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A note on $|N, p, q|_k$, $(1 \leq k \leq 2)$ summability of orthogonal series

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Abstract. In this paper we present some results on $|N, p, q|_k$, $(1 \leq k \leq 2)$ summability of orthogonal series.

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1 Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$. Then, let p denotes the sequence $\{p_n\}$. For two given sequences p and q, the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^{n} p_m q_{n-m} = \sum_{m=0}^{n} p_{n-m} q_m.$$

When $(p * q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{m=0}^{n} p_{n-m} q_m s_m.$$

The infinite series $\sum_{n=0}^{\infty} a_n$ is absolutely summable $(N, p, q)_k$ of order k, if for $k \geq 1$ the series

$$\sum_{n=0}^{\infty} n^{k-1} |t_n^{p,q} - t_{n-1}^{p,q}|^k$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|_k.$$

We note that for $k = 1$, $|N, p, q|_k$ summability is the same as $|N, p, q|$ summability introduced by Tanaka [3].

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \tag{1}$$

where $a_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.

We write (see [4])

$$R_n := (p * q)_n,\ R_n^j := \sum_{m=j}^{n} p_{n-m} q_m$$

and

$$R_n^{n+1} = 0,\ R_n^0 = R_n.$$

Also we put

$$P_n := (p * 1)_n = \sum_{m=0}^{n} p_m \quad \text{and} \quad Q_n := (1 * q)_n = \sum_{m=0}^{n} q_m.$$

Our main purpose of the present paper is to study the $|N, p, q|_k$ summability of the orthogonal series (1), for $1 \le k \le 2$, and to deduce as corollaries all results of Y. Okuyama [4].

Throughout this paper K denotes a positive constant that it may depends only on k, and be different in different relations.

2 Main Results

We prove the following theorem.

Theorem 2.1 *If for $1 \le k \le 2$ the series*

$$\sum_{n=0}^{\infty} \left\{ n^{2-\frac{2}{k}} \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is summable $|N, p, q|_k$ almost everywhere.

Proof. For the generalized Nörlund transform $t_n^{p,q}(x)$ of the partial sums of the orthogonal series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ we have that

$$\begin{aligned} t_n^{p,q}(x) &= \frac{1}{R_n} \sum_{m=0}^{n} p_{n-m} q_m \sum_{j=0}^{m} a_j \varphi_j(x) \\ &= \frac{1}{R_n} \sum_{j=0}^{n} a_j \varphi_j(x) \sum_{m=j}^{n} p_{n-m} q_m \\ &= \frac{1}{R_n} \sum_{j=0}^{n} R_n^j a_j \varphi_j(x) \end{aligned}$$

where $\sum_{j=0}^{m} a_j \varphi_j(x)$ are partial sums of order k of the series (1).

As in [4] page 163 one can find that

$$\triangle t_n^{p,q}(x) := t_n^{p,q}(x) - t_{n-1}^{p,q}(x) = \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right) a_j \varphi_j(x).$$

Using the Hölder's inequality and orthogonality to the latter equality, we have that

$$\int_a^b |\triangle t_n^{p,q}(x)|^k dx \le (b-a)^{1-\frac{k}{2}} \left(\int_a^b |t_n^{p,q}(x) - t_{n-1}^{p,q}(x)|^2 dx \right)^{\frac{k}{2}}$$
$$= (b-a)^{1-\frac{k}{2}} \left[\sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right]^{\frac{k}{2}}.$$

Hence, the series

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\triangle t_n^{p,q}(x)|^k dx \le (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} n^{k-1} \left[\sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right]^{\frac{k}{2}} \tag{2}$$

converges by the assumption. According to the Lemma of Beppo-Levi the proof of the theorem ends. QED

For $k = 1$ in theorem 2.1 we have the following result.

Corollary 2.2 [4] *If the series*

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is summable $|N,p,q|$ almost everywhere.

Let us prove now another two corollaries of the Theorem 2.1.

Corollary 2.3 *If for $1 \le k \le 2$ the series*

$$\sum_{n=0}^{\infty} \left(\frac{n^{1-\frac{1}{k}} p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{j=1}^{n} p_{n-j}^2 \left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 |a_j|^2 \right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is summable $|N,p|_k$ almost everywhere.

Proof. After some elementary calculations one can show that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = \frac{p_n}{P_n P_{n-1}} \left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right) p_{n-j}$$

for all $q_n = 1$, and the proof follows immediately from Theorem 2.1. QED

Remark 2.4 *We note that:*

1. *If $p_n = 1$ for all values of n then $|N,p|_k$ summability reduces to $|C,1|_k$ summability*
2. *If $k = 1$ and $p_n = 1/(n+1)$ then $|N,p|_k$ is equivalent to $|R, \log n, 1|$ summability.*

Corollary 2.5 *If for $1 \leq k \leq 2$ the series*

$$\sum_{n=0}^{\infty}\left(\frac{n^{1-\frac{1}{k}}q_n}{Q_nQ_{n-1}}\right)^k\left\{\sum_{j=1}^{n}Q_{j-1}^2|a_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|\overline{N},q|_k$ almost everywhere.

Proof. From the fact that

$$\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}=-\frac{q_nQ_{j-1}}{Q_nQ_{n-1}}$$

for all $p_n=1$, the proof follows immediately from Theorem 2.1. QED

Also, putting $k=1$ in Corollaries 2.3 and 2.5 we obtain
Corollary 2.6 [1] *If the series*

$$\sum_{n=0}^{\infty}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{j=1}^{n}p_{n-j}^2\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2|a_j|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|N,p|$ almost everywhere.
Corollary 2.7 [2] *If the series*

$$\sum_{n=0}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\left\{\sum_{j=1}^{n}Q_{j-1}^2|a_j|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|\overline{N},q|$ almost everywhere.
If we put

$$w^{(k)}(j):=\frac{1}{j^{\frac{2}{k}-1}}\sum_{n=j}^{\infty}n^{\frac{2}{k}}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2 \tag{3}$$

then the following theorem holds true.
Theorem 2.8 *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty}|a_n|^2\Omega^{\frac{2}{k}-1}(n)w^{(k)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty}a_n\varphi_n(x)\in|N,p,q|_k$ almost everywhere, where $w^{(k)}(n)$ is defined by (3).*

Proof. Applying Hölder's inequality to the inequality (2) we get that

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\triangle t_n^{p,q}(x)|^k dx \le$$

$$\le K \sum_{n=1}^{\infty} n^{k-1} \left[\sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right]^{\frac{k}{2}}$$

$$= K \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{\frac{2-k}{2}}} \left[n\Omega^{\frac{2}{k}-1}(n) \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right]^{\frac{k}{2}}$$

$$\le K \left(\sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))} \right)^{\frac{2-k}{2}} \left[\sum_{n=1}^{\infty} n\Omega^{\frac{2}{k}-1}(n) \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right]^{\frac{k}{2}}$$

$$\le K \left\{ \sum_{j=1}^{\infty} |a_j|^2 \sum_{n=j}^{\infty} n\Omega^{\frac{2}{k}-1}(n) \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 \right\}^{\frac{k}{2}}$$

$$\le K \left\{ \sum_{j=1}^{\infty} |a_j|^2 \left(\frac{\Omega(j)}{j} \right)^{\frac{2}{k}-1} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 \right\}^{\frac{k}{2}}$$

$$= K \left\{ \sum_{j=1}^{\infty} |a_j|^2 \Omega^{\frac{2}{k}-1}(j) w^{(k)}(j) \right\}^{\frac{k}{2}},$$

which is finite by assumption, and this completes the proof. QED

Finally, as a direct consequence of the theorem 2.8 is the following $(k = 1)$.

Corollary 2.9 [4] *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |a_n|^2 \Omega(n) w^{(1)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} a_n \varphi_n(x) \in |N, p, q|$ almost everywhere, where $w^{(1)}(n)$ is defined by*

$$w^{(1)}(j) := \frac{1}{j} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2.$$

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