_0}|} \\
&= \frac{|\omega_{n,1}(x)|}{2} + |\omega_{n,k_0}(x)| \qquad (21)
\end{aligned}$$

where, in the last inequality, we have used the fact that the following function

$$t \in [0,\pi] \to \frac{1}{\cos\theta - \cos t}$$

is monotone decreasing.
Let us observe that for $k = 1, \ldots, k_0$ we have

$$\begin{aligned}
|\omega_{n,k}(x)| &= \left|\frac{\sin(n-1)\theta\sin\theta}{(n-1)(\cos\theta-\cos\theta_{n,k})}\right| = \left|\frac{\sin\left((n-1)(\theta-\theta_{n,k})+(k-1)\pi\right)\sin\theta}{2(n-1)\sin\left(\frac{\theta-\theta_{n,k}}{2}\right)\sin\left(\frac{\theta+\theta_{n,k}}{2}\right)}\right| \\
&= \left|\frac{\sin\left((n-1)(\theta-\theta_{n,k})\right)\sin\theta}{2(n-1)\sin\left(\frac{\theta-\theta_{n,k}}{2}\right)\sin\left(\frac{\theta+\theta_{n,k}}{2}\right)}\right| \\
&= \frac{2\sin\theta}{\theta+\theta_{n,k}}\,\frac{|\sin\left((n-1)(\theta-\theta_{n,k})\right)|}{(n-1)|\theta-\theta_{n,k}|}\,\frac{\frac{|\theta-\theta_{n,k}|}{2}}{\left|\sin\left(\frac{\theta-\theta_{n,k}}{2}\right)\right|}\,\frac{\frac{\theta+\theta_{n,k}}{2}}{\left|\sin\left(\frac{\theta+\theta_{n,k}}{2}\right)\right|} \\
&\le \frac{2\sin\theta}{\theta}\,\frac{|\sin\left((n-1)(\theta-\theta_{n,k}|\right)|}{(n-1)|\theta-\theta_{n,k}|}\,\frac{\frac{|\theta-\theta_{n,k}|}{2}}{\sin\left|\frac{\theta-\theta_{n,k}}{2}\right|}\,\frac{\frac{\theta+\theta_{n,k}}{2}}{\left|\sin\left(\frac{\theta+\theta_{n,k}}{2}\right)\right|} \\
&\le 2C \qquad (22)
\end{aligned}$$

where the existence of the constant $C \ge 1$ is a consequence of the boundedness of the functions $\sin\alpha/\alpha$ on $[0, n\pi]$ for all $n \ge 1$ and $\alpha/\sin\alpha$ on $[a,b] \subset [0,\pi[$; in particular, observe that $\frac{\sin\alpha}{\alpha} \le 1$ while $\frac{\alpha}{\sin\alpha} \ge 1$ and $\left|\frac{\theta-\theta_{n,k}}{2}\right|, \left|\frac{\theta+\theta_{n,k}}{2}\right| \ne \pi$, since $0 < \theta_{n,k} < \pi$ for all k; notice also that (22) does not depend on the particular choice of $k \ge 1$ and it holds for all $x \in [-1,1]$. Then (21) becomes

$$|L_n h_1(x)| \le 3C \tag{23}$$

and this estimate is uniform with respect to $n \ge 1$ and $x \in [-1,1]$.

We can conclude that

$$|L_{n,m}h(x,y)| \le 3C|L_m h_2(y)|$$

where, by (19), the last term converges to 0 as $m \to \infty$. So we can conclude that

$$\lim_{n,m\to\infty} L_{n,m}h = h \qquad \text{uniformly in } [-1,1]\times[-1,b_1].$$

Arguing in a similar way, we can get the uniform convergence of $(L_{n,m}h)_{n,m\ge1}$ to h in $[-1,a_1]\times[-1,1]$.

If $(x,y)\in[a_2,1]\times[b_2,1]$, then

$$|L_{n,m}h(x,y)-1| = |L_n h_1(x)L_m h_2(y)-1|$$

which is uniformly convergent to 0 as $n,m\to\infty$. Therefore we can conclude that

$$\lim_{n,m\to\infty} L_{n,m}h = h \qquad \text{uniformly in } [a_2,1]\times[b_2,1].$$

We start with the proof of property 1). Let $y\in]y_0,1]$. From equation (17) we have

$$L_{n,m}h(x_0,y) = L_n h_1(x_0)L_m h_2(y),$$

and thank to (19) we have

$$\lim_{m\to\infty} L_{n,m}h(x,y_0) = L_n h_1(x_0).$$

From (23) we have that $L_n h_1(x_0)$ is bounded, then the previous limit is uniform with respect $n\in\mathbb{N}$. Therefore we can apply Proposition 3 with $k(m)=m$ (and consequently $\alpha=1$) and Theorem 3 with h replaced by h_1, and conclude the proof of 1).

The proof of property 2) is at all similar to that of property 1) interchanging the role of x and y.

Now we prove 3). From (17) we have

$$L_{n,m}h(x_0,y_0) = L_n h_1(x_0)L_m h_2(y_0).$$

Arguing as in Theorem 3 and taking into account that the value d is set to 1 and $g(0)=1$, we can consider $\left(L_{r_i(n)}h_1(x_0)\right)_{n\ge1}$, $i=0,\dots,q_1-1$ and $\left(L_{s_j(m)}h_2(y_0)\right)_{m\ge1}$, $j=0,\dots,q_2-1$, subsequences respectively of $(L_n h_1(x_0))_{n\ge1}$ and $(L_m h_2(y_0))_{m\ge1}$ with density respectively $1/q_1$ and $1/q_2$ such that

$$\lim_{n\to\infty} L_{r_i(n)}h_1(x_0) = g\left(\frac{i}{q_1}\right), \quad i=0,\dots,q_1-1,$$

and

$$\lim_{n\to\infty} L_{s_j(m)}h_1(x_0) = g\left(\frac{j}{q_2}\right), \quad j=0,\dots,q_2-1.$$

Therefore we can consider q_1q_2 subsequences of $(L_{n,m}h(x_0,y_0))_{n,m\ge1}$, let us say $\big(L_{r_i(n),s_j(m)}$ $h(x_0,y_0)\big)_{n,m\ge1}$ with $i=0,\dots,q_1-1$ and $j=0,\dots,q_2-1$ such that $\delta(\{\,(r_i(n),s_j(m))|n,m\in\mathbb{N}\,\}) = \frac{1}{q_1q_2}$ and

$$\lim_{n,m\to\infty} L_{r_i(n),s_j(m)}h(x_0,y_0) = G\left(\frac{i}{q_1},\frac{j}{q_2}\right)$$

where $i=0,\dots,q_1-1$, $j=0,\dots,q_2-1$. From Propositions 1 and 2 we have the result.

Now, let us prove 3) case ii). Suppose that $\frac{\theta_0}{\pi}=\frac{p_1}{q_1}$, $p_1,\,q_1\in\mathbb{N}$, $q_1\neq0$, $GCD(p_1,q_1)=1$ and $\frac{\gamma_0}{\pi}$ is irrational, from (17) we have

$$L_{n,m}h(x_0,y_0) = L_n h_1(x_0)L_m h_2(y_0).$$

We can consider $\left(L_{r_i(n)}h_1(x_0)\right)_{n\geq 1}$, $i = 0, \ldots, q_1 - 1$ subsequences of $(L_n h_1(x_0))_{n\geq 1}$ with density $1/q_1$ such that

$$\lim_{n\to\infty} L_{r_j(n),m}h(x_0, y_0) = g\left(\frac{i}{q_1}\right) L_m h_2(y_0), \quad i = 0, \ldots, q_1 - 1.$$

Applying Proposition 3 we have

$$i\left(L_{n,m}h(x_0, y_0); A\right) \geq \frac{1}{q_1} i\left(d_2 L_m h_2(y_0); A\right) = \frac{|(d_2 g)^{-1}(A)|}{q_1}. \tag{24}$$

and

$$\begin{aligned} i\left(L_{n,m}h(x_0, y_0); A\right) &\geq \frac{1}{q_1} i\left(g\left(\frac{i}{q_1}\right) L_m h_2(y_0); A\right) \\ &= \frac{|G_i^{-1}(A)|}{q_1}, \qquad i = 1, \ldots, q_1 - 1. \end{aligned} \tag{25}$$

where $G_i(t) = G\left(\frac{i}{q_1}, t\right) = g\left(\frac{i}{q_1}\right) g(t)$. Since the sum of indices can't exceed 1 in inequalities (24) and (25) we have equalities.

The proof of 3) case iii) is at all similar to the previous one interchanging the role of x and y.

Let us conclude the proof of our theorem, considering the case in which both $\frac{\theta_0}{\pi}$ and $\frac{\gamma_0}{\pi}$ are irrational. In this case, we have $i(L_n h_1(x_0); A) = |g^{-1}(A)|$ and $i(L_m h_2(y_0); A) = |g^{-1}(A)|$ for every A Peano-Jordan measurable set, apply Theorem 4 to the sequences $(L_n h_1(x))_{n\geq 1}$ and to $(L_m h_2(y_0))_{m\geq 1}$ and taking into account that $L_{n,m}h(x_0, y_0) = L_n h_1(x_0) L_m h_2(y_0)$ the claim easily follows. $\boxed{QED}$

3 Bivariate Shepard operators on discontinuous functions

Among all different kinds of bivariate Shepard operators (see e.g. [9]), for the sake of simplicity we concentrate our attention to the bivariate Shepard operators obtained as tensor product of univariate Shepard operators

$$S_{n,m,s}f(x, y) := \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{|x - x_i|^{-s}}{\sum_{k=0}^{n} |x - x_k|^{-s}} \frac{|y - y_j|^{-s}}{\sum_{k=0}^{m} |y - y_k|^{-s}} f(x_i, y_j), \tag{26}$$

where f is a suitable function defined in $[0, 1] \times [0, 1]$, $s \geq 1$, $n,\ m \geq 1$ and $((x_i, y_j))_{i,j}$ is the matrix $(n + 1) \times (m + 1)$ of equispaced nodes in $[0, 1] \times [0, 1]$, that is

$$x_i := \frac{i}{n},\ i = 0, \ldots, n, \qquad y_j := \frac{j}{m},\ j = 0, \ldots, m.$$

The aim of this section is the study of their behavior on a particular class of bivariate functions having suitable discontinuities defined as follows.

Consider $z_0 = (x_0, y_0) \in [0, 1] \times [0, 1]$ and define the following function $h_{z_0,d} : [0, 1] \times [0, 1] \to \mathbb{R}$,

$$h_{z_0,d}(x, y) := \begin{cases} 1, & (x, y) \in [0, x_0] \times [0, y_0], \\ 0, & \text{otherwise}, \end{cases} \tag{27}$$

In order to state the convergence properties of the sequence $(S_{n,m,s}h_{z_0,d})_{n\geq 1}$, for every $s>1$ we consider the function $g_s : [0,1[\to \mathbb{R}$ defined as follows

$$g_s(t) := \begin{cases} \dfrac{\zeta(s,t)}{\zeta(s,t)+\zeta(s,1-t)}, & t \in]0,1[, \\ 1, & t=0, \end{cases}$$

where ζ denotes the Hurwitz zeta function:

$$\zeta(s,a) := \sum_{n=0}^{+\infty} \frac{1}{(n+a)^s} \tag{28}$$

for all $s,a \in \mathbb{C}$ such that $\Re[s] > 1$ and $\Re[a] > 0$. The previous series is absolutely convergent and its sum can be extended to a meromorphic function defined for all $s \neq 1$.

We consider also $G_s : [0,1[\times[0,1[\to \mathbb{R}$ defined as follow

$$G_s(x,y) := g_s(x)g_s(y).$$

We have the following result.

Theorem 4. *Let $z_0 = (x_0,y_0) \in [0,1]\times[0,1]$ and $h := h_{z_0,d}$ be defined by (27). Then for every $s \geq 1$ the sequence $(S_{n,m,s}h)_{n\geq 1}$ converges uniformly to h on every compact subset of $[0,1]\times[0,1]\setminus Q$, where $Q := ([0,x_0]\times\{\,y_0\,\}) \cup (\{\,x_0\,\}\times[0,y_0])$.*

As regards the behavior of the sequence $(S_{n,m,s}h(x,y))_{n,m\geq 1}$ where $(x,y)\in Q$, we have:

1) if $x \in [0,x_0[$ and $y = y_0$, we have to consider the following cases:

i) if $y_0 = \frac{p_2}{q_2}$ with $p_2,q_2 \in \mathbb{N}$, $q_2 \neq 0$, $GCD(p_2,q_2) = 1$, then

$$i\left(S_{n,m,s}h(x,y_0); g_s\left(\frac{m_2}{q_2}\right)\right) = \frac{1}{q_2} \quad m_2 = 0,\dots,q_2-1$$

if $s > 1$; while

$$i\left(S_{n,m,s}h(x,y_0);1\right) = \frac{1}{q_2}, \quad i\left(S_{n,m,s}h(x,y_0);\frac{1}{2}\right) = 1-\frac{1}{q_2}$$

if $s = 1$;

ii) if y_0 is irrational, we have

$$i\left(S_{n,m,s}h(x,y_0);A\right) = |g_s^{-1}(A)|$$

for every Peano-Jordan measurable set $A \subset \mathbb{R}$ if $s > 1$; while

$$i\left(S_{n,m,s}h(x,y_0);\frac{1}{2}\right) = 1$$

if $s = 1$.

2) If $x = x_0$ and $y \in [0,y_0[$, we have to consider the following cases:

i) if $x_0 = \frac{p_1}{q_1}$ with $p_1,q_1 \in \mathbb{N}$, $q_1 \neq 0$, $GCD(p_1,q_1) = 1$, then

$$i\left(S_{n,m,s}h(x_0,y); g_s\left(\frac{m_1}{q_1}\right)\right) = \frac{1}{q_1} \quad m_1 = 0,\dots,q_1-1,$$

if $s > 1$; while

$$i\left(S_{n,m,s}h(x_0,y);1\right) = \frac{1}{q_1}, \quad i\left(S_{n,m,s}h(x_0,y);\frac{1}{2}\right) = 1-\frac{1}{q_1}$$

if $s = 1$;

ii) if x_0 is irrational, we have

$$i\left(S_{n,m,s}h(x_0,y);A\right)=|g_s^{-1}(A)|$$

for every Peano-Jordan measurable set $A\subset\mathbb{R}$ if $s>1$; while

$$i\left(S_{n,m,s}h(x_0,y);\frac{1}{2}\right)=1$$

if $s=1$.

3) If $x=x_0$ and $y=y_0$, we have to consider the following cases:

i) if $x_0=\frac{p_1}{q_1}$ and $y_0=\frac{p_2}{q_2}$ with $p_i,q_i\in\mathbb{N}$, $q_i\neq 0$, $GCD(p_i,q_i)=1$, $i=1,2$, then

$$i\left(S_{n,m,s}h(x_0,y_0);G_s\left(\frac{m_1}{q_1},\frac{m_2}{q_2}\right)\right)=\frac{1}{q_1q_2},$$

where $m_1=0,\dots,q_1-1$, $m_2=0,\dots q_2-1$, if $s>1$. While, if $s=1$,

$$i\left(S_{n,m,s}h(x_0,y_0);\frac{1}{2}\right)=\frac{1}{q_1q_2},\quad i\left(S_{n,m,s}h(x_0,y_0);\frac{1}{4}\right)=1-\frac{1}{q_1q_2};$$

ii) if $x_0=\frac{p_1}{q_1}$ with $p_1,q_1\in\mathbb{N}$, $q_1\neq 0$, $GCD(p_1,q_1)=1$ and y_0 is irrational, then

$$i\left(S_{n,m,s}h(x_0,y_0);\left[0,g\left(\frac{j}{q_1}\right)\right]\right)\geq\frac{1}{q_1},\qquad j=0,\dots,q_2-1$$

while

$$i\left(S_{n,m,s}h(x_0,y_0);\frac{1}{2}\right)=\frac{1}{q_1},\quad i\left(S_{n,m,s}h(x_0,y_0);\frac{1}{4}\right)=1-\frac{1}{q_1}$$

if $s=1$;

iii) if x_0 is irrational and $y_0=\frac{p_2}{q_2}$ with $p_2,q_2\in\mathbb{N}$, $q_2\neq 0$, $GCD(p_2,q_2)=1$, then

$$i\left(S_{n,m,s}h(x_0,y_0);\left[0,g\left(\frac{j}{q_2}\right)\right]\right)\geq\frac{1}{q_2},\qquad j=0,\dots,q_2-1;$$

while

$$i\left(S_{n,m,s}h(x_0,y_0);\frac{1}{2}\right)=\frac{1}{q_2},\quad i\left(S_{n,m,s}h(x_0,y_0);\frac{1}{4}\right)=1-\frac{1}{q_2}$$

if $s=1$;

iv) if x_0 and y_0 are both irrational, then

$$i\left(S_{n,m,s}h(x_0,y_0);A\right)=|G_s^{-1}(A)|$$

for every Peano-Jordan measurable set $A\subset\mathbb{R}$ if $s>1$, while

$$i\left(S_{n,m,s}h(x_0,y_0);\frac{1}{4}\right)=1$$

if $s=1$.

Proof. We define the functions $h_1, h_2 : [0,1] \to \mathbb{R}$ as follows

$$h_1(t) := \begin{cases} 1 & \text{if } t < x_0, \\ 0 & \text{if } t \geq x_0, \end{cases} \qquad h_2(t) := \begin{cases} 1 & \text{if } t < y_0, \\ 0 & \text{if } t \geq y_0, \end{cases} \tag{29}$$

then we can write

$$h(x,y) = h_1(x)h_2(y),$$

and consequently we have

$$S_{n,m}h(x,y) = S_n h_1(x) S_m h_2(y), \tag{30}$$

moreover using [2, Theorem 3.1], it follows that

$$\lim_{n\to\infty} S_n h_1 = h_1 \text{ uniformly on } [0,1] \setminus \{ x_0 \} \tag{31}$$

and

$$\lim_{m\to\infty} S_m h_2 = h_2 \text{ uniformly on } [0,1] \setminus \{ y_0 \}. \tag{32}$$

Let us consider a compact set $K \subset [0,1] \times [0,1] \setminus Q$; there exist $0 \leq a_1 < x_0 < a_2 \leq 1$ and $0 \leq b_1 < y_0 < b_2 \leq 1$ such that

$$K \subset [0,a_1] \times [0,b_1] \cup [a_2,1] \times [0,1] \cup [0,a_2] \times [b_2,1].$$

Firstly, let us consider $(x,y) \in [0,a_1] \times [0,b_1]$; then

$$|S_{n,m}h(x,y) - h(x,y)| = |S_{n,m}h(x,y) - 1|$$

which is uniformly convergent to 0 as $n,m \to \infty$ by (31) and (32). Therefore we can conclude that

$$\lim_{n,m\to\infty} S_{n,m}h = h \qquad \text{uniformly in } [0,a_1] \times [0,b_1].$$

Let us consider now $(x,y) \in [a_2,1] \times [0,1]$; for sufficiently large $n,m \geq 1$ there exist $k_0, \ell_0 \geq 1$ such that

$$\frac{k_0}{n} \leq x_0 < \frac{k_0+1}{n}, \qquad \frac{\ell_0}{m} \leq y_0 < \frac{\ell_0+1}{m}.$$

Notice that

$$|S_m h_2(y)| = \frac{\sum_{j=0}^{\ell_0} |y - \frac{j}{m}|^{-s}}{\sum_{j=0}^{m} |y - \frac{j}{m}|^{-s}} \leq \frac{\sum_{j=0}^{\ell_0} |y - \frac{j}{m}|^{-s}}{\sum_{j=0}^{\ell_0} |y - \frac{j}{m}|^{-s} + \sum_{j=\ell_0+1}^{m} |y - \frac{j}{m}|^{-s}} \leq 1 \tag{33}$$

and this estimate is independent of $y \in [0,1]$ and $n \geq 1$. Then

$$|S_{n,m}h(x,y)| \leq |S_n h_1(x)|$$

where, by (31), the last term converges uniformly to 0 as $n \to \infty$. So we can conclude that

$$\lim_{n,m\to\infty} S_{n,m}h = h \qquad \text{uniformly in } [a_2,1] \times [0,1].$$

Arguing similarly, exchanging the role between x and y, we get also that

$$\lim_{n,m\to\infty} S_{n,m}h = h \qquad \text{uniformly in } [0,a_2] \times [b_2,1].$$

The proof of the claims 1)-3) is at all similar to the one of Theorem 3, we have only to use the decomposition (30) and [2, Theorem 3.1] in place of Theorem 1.

In particular, in the case 3)–i), if $s = 1$, using [2, Theorem 3.1], we can consider two subsequences $(S_{k_1(n),1}h_1(x_0))_{n\geq 1}$ and $(S_{k_2(n),1}h_1(x_0))_{n\geq 1}$ of $(S_{n,1}h_1(x_0))_{n\geq 1}$, converging respectively to 1 and $\frac{1}{2}$, with density respectively $\frac{1}{q_1}$ and $1-\frac{1}{q_1}$. Then

$$\lim_{n\to\infty} S_{k_1(n),m,1}h(x_0,y_0) = S_m h_2(y_0),$$

$$\lim_{n\to\infty} S_{k_2(n),m,1}h(x_0,y_0) = \frac{1}{2}S_m h_2(y_0)$$

and the previous limits are uniform with respect to $m \geq 1$ by (33). So we can apply Proposition 3 and we get

$$i\left(S_{n,m,1}h(x_0,y_0);\frac{1}{2}\right) \geq \frac{1}{q_1} i\left(S_{m,1}h_2(y_0);\frac{1}{2}\right) = \frac{1}{q_1}$$

and

$$i\left(S_{n,m,1}h(x_0,y_0);\frac{1}{4}\right) \geq \left(1-\frac{1}{q_1}\right) i\left(\frac{1}{2}S_{m,1}h_2(y_0);\frac{1}{4}\right) = 1-\frac{1}{q_1}.$$

Finally, the previous inequalities become equalities by Proposition 2.

Arguing similarly the others cases of our claim can be proved. $\boxed{QED}$

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