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Groups with few non-normal cyclic subgroups

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Abstract. R. Brandl in [2] and H. Mousavi in [6] classified finite groups which have respectively just one or exactly two conjugacy classes of non-normal subgroups. In this paper we determine finite groups which have just one or exactly two conjugacy classes of non-normal cyclic subgropus. In particular, in a nilpotent group if all non-normal cyclic subgroups are conjugate, then any two non-normal subgroups are conjugate. In general, if a group has exactly two conjugacy classes of non-normal cyclic subgroups, there is no upper bound for the number of conjugacy classes of non-normal subgroups.

Keywords: conjugacy classes, cyclic subgroup

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1 Preliminaries

Notations
$\nu(G)$ will denote the number of conjugacy classes of non-normal subgroups of G.
$\nu_c(G)$ will denote the number of conjugacy classes of non-normal cyclic subgroups of G.
$G = [A]B$ will denote that G is the semidirect product of A and B, with A normal in G.
By $A \sim_G B$, with A and B subgroups of G, we mean that A and B are conjugate in G.
Let P be a p-group: then $\Omega_i(P) = \langle x \in P | x^{p^i} = 1\rangle$, $\mho_i(P) = \langle a^{p^i} | a \in P\rangle$.
All groups considered in this paper are finite.

As in [7] Propositions 2.2 and 2.6 one can prove the following

Proposition 1. *Let G be a group.*

(1) If N is a normal subgroup of G, then $\nu_c\left(\frac{G}{N}\right) \leq \nu_c(G)$

(2) If $G = A \times B$, then $\nu_c(G) \geq \nu_c(A)\nu_c(B) + \nu_c(A)\mu_c(B) + \mu_c(A)\nu_c(B)$, where $\mu_c(G)$ denotes the number of normal cyclic subgroups of G; if $(|A|, |B|) = 1$, then equality holds.

Proposition 2. *If P is a non-abelian p-group with $|P| = p^n$ and $\exp P = p^{n-1}$, then $\nu_c(P) \leq 2$.*

It is $\nu_c(P) = 1$ if and only if $P \simeq M_n(p) = \langle a, b | a^{p^{n-1}} = b^p = 1, [a,b] = a^{p^{n-2}} \rangle$ where $n \geq 4$ if $p = 2$. Moreover, $\nu(P) = 1$.

It is $\nu_c(P) = 2$ if and only if $p = 2$ and P is isomorphic to one of the following groups:

i. $D_n = \langle a, b | a^{2^{n-1}} = b^2 = 1, [a,b] = a^{-2} \rangle$ *where $n \geq 3$; the non-normal cyclic subgroups of D_n have order 2. It is $\nu(D_n) = 2n - 4$.*

ii. $S_n = \langle a, b | a^{2^{n-1}} = b^2 = 1, [a,b] = a^{-2+2^{n-2}} \rangle$ *where $n \geq 4$; the non-normal cyclic subgroups of S_n have order 2 or 4. It is $\nu(S_n) = 2n - 5$.*

iii. $Q_n = \langle a, b | a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, [a,b] = a^{-2} \rangle$ *where $n \geq 4$; the non-normal cyclic subgroups of Q_n have order 4. It is $\nu(Q_n) = 2n - 6$.*

The proof is a straightforward check on the groups Q_n, S_n and D_n, bearing in mind [2] and [7], Prop.2.5.

Proposition 3. *Let P be a p-group;*

(1) if there exists a subgroup N contained in $Z(P)$ such that $\nu\left(\frac{P}{N}\right) = 1$, then $[P : Z(P)] = p^2$ and $|P'| = p$;

(2) if $\nu\left(\frac{P}{Z(P)}\right) = 0$, then either $\frac{P}{Z(P)}$ is abelian or $\nu_c(P) \geq 3$.

Proof. (1) By [1] it is $\frac{P}{N} \simeq M_r(p)$, where $r \geq 3$ if p odd, $r \geq 4$ if $p = 2$; so $\frac{P}{N}$ has two cyclic maximal subgroups and therefore P has two abelian maximal subgroups. It follows $[P : Z(P)] = p^2$. Since P has an abelian maximal subgroup, one has $|P'||Z(P)| = p^{n-1}$, so $|P'| = p$.

(2) if $\frac{P}{Z(P)}$ is non-abelian, then

$$\frac{P}{Z(P)} = \langle \bar{x}, \bar{y} | \bar{x}^4 = 1, \bar{x}^2 = \bar{y}^2 = [\bar{x}, \bar{y}] \rangle \times \bar{E}$$

where $\bar{E}$ is an elementary abelian 2-group. The subgroups $\langle x \rangle$, $\langle y \rangle$ and $\langle xy \rangle$ represent three different conjugacy classes in P; hence, if $\nu_c(P) \leq 2$, we can assume $\langle x \rangle \lhd P$. Then $|x| \geq 2^3$; since $[x,y] \in \langle x \rangle$, it is $[x,y]^2 = [x^2, y] = 1$, so $[x,y] \in \langle x^4 \rangle \subseteq Z(P)$, a contradiction.

QED

A check on minimal non-abelian groups (see [5]) proves the following

Proposition 4. *Let G be a minimal non-abelian group; then*

(1) $\nu_c(G) = 1$ if and only if $\nu(G) = 1$;

(2) if G is non-nilpotent (so that $G = [Q]P \simeq G(q,p,n)$ with $|Q| = q^m$, $|P| = p^n$), one has $\nu_c(G) = 2$ if and only if $n = 1$, $m \geq 2$ and $p = \frac{q^m - 1}{q - 1}$;

(3) if G is a p-group, one has $\nu_c(G) = 2$ if and only if $\nu(G) = 2$; hence either $G = [C_n]C_4 = [\langle x \rangle] \langle y \rangle$ with $[x, y^2] = 1$ or $G = [C_4]C_2 \simeq D_3$.

Later we shall use the following Proposition on power-automorphisms. An automorphism ϕ of a group G is said to be a power-automorphism if $\phi(H) = H$ for every subgroup H of G.

Proposition 5. *([4], Hilfsatz 5) Let P be a p-group and $\alpha \neq 1$ a p'-power automorphism of P. Then P is abelian and for every $a \in P$ it is $\alpha(a) = a^k$ where $k \in Z$ does not depend on a.*

2 $\nu_c = 1$

Theorem 1. *Let G be a non-nilpotent group; one has $\nu_c(G) = 1$ if and only if $G = [N]P$, where N is an abelian group of odd order, $P \in \mathrm{Syl}_p(G)$ is cyclic and P induces on N a group of fixed-point-free power-automorphisms of order p.*

Proof. Assume $\nu_c(G) = 1$ and $P \in \mathrm{Syl}_p(G)$ with $P \ntriangleleft G$; let $s \in P$ be such that $\langle s \rangle \ntriangleleft G$.

For each prime $q \neq p$ and for each q-element $a \in G$ we have $\langle a \rangle \triangleleft G$ and therefore for $Q \in \mathrm{Syl}_q(G)$ it is $Q \triangleleft G$. Furthermore $[a, s] \neq 1$ and $q \neq 2$, so s induces a fixed-point-free automorphism on Q; by Proposition 5 the subgroup Q is abelian and s induces a power-automorphism on Q.

Consequently $G = [N]P$ with N abelian and $C_N(s) = \langle 1 \rangle$.

For every $b \in P \setminus C_P(N)$ it is $\langle b \rangle \ntriangleleft G$ so that $\langle b \rangle = \langle s^g \rangle$ for some $g \in G$; if $g = ny$ with $n \in N$, $y \in P$ and $b = g^{-1}s^l g$ for some $l \in \mathbb{N}$, then $b = [y, s^{-l}]s^l \in P'\langle s \rangle$. Since $P \neq C_P(N)$, it is $P = P'\langle s \rangle = \langle s \rangle$.

Vice versa, since s induces on N a fixed-point-free automorphism of order p, the conjugates of P are the only non-normal cyclic subgroups of G. QED

Lemma 1. *Let G be a nilpotent group with $\nu_c(G) = 1$; then*

(1) G is a p-group with $|\Omega_1(G)| \geq p^2$;

(2) if $|G'| = p$, it is $|\Omega_1(G)| = p^2$ and $\Phi(G) \subseteq Z(G)$;

(3) if $p = 2$ and $G' \subseteq Z(G)$, for any $a \in G \setminus Z(G)$ with $|a| = 2$ it is $\Omega_2(G) \subseteq C_G(a)$ and $\exp G \geq 2^3$

Proof. (1) It follows from Propositions 1 and 2.

(2) Consider $\langle a \rangle \ntriangleleft G$.

If $|a| = p$, then for any $b \in G$ with $|b| = p$ we have either $\langle b \rangle \ntriangleleft G$ or $\langle ab \rangle \ntriangleleft G$, so that respectively either $\langle b \rangle$ or $\langle ab \rangle$ is conjugate to $\langle a \rangle$; it follows $b \in \langle a \rangle G'$. Therefore $\Omega_1(G) \subseteq \langle a \rangle \times G'$ and $|\Omega_1(G)| = p^2$.

If $|a| = p^r$ with $r \geq 2$, then $\Omega_1(G) \subseteq Z(G)$. If it were $|\Omega_1(G)| \geq p^3$, then for any $c \in G \setminus \langle a \rangle G'$ such that $|c| = p$ it would be $\langle ac \rangle \nsim_G \langle a \rangle$ and so $\langle ac \rangle \triangleleft G$. If $g \notin N_G(\langle a \rangle)$, then $1 \neq [a, g] = [ac, g] \in \langle ac \rangle \cap G'$ and therefore $G' \subseteq \Omega_1(\langle ac \rangle) \subseteq \langle a \rangle$, a contradiction.

For any $a, b \in G$ it is $1 = [a, b]^p = [a^p, b]$ and so $\Phi(G) = G'\mho_1(G) \subseteq Z(G)$.

(3) As above we prove $\Omega_1(G) \subseteq \langle a \rangle G'$. If $b \in G$ is such that $|b| = 4$, then $\langle b \rangle \triangleleft G$; if it were $[a, b] \neq 1$, it would be $(ab)^2 = 1$ and $ab \in \Omega_1(G) \subseteq C_G(a)$, a contradiction. So $\Omega_2(G) \subseteq C_G(a)$ and $\exp G \geq 2^3$.

QED

Theorem 2. *Let P be a p-group; it is $\nu_c(P) = 1$ (if and) only if it is $\nu(P) = 1$, that means $P \simeq M_n(p) = \langle a, b | a^{p^{n-1}} = b^p = 1, [a, b] = a^{p^{n-2}} \rangle$, where $n \geq 4$ if $p = 2$.*

Proof. Suppose $\nu_c(P) = 1$ and let P be a minimal counterexample with $|P| = p^n$; by Proposition 2 it is $\exp P \leq p^{n-2}$.

Let us consider first the case $p \neq 2$.

Let N be a minimal normal subgroup of P; then $\nu_c\left(\frac{P}{N}\right) \leq \nu_c(P) = 1$.

If $\nu_c\left(\frac{P}{N}\right) = 0$, then $\frac{P}{N}$ is abelian and $P' = N$ has order p. If $\nu_c\left(\frac{P}{N}\right) = 1$, the minimality of P implies $\nu\left(\frac{P}{N}\right) = 1$. By Proposition 3 it is again $|P'| = p$.

By Lemma 1,2) it is $|\Omega_1(P)| = p^2$; as P is regular, $|\mho_1(P)| = p^{n-2}$ and $\phi(P) = \mho_1(P) = Z(P)$. Then P is a minimal non-abelian group; this contradicts Proposition 4.

Let us suppose now $p = 2$.

Step 1: $P' \subseteq Z(P)$

By Proposition 3 it is $\nu\left(\frac{P}{Z(P)}\right) \neq 1$ and therefore $\nu_c\left(\frac{P}{Z(P)}\right) \neq 1$; then $\nu_c\left(\frac{P}{Z(P)}\right) = \nu\left(\frac{P}{Z(P)}\right) = 0$ and $\frac{P}{Z(P)}$ is abelian.

Step 2: $\Omega_1(P) \nsubseteq Z(P)$

Suppose $\Omega_1(P) \subseteq Z(P)$. Since $\nu_c(Q_n) = 2$, P has at least two different minimal normal subgroups N_1 and N_2 and we may suppose $\frac{P}{N_1}$ and $\frac{P}{N_2}$ non-abelian.

If $\nu_c\left(\frac{P}{N_1}\right) = 1$, then $\nu\left(\frac{P}{N_1}\right) = 1$; by Proposition 3 one has $|P'| = 2$ and

$$\frac{P}{N_1} = \langle \overline{a}, \overline{b} | \overline{a}^{2^{n-2}} = \overline{b}^2 = 1, [\overline{a}, \overline{b}] = \overline{a}^{2^{n-3}} \rangle$$

where $n \geq 5$.

Then $\langle b \rangle \ntriangleleft P$ so that $|b| = 4$ and $N_1 \subseteq \langle b \rangle$; since $\exp P \leq 2^{n-2}$, it is $|a| = 2^{n-2} \geq 2^3$ and therefore $\langle a \rangle \triangleleft P$. So $P = \langle a, b | a^{2^{n-2}} = b^4 = 1, [a, b] = a^{2^{n-3}} \rangle$ would be a minimal non-abelian group; this contradicts Proposition 4.

So it must be $\nu_c\left(\frac{P}{N_1}\right) = \nu_c\left(\frac{P}{N_2}\right) = 0$ and $\frac{P}{N_1} \simeq \frac{P}{N_2} \simeq Q_3 \times E$, where E is elementary abelian; it follows $\exp P = 4$. Let

$$\frac{P}{N_1} = \langle \overline{x}, \overline{y} | \overline{x}^4 = 1, \overline{x}^2 = \overline{y}^2 = [\overline{x}, \overline{y}] \rangle \times \left(\times_{i=1}^{n-4} \langle \overline{e}_i \rangle \right)$$

Any two of the subgroups $\langle x \rangle$, $\langle y \rangle$, $\langle xy \rangle$ are not conjugate; we may suppose $\langle x \rangle \triangleleft P$, $\langle y \rangle \triangleleft P$ and $\langle x, y \rangle \simeq Q_3$.

Since P is non-hamiltonian, we may suppose $|e_1| = 4$, $N \subseteq \langle e_1 \rangle \triangleleft P$ and $[x, e_1] = [y, e_1] = 1$; it follows $|xe_1| = 4$ and $[xe_1, y] = [x, y] = x^2 \notin \langle xe_1 \rangle$. Therefore $\langle xe_1 \rangle \ntriangleleft P$ and $\langle ye_1 \rangle \ntriangleleft P$; since $\nu_c(P) = 1$, then $ye_1 \in \langle xe_1 \rangle P' \subseteq \langle x, e_1 \rangle$, a contradiction.

Step 3: $|Z(P)| \leq 2^{n-3}$

If $[P : Z(P)] = 4$, then $|P'| = 2$ and $|\Omega_1(P)| = 4$ by Lemma 1,2).

Since $\Omega_1(P) \nsubseteq Z(P)$, we have $|\Omega_1(Z(P))| = 2$ and $Z(P) = \langle z \rangle$ is cyclic; furthermore there exists $a \in P$ such that $|a| = 2$ and $\langle a \rangle \ntriangleleft P$. By Proposition 4 there exists a non-abelian maximal subgroup of P and $\phi(P) \subseteq \langle z^2 \rangle$. Let $x \in P \setminus \langle a, z \rangle$; it is $x^2 = z^{2r}$ so that $xz^{-r} \in \Omega_1(P) \subseteq \langle a, z \rangle$, a contradiction.

Step 4: $Z(P)$ is cyclic.

If $Z(P)$ were not cyclic, there would be at least two minimal normal subgroups N_1 and N_2 with $\frac{P}{N_1}$ and $\frac{P}{N_2}$ non-abelian. By Proposition 3 one would have $\nu\left(\frac{P}{N_1}\right) \neq 1$ and $\nu\left(\frac{P}{N_2}\right) \neq 1$; since P is a minimal counterexample, then $\nu_c\left(\frac{P}{N_1}\right) = \nu_c\left(\frac{P}{N_2}\right) = 0$. ¿From $\exp \frac{P}{N_1} = \exp \frac{P}{N_2} = 4$ it would follow $\exp P = 4$; this contradicts Lemma 1,3).

Step 5: $|P'| = 2$.

Let N be the only minimal normal subgroup of P; one has $\nu_c\left(\frac{P}{N}\right) = 0$ by Proposition 3. If it were $N \neq P'$, it would be

$$\frac{P}{N} = \langle \overline{x}, \overline{y} | \overline{x}^4 = 1, \overline{x}^2 = \overline{y}^2 = [\overline{x}, \overline{y}] \rangle \times \left(\times_{i=1}^{m} \langle \overline{e}_i \rangle \right);$$

then $[x, y] \in \langle x^2 \rangle N = \langle y^2 \rangle N = P' \simeq C_4$ and $[x, y]^2 = [x^2, y] = 1$, a contradiction.

Conclusion : Since $|P'| = 2$, for any $a, b \in P$ it is $[a^2, b] = [a, b]^2 = 1$ and $\mho_1(G) \subseteq Z(P)$; from $|Z(P)| \leq 2^{n-3}$ it follows that every maximal subgroup of P is non-abelian.

Since $\Omega_1(P) \nsubseteq Z(P)$, there exists an element $a \in P$ such that $\langle a \rangle \ntrianglelefteq P$, $|a| = 2$ and $\Omega_1(P) = \langle a \rangle \times P' \subseteq C_P(a)$.

For every $b \in P \setminus C_P(a)$ it is $[a, b] = c$, where $P' = \langle c \rangle$; then it follows $[P : C_P(a)] = 2$, $P = \langle b \rangle C_P(a)$ and $C_P(a) \neq \langle a \rangle Z(P)$.

Let $d \in C_P(a) \setminus \langle a \rangle Z(P)$: from $b^2, d^2 \in Z(P)$ it follows either $\langle b^2 \rangle \subseteq \langle d^2 \rangle$ or $\langle d^2 \rangle \subseteq \langle b^2 \rangle$.

If $b^2 = d^{2r}$, then $(bd^{-r})^4 = 1$ and $bd^{-r} \in \Omega_2(P) \subseteq C_P(a)$ by Lemma 1,3); this contradicts $b \notin C_P(a)$.

If $d^2 = b^{4s}$, then $db^{-2s} \in \Omega_1(P) \subseteq \langle a \rangle Z(P)$, whence $d \in \langle a \rangle Z(P)$, a contradiction.

QED

3 $\nu_c = 2$

3.1 Direct products

Proposition 6. *If G is a direct product of proper subgroups and $\nu_c(G) = 2$, then $G = A \times B$ where $\nu_c(A) = 1$ and $|B| = q$ for some prime q.*

Moreover, if q divides $|A|$, then $q = 2$.

Proof. Suppose $G = A \times B$ with A and B proper subgroups of G. Proposition 1 implies either $\nu_c(A) = 0$ or $\nu_c(B) = 0$. Suppose $\nu_c(B) = 0$; then $\mu_c(B) \geq 2$ and either $\nu_c(A) = 0$ or $\nu_c(A) = 1$.

If $\nu_c(A) = 0$, we may suppose A non-abelian, so that $A \simeq Q_3 \times E$, where E is an elementary abelian 2-group. Since $\nu_c(Q_3 \times Q_3) > 2$, B is abelian and there exists $b \in B$ such that $|b| = 4$. Let $A = \langle x, y | x^4 = 1, x^2 = y^2 = [x, y] \rangle \times E$: the three subgroups $\langle xb \rangle$, $\langle yb \rangle$ and $\langle xyb \rangle$ are non-normal and non-conjugate, a contradiction.

It must be $\nu_c(A) = 1$ and $\mu_c(B) = 2$. So $B = \langle b \rangle \simeq C_q$ for some prime q.

Suppose q divides $|A|$. If A is non-nilpotent, then $A = [N]P$ as in Theorem 1 and the subgroups $P = \langle x \rangle$ and $\langle xb \rangle$ are non-normal and non-conjugate. If q divides $|N|$, for any $a \in N$ such that $|a| = q$ we would have $\langle ab \rangle \ntrianglelefteq G$ so that $\nu_c(G) \geq 3$; therefore $q = p$.

In any case, if $q \neq 2$, there exists $y \in A$ such that the subgroups $\langle y \rangle$, $\langle yb \rangle$ and $\langle y^2 b \rangle$ are non-normal and pairwise non-conjugate, so that $\nu_c(G) \geq 3$. QED

Corollary 1. *If G is nilpotent and it is not a p-group, then $\nu_c(G) = 2$ if and only if $G \simeq M_n(p) \times C_q$, where p, q are distinct primes and $n \geq 4$ if $p = 2$.*

3.2 p-groups with $\nu_c = 2$

Lemma 2. *Let P be a non-abelian p-group; if $\exp P = p$, then $\nu_c(P) \geq 4$.*

Proof. Let $|P| = p^n$. If P' is the only minimal normal subgroup of P, P has $\frac{p^n - p}{p-1}$ non-normal cyclic subgroups of order p and each of them has p conjugates. Then $\nu_c(P) \geq \frac{p^n - p}{(p-1)p} \geq p + 1 \geq 4$.

Otherwise, if N is a minimal normal subgroup of P with $N \neq P'$, by induction $\nu_c(P) \geq \nu_c\left(\frac{P}{N}\right) \geq 4$. QED

Proposition 7. *If P is a p-group with $\nu_c(P) = 2$, then $p = 2$.*

Proof. Suppose $p \neq 2$, P a minimal counterexample. If $|P| = p^n$, then $n \geq 4$ and $p^2 \leq \exp P \leq p^{n-2}$.

For every minimal normal subgroup N of P we have $\nu_c\left(\frac{P}{N}\right) \leq 1$.

If P' were the only minimal normal subgroup of P, one would have $\mho_1(P) \subseteq Z(P)$. Therefore every non-normal cyclic subgroup would have order p and would have p conjugates; moreover P would be regular and $\exp \Omega_1(P) = p$. Then $\Omega_1(P)$ would contain exactly $2p+1$ cyclic subgroups; a calculation on the order of $\Omega_1(P)$ shows that this is impossible.

So there exists a minimal normal subgroup N of P with $\nu_c\left(\frac{P}{N}\right) = 1$; that means

$$\frac{P}{N} = \langle \bar{a}, \bar{b} | \bar{a}^{p^{n-2}} = \bar{b}^p = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^{n-3}} \rangle,$$

$|P'| = p$, $|Z(P)| = p^{n-2}$ and $Z(P) = \langle a^p, N \rangle$ with $|a| = p^{n-2} = \exp P$.

If $|b| = p^2$, then $Z(P) = \langle a^p, b^p \rangle = \Phi(P)$ and P would be a minimal non-abelian group; this contradicts Proposition 4. So $|b| = p$.

If $N = \langle y \rangle$, the subgroups $\langle b \rangle$, $\langle by \rangle$ and $\langle b^2 y \rangle$ are non-normal and pairwise non-conjugate, a contradiction. QED

If G is not a Dedekind group, let $R(G)$ denote the intersection of all non normal subgroups of G; the groups with $R(G) \neq \langle 1 \rangle$ are determined in [1].

Lemma 3. *Let P be a non-Dedekind 2-group with $R(P) \neq \langle 1 \rangle$, $|P| = 2^n$ and $\exp P \leq 2^{n-2}$. If $\nu_c(P) = 2$, then $P = [\langle a \rangle]\langle b \rangle$ with $|a| = 2^{n-2}$, $|b| = 4$.*

Proof. It is $\nu_c(Q_3 \times C_4) = 3$ and $\nu_c(Q_3 \times Q_3) = 9$. Then by [1], Theorem 1 it is

$$P = \langle A, x | A \text{ is abelian}, x^4 = 1, 1 \neq x^2 \in A, [x, a] = a^2 \text{ for any } a \in A \rangle.$$

For any $y \in A$ it is $(yx)^2 = x^2$. If $\langle yx \rangle \triangleleft P$, for any $a \in A$ one would have $a^2 = [yx, a] = x^2$, so $\Phi(A) = \langle x^2 \rangle$; it would follow $A \simeq C_4 \times E$ with $\Phi(E) = 1$ and $P \simeq Q_3 \times E$ hamiltonian. So $\langle yx \rangle \ntriangleleft P$ for every $y \in A$.

Since $P' = \Phi(A) = \mho_1(A)$, then for any $y_1, y_2 \in A$ the subgroups $\langle y_1 x \rangle$ and $\langle y_2 x \rangle$ are conjugate if and only if $y_1^{-1} y_2 \in \langle x^2 \rangle \Phi(A)$, so $[A : \langle x^2 \rangle \Phi(A)] = 2$. If $a \in A \setminus \langle x^2 \Phi(A) \rangle$ it is $A = \langle a, x^2, \Phi(A) \rangle = \langle a, x^2 \rangle$.

Since $\exp A \leq \exp P \leq 2^{n-2}$, we have $x^2 \notin \langle a \rangle$, hence $P = [\langle a \rangle]\langle x \rangle$ with $|a| = 2^{n-2}$, $|x| = 4$. QED

Remark 1. If $P = [\langle a \rangle]\langle b \rangle$, with $|a| = 2^{n-2}$, $|b| = 4$, $[a, b^2] = 1 \neq [a, b]$ and $n \geq 4$, then $\nu_c(P) = 2$: if $[a, b] = a^{2^{n-3}}$, the non normal cyclic subgroups not conjugate to $\langle b \rangle$ are conjugate to $\langle a^{2^{n-4}} b \rangle$, otherwise they are conjugate to $\langle ab \rangle$.

Theorem 3. *Let P be a non abelian 2-group, which is not a direct product of proper subgroups, with $|P| = p^n$, $\exp P \leq 2^{n-2}$. Then $\nu_c(P) = 2$ if and only if $P = [\langle a \rangle]\langle b \rangle$, with $|a| = 2^{n-2}$, $|b| = 4$, $[a, b^2] = 1$ and $n \geq 4$.*

Proof. We see that if $P = [\langle a \rangle]\langle b \rangle$ with $|a| = 2^{n-2}$, $|b| = 4$ and $[a, b^2] \neq 1$, then $b^{-1}ab = a^{1+2^{n-4}}$ or $b^{-1}ab = a^{-1+2^{n-4}}$; the subgroups $\langle b \rangle$, $\langle b^2 \rangle$ and respectively $\langle a^{2^{n-3}} b \rangle$ in the first case and $\langle ba \rangle$ in the second case are non normal and pairwise non-conjugate. So it will suffice to prove that, if $\nu_c(P) = 2$, then $P = [\langle a \rangle]\langle b \rangle$ with $|a| = 2^{n-2}$ and $|b| = 4$.

First of all, note that for any minimal normal subgroup T of P we may suppose $\nu_c\left(\frac{P}{T}\right) \neq 1$. Indeed, if it were

$$\frac{P}{T} = \langle \bar{a}, \bar{b} | \bar{a}^{2^{n-2}} = \bar{b} = 1, [\bar{a}, \bar{b}] = \bar{a}^{2^{n-5}} \rangle$$

with $n \geq 5$, then $P = \langle a, b\rangle \times T$ if $|b| = 2$ and $P = [\langle a\rangle]\langle b\rangle$ if $|b| = 4$.

By Lemma 3, we may suppose $R(P) = \langle 1\rangle$. Let P be a minimal counterexample and let $H = \langle h\rangle$, $K = \langle k\rangle$ be two non-conjugate non-normal cyclic subgroups of P, with $|H| \leq |K|$.

If $H \not\leq K$, since $\nu_c(P) = 2$, then $H \cap K \lhd P$. It follows $H \cap K \leq H_P \cap K_P = R(P) = \langle 1\rangle$. Then either $H \leq K$ or $H^g \cap K = \langle 1\rangle$ for every $g \in P$.

If $H \leq K$, then $|H| = 2$, $|K| = 4$ and $h = k^2$.

Let T be a minimal normal subgroup of P. It is $\frac{KT}{T} \ntriangleleft \frac{P}{T}$ with $\frac{KT}{T} = 4$ and $\nu_c\left(\frac{P}{T}\right) = 2$. If $\frac{S}{T} = \langle sT\rangle \ntriangleleft \frac{P}{T}$ with $\frac{S}{T}$ non-conjugate to $\frac{KT}{T}$, then $\langle s\rangle$ is conjugate to H and $\frac{HT}{T} \ntriangleleft \frac{P}{T}$. So $\frac{P}{T}$ has two non-normal cyclic subgroups $\frac{HT}{T}$ and $\frac{KT}{T}$ of order respectively 2 and 4, with $\frac{HT}{T} \leq \frac{KT}{T}$. This implies that $\frac{P}{T}$ is not a direct product and $\exp\frac{P}{T} \leq 2^{n-3}$.

By minimality of P, $\frac{P}{T} = [\langle aT\rangle]\langle bT\rangle \simeq [C_{2^{n-3}}]C_4$ with $n \geq 5$. But $\langle b\rangle \ntriangleleft P$, so $|b| = 4$, $|a| = 2^{n-3} \geq 4$. Since $\langle a\rangle$ is not conjugate to $\langle b\rangle$, then $\langle a\rangle \lhd P$ and $P = \langle a, b\rangle \times T$, contrary to the hypothesis.

So $H^g \cap K = \langle 1\rangle$ for every $g \in P$. We distinguish two cases.

Case 1: $|K| \geq 4$.

Every proper subgroup of K is normal in P; for $T \leq K$ with $|T| = 2$ one has $\nu_c\left(\frac{P}{T}\right) = 2$. If $\exp\frac{P}{T} = 2^{n-2}$, then $\frac{P}{T}$ is isomorphic to D_{n-1} or S_{n-1}, because the non-normal cyclic subgroups of Q_{n-1} have non-trivial intersection. Then $\frac{P}{T}$ has a non-normal cyclic subgroup $\frac{S}{T} = \langle sT\rangle$ of order 2, and $P = \langle a, s, T\rangle$ with $|a| = 2^{n-2}$. Since P is not a direct product, we have $P = \langle a, s\rangle = [\langle a\rangle]\langle s\rangle$ and $|s| = 4$, against our hypothesis.
Then $\exp\frac{P}{T} \leq 2^{n-3}$.

If $\frac{P}{T}$ were a direct product of proper subgroups, we would have

$$\frac{P}{T} = \langle \overline{x}, \overline{y} | \overline{x}^{2^{n-3}} = \overline{y}^2 = 1, [\overline{x}, \overline{y}] = \overline{x}^{2^{n-4}}\rangle \times \langle \overline{z} | \overline{z}^2 = 1\rangle$$

where $n \geq 6$; we can suppose $\frac{K}{T} = \langle \overline{y}\rangle$ and $\frac{HT}{T} = \langle \overline{yz}\rangle$ and so $K = \langle y\rangle$, $H = \langle yz\rangle$ with $|K| = 4$, $|H| = 2$.
The subgroup $\langle z\rangle$ is conjugate neither to K nor to H, so $\langle z\rangle \lhd P$. Since P is not a direct product, it is $|z| = 4$. It follows $y^2 = z^2$ and $[y, z] = 1$, so $P' \leq \langle x\rangle$.
If it were $T \leq \langle x\rangle$, it would be $T \leq P' = \langle x^{2^{n-4}}\rangle \leq \langle x^4\rangle$ and $1 = [x, y]^4 = [x^4, y]$; then $1 = [x, y^2] = [x, y]^2$, so $[x, y] \in T$, a contradiction.
So $T \not\leq \langle x\rangle$ and $[x, z] = 1$; it would follow $P = \langle x, yz\rangle \times \langle z\rangle$, a contradiction.

By the minimality of P one has $\frac{P}{T} = [\langle aT\rangle]\langle bT\rangle \simeq [C_{2^{n-3}}]C_4$ and $[a, b^2] \in T$. By the previous Remark we may suppose $K = \langle b\rangle$ and $|K| = 8$, $H = \langle ab\rangle$ with $|H| = 4$. It follows $\langle a\rangle \lhd P$, so $[a, b^2] \in \langle a\rangle \cap T$.
If $T \leq \langle a\rangle$ then $P = [\langle a\rangle]\langle ab\rangle \simeq [C_{2^{n-2}}]C_4$.
If $T \not\leq \langle a\rangle$, $[a, b^2] = 1$ and $P = [\langle a\rangle]\langle b\rangle$. Since $\langle \overline{a}\overline{b}^2\rangle \lhd \frac{P}{T}$, the subgroup $\langle ab^2\rangle$ is conjugate neither to K nor to H, hence $\langle ab^2\rangle \lhd P$, a contradiction.

Case 2: $|K| = |H| = 2$

Let T be a minimal normal subgroup of P.
If $\nu_c\left(\frac{P}{T}\right) = 2$, the non normal cyclic subgroups of $\frac{P}{T}$ are conjugate either to $\frac{HT}{T}$ or to $\frac{KT}{T}$, so their order is 2. By the minimality of P one has $\frac{P}{T} \simeq D_{n-1}$ or $\frac{P}{T} \simeq M_{n-2}(2) \times C_2$.
If $\frac{P}{T} = \langle \overline{a}, \overline{b} | \overline{a}^{2^{n-2}} = \overline{b}^2 = 1, [\overline{b}, \overline{a}] = \overline{a}^2\rangle$, then $P = \langle a, b\rangle \times T$.

Let $\frac{P}{T} = \langle \overline{a}, \overline{b} | \overline{a}^{2^{n-3}} = \overline{b}^2 = 1, [\overline{a}, \overline{b}] = \overline{a}^{2^{n-4}} \rangle \times \langle \overline{c} | \overline{c}^2 = 1 \rangle$ with $n \geq 6$. We can suppose $K = \langle b \rangle$, $H = \langle bc \rangle$. Since $\langle \overline{c} \rangle$ is conjugate neither to $\langle \overline{b} \rangle$ nor to $\langle \overline{bc} \rangle$, it is $\langle c \rangle \triangleleft P$
If $|c| = 2$, then $P = \langle a, b, T \rangle \times \langle c \rangle$.
So $|c| = 4$ and $T = \langle c^2 \rangle$; from $1 = (bc)^2 = c^2[b, c]$ it follows $[b, c] = c^2$.
If $T \nleq \langle a, b \rangle$, we have $[a, c] = 1$ and $|a| = 2^{n-3}$; thus $\left| ca^{2^{n-5}} \right| = 4$ and $\langle ca^{2^{n-5}} \rangle \triangleleft P$, but $[b, ca^{2^{n-5}}] = [b, a^{2^{n-5}}][b, c] = c^2 \notin \langle ca^{2^{n-5}} \rangle$.
If $T \leq \langle a, b \rangle$, then $T \leq \langle a \rangle$ and $|a| = 2^{n-2}$. ¿From $[a, c] \in T$ it follows $[a, c^2] = 1$, so $(a^{2^{n-4}} c)^2 = 1$. Since $[a^4, b] = 1$ and $n \geq 6$, then $[a^{2^{n-4}} c, b] = c^2 \neq 1$, so $\langle a^{2^{n-4}} c \rangle \ntriangleleft P$, but $\langle a^{2^{n-4}} c \rangle$ is conjugate neither to $\langle a \rangle$ nor to $\langle bc \rangle$.

So $\nu_c\left(\frac{P}{T}\right) = 0$ for every minimal normal subgroup T of P. Since $HT \triangleleft P$, then $T = \langle [h, g] \rangle$ for any $g \in P \setminus N_P(H)$; it follows that P has just one minimal normal subgroup T.
For $\frac{P}{T} = \langle xT, yT \rangle \times E$ with $\langle xT, yT \rangle \simeq Q_3$ and $\Phi(E) = \langle 1 \rangle$ we may suppose $\langle x \rangle \triangleleft P$, hence $[x, y]^2 = [x^2, y] = 1$, so $[x, y] \in T$, a contradiction.
Then $\frac{P}{T}$ is abelian, $P' = T = \langle t \rangle \leq Z(P)$ and $Z(P)$ is cyclic. Since H and K are not conjugate, it is $hk \neq t$. If $[h, k] = 1$, then $|hk| = 2$, $\langle hk \rangle \ntriangleleft P$ and so $\nu_c(P) \geq 3$. Therefore $(hk)^2 = [h, k] = t$.
Let $Z(P) = \langle z \rangle$ with $|Z(P)| = 2^s$. If $s \geq 2$, it is $(z^{2^{s-2}} hk)^2 = 1$, thus $L = \langle z^{2^{s-2}} hk \rangle \ntriangleleft P$ with L neither conjugate to H nor to K, a contradiction.
We conclude that $Z(P) = P' = \langle t \rangle$; P is an extraspecial group and P is a central product $P = S_1 * S_2 * \ldots * S_r$, with S_1 isomorphic either to D_3 or to Q_3, $S_i \simeq D_3$ for $2 \leq i \leq r$ and $|S_i \cap S_j| = 2$ for $i \neq j$ (see [8], 5.3).
Let $S_2 = \langle a, b | a^4 = b^2 = 1, [a, b] = a^2 \rangle$. If $S_1 = \langle c, d | c^4 = d^2 = 1, [c, d] = c^2 \rangle$, the subgroups $\langle b \rangle$, $\langle d \rangle$ and $\langle bd \rangle$ are non-normal and pairwise non-conjugate. If $S_1 = \langle c, d | c^4 = 1, c^2 = d^2 = [c, d] \rangle$, the subgroups $\langle b \rangle$, $\langle ac \rangle$ and $\langle ad \rangle$ are non-normal and pairwise non-conjugate.

QED

Remark 2. If $P = [\langle a \rangle]\langle b \rangle$, with $|a| = 2^{n-2}$, $|b| = 4$, $[a, b^2] = 1$ and $n \geq 4$ it is $\nu(P) = 2$ if and only if $a^b = a^{1+2^{n-3}}$; this follows from Theorem I in [6].

3.3 Non nilpotent groups with $\nu_c = 2$

Proposition 8. *Let G be a non-nilpotent group such that $\nu_c(G) = 2$. If H and K are non-normal, non-conjugate cyclic subgroups, then one of the following cases holds:*

(1) $|H| = p^\alpha$, $|K| = p^\beta$

(2) $|H| = p^\alpha$, $|K| = q^\beta$

(3) $|H| = p^\alpha$, $|K| = p^\alpha q$

where p, q are distinct primes.

The third case holds only if $G = A \times B$, where A and B are proper subgroups of G and $(|A|, |B|) = 1$.

Proof. Obviously there exists a cyclic subgroup $H = \langle h \rangle$ such that $H \ntriangleleft G$ and $|H| = p^\alpha$, where p is a prime. If $|K|$ is not a prime power, then $K = R \times S$, where $(|R|, |S|) = 1$ and $R \ntriangleleft G$, so that R is conjugate to H. Let q be a prime and T a subgroup of S such that $|T| = q$;

since $R \times T \ntriangleleft G$ and $\nu_c(G) = 2$, then $R \times T = K$ and $|K| = p^\alpha q$. For any prime $w \neq p$ and any w-element $y \in G$ one has $\langle y \rangle \triangleleft G$ and any Sylow w-subgroup is normal.

Since $[H, T] = \langle 1 \rangle$, by Proposition 5 it is $[H, Q] = \langle 1 \rangle$. Suppose $H \subseteq P \in \mathrm{Syl}_p(G)$; if it were $[P, Q] \neq \langle 1 \rangle$, for any $a \in P$ such that $[a, Q] \neq \langle 1 \rangle$ it would be $\langle a \rangle \ntriangleleft G$, consequently $\langle a \rangle \sim_G H$, a contradiction.

So $Q \subseteq Z(G)$ and Q is a direct factor of G. QED

Lemma 4. *Let G be a non-nilpotent group with $\nu_c(G) = 2$. If there exists a unique prime p such that the Sylow p-subgroups are non-normal and there are two non-normal cyclic subgroups H and K whose orders are coprime (i.e. $|H| = p^\alpha$, $|K| = q^\beta$), then G is a minimal non-abelian group.*

Proof. By Proposition 6 G is not a direct product of proper subgroups.

Let $H = \langle h \rangle \subseteq P \in \mathrm{Syl}_p(G)$ and $K = \langle k \rangle \subseteq Q \in \mathrm{Syl}_q(G)$ and $Q \triangleleft G$. For any prime r different from p and q and for any non trivial r-element $a \in G$ one would have $\langle a \rangle \triangleleft G$ and $\langle ka \rangle = \langle k \rangle \times \langle a \rangle \ntriangleleft G$, against the hypothesis $\nu_c(G) = 2$; so $G = [Q]P$.

In a similar way one proves $C_Q(H) = \langle 1 \rangle = C_P(K)$ and therefore $N_G(P) = P$ and $C_P(Q) = \langle 1 \rangle$. So for any $b \in P \setminus \langle 1 \rangle$ it is $\langle b \rangle \ntriangleleft G$ and $\langle b \rangle$ is conjugate to H; this means $|H| = p = \exp P$. We may suppose $H \subseteq Z(P)$; then any $\langle b \rangle$ would be conjugated to H by an element of Q and therefore $G = QH$ with $H = P$. It will suffice to prove that Q is a minimal normal subgroup of G.

By the Frattini Argument $P\Phi(Q) \ntriangleleft G$, so that $\nu_c\left(\frac{G}{\Phi(Q)}\right) \neq 0$. We distinguish two cases: $\nu_c\left(\frac{G}{\Phi(Q)}\right) = 1$ and $\nu_c\left(\frac{G}{\Phi(Q)}\right) = 2$.

Case 1 : $\nu_c\left(\frac{G}{\Phi(Q)}\right) = 1$

Every q-subgroup of $\frac{G}{\Phi(Q)}$ is normal.

It is $\nu_c\left(\frac{G}{\Phi(K)}\right) = 2$. If $\Phi(K) \neq \langle 1 \rangle$, by induction on the order of the group, $\frac{G}{\Phi(K)}$ would be a minimal non-abelian group, $\frac{Q}{\Phi(K)}$ would be elementary abelian and $\Phi(Q) = \Phi(K)$, which contradicts $\nu_c\left(\frac{G}{\Phi(Q)}\right) = 1$. Therefore $|K| = q$.

Since $K\Phi(Q) \triangleleft G$, $K^G \neq Q$. Let $y \in Q \setminus K^G$, then $\langle y \rangle \triangleleft G$ and $\langle ky \rangle \triangleleft G$. Hence $\langle y, k \rangle \triangleleft G$, so that $K^G \subseteq \Omega_1(\langle y, k \rangle) \triangleleft G$.

By Theorem 1, $q \neq 2$ and either $\langle y, k \rangle \simeq M_s(q)$ or $\langle y, k \rangle$ is abelian; this implies $|\Omega_1(\langle y, k \rangle)| = q^2$ and $K^G = \Omega_1(\langle y, k \rangle)$. It follows that K has q conjugates.

We may suppose $H = P \subseteq N_G(K)$.

Since $\nu_c(Q) = 1$, it is $Q = \langle a, k | a^{q^l} = k^q = 1, [a, k] = a^{q^{l-1}} \rangle$ and $K^G = \langle k, a^{q^{l-1}} \rangle$.

Since $\nu_c\left(\frac{G}{\Phi(Q)}\right) = 1$, one has $h^{-1}ah = a^r f_1$ and $h^{-1}kh = k^r f_2$ where $f_1, f_2 \in \Phi(Q)$.

Since $h \in N_G(K)$, it is $f_2 = 1$; furthermore $h^{-1}a^{q^{l-1}}h = a^{rq^{l-1}}$ and the automorphism induced by h on Q fixes every non-normal subgroup of Q. From $[h, k] \neq 1$ it follows that h induces a non-identity q'-automorphism which fixes every subgroup of Q; a contradiction by Proposition 5.

Case 2 : $\nu_c\left(\frac{G}{\Phi(Q)}\right) = 2$

Suppose $\Phi(Q) \neq \langle 1 \rangle$.

The subgroups $\frac{H\Phi(Q)}{\Phi(Q)}$ and $\frac{K\Phi(Q)}{\Phi(Q)}$ are non-normal in $\frac{G}{\Phi(Q)}$. By induction $\frac{G}{\Phi(Q)}$ is a minimal non-abelian group of order $q^m p$, with $m \geq 2$.

There is no normal subgroup of G of order q. Indeed, if $\langle y\rangle \triangleleft G$ with $|y| = q$, then the subgroup $H\langle y\rangle$ cannot be cyclic, so that $q \equiv 1(\operatorname{mod} p)$, but m is the least integer such that $q^m \equiv 1(\operatorname{mod} p)$.

It follows $|K| = q$ and $\exp Q = q$.

The proper cyclic subgroups of Q are conjugate in G, then they are contained in $Z(Q)$ and Q is elementary abelian, a contradiction.

So $\Phi(Q) = \langle 1\rangle$.

If $Q = K^G \times T$, for any $1 \neq t \in T$ it would be $h^{-1}th = t^r$ and $h^{-1}(kt)h = k^s t^s$; it would follow $h^{-1}kh = k^s$ and $\langle k\rangle \triangleleft G$. So $Q = K^G$.

Let A be a proper subgroup of Q normal in G. Since $Q = K^G$, $\langle a\rangle \triangleleft G$ for any $a \in A$. By Maschke's Theorem $Q = A \times B$, where $B \triangleleft G$. Then $h^{-1}ah = a^r$ for any $a \in A$ and $h^{-1}bh = b^s$ for any $b \in B$, with $r \not\equiv s(\operatorname{mod} q)$; this means $\langle ab\rangle \ntriangleleft G$ for $a \neq 1$ and $b \neq 1$. Let $k = \bar{a}\bar{b}$, $\bar{a} \in A$ and $\bar{b} \in B$. Since $\langle ab\rangle$ is conjugate to $\langle k\rangle$, it is $\langle a\rangle = \langle\bar{a}\rangle$ and $\langle b\rangle = \langle\bar{b}\rangle$, therefore $|Q| = q^2$ and K has $q-1$ conjugates. Then $q = 3$ and either $r \equiv 1(\operatorname{mod} q)$ or $s \equiv 1(\operatorname{mod} q)$; it follows respectively $A \subseteq C_Q(H)$ or $B \subseteq C_Q(H)$, which contradicts $C_Q(H) = \langle 1\rangle$.

So Q is a minimal normal subgroup of G.

QED

Proposition 9. *If G is a non-nilpotent group with $\nu_c(G) = 2$, then there exists just one prime p such that the Sylow p-subgroups of G are not normal.*

Proof. Let G be a minimal counterexample.

Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ be non-normal in G. There exist two non-normal subgroups $H = \langle h\rangle \subseteq P$ and $K = \langle k\rangle \subseteq Q$; since $\nu_c(G) = 2$, one has $[h,k] \neq 1$, thus $[P,Q] \neq \langle 1\rangle$.

For any prime $r \notin \{p,q\}$ and for any r-element $x \in G$ we have $\langle x\rangle \triangleleft G$; therefore if $R \in \mathrm{Syl}_r(G)$, then $R \triangleleft G$ and R is abelian.

Let N be the product of all the normal Sylow subgroups of G; N is abelian, P and Q induce on N power automorphisms, so that $[P,Q] \subseteq C_G(N)$ and $[P,Q] \triangleleft G$.

We can suppose $G = [N](PQ)$ with $C_{NP}(Q) = C_{NQ}(P) = \langle 1\rangle$. If $N \neq \langle 1\rangle$ then $P[P,Q] \ntriangleleft G$ and $Q[P,Q] \ntriangleleft G$ and $\nu_c\left(\frac{G}{[P,Q]}\right) = 2$ against the minimality of G.

So $N = \langle 1\rangle$ and $G = PQ$.

Suppose $P_G \neq \langle 1\rangle$; by minimality of G it is $\frac{QP_G}{P_G} \triangleleft \frac{G}{P_G}$, so $G = [P_G]N_G(Q)$. Then $P = [P_G]N_P(Q)$ and without loss of generality we may suppose $H \leq N_P(Q)$.

Every subgroup of P is normal in G; since $C_{P_G}(Q) = \langle 1\rangle$, it is $p \neq 2$, P_G is abelian and Q induces on P_G a group of power-automorphisms. If $a \in P_G$, $|a| = \exp P_G$, then $C_Q(P_G) = C_Q(a)$ and $\frac{Q}{C_Q(P_G)}$ is cyclic; we may suppose $Q = KC_Q(P_G)$. Now, $\langle k^q\rangle \triangleleft G$, so $[Q : C_Q(P_G)] = q$ and q divides $p-1$. Since $N_P(Q) \leq N_G(C_Q(P_G))$, then $C_Q(P_G) \triangleleft G$.

If $\nu_c\left(\frac{G}{P_G}\right) = 2$, by Lemma 4 it is $N_G(Q) \simeq \frac{G}{P_G} \simeq G(q,p,1)$ and $|Q| \geq q^2$. Q would be a minimal normal subgroup of $N_G(Q)$. Since $[Q, P_G] \neq \langle 1\rangle$, one has $C_Q(P_G) = \langle 1\rangle$, against $[Q : C_Q(P_G)] = q$.

If $\nu_c\left(\frac{G}{P_G}\right) = 1$, then $\frac{G}{P_G} = \left[\frac{QP_G}{P_G}\right]\frac{P}{P_G}$ with $Q \simeq \frac{QP_G}{P_G}$ abelian and $\frac{P}{P_G}$ cyclic. For $\frac{P}{P_G} = \langle xP_G\rangle$ we have $\langle x\rangle \ntriangleleft G$, hence $\langle x^p\rangle \triangleleft G$; this means $\left|\frac{P}{P_G}\right| = p = |N_P(Q)|$ and $H = N_P(Q)$.

We have $\left|\frac{QH}{C_Q(P_G)}\right| = pq$; as $q < p$, one has $Q \leq N_G(HC_Q(P_G))$, so $[Q,H] \subseteq HC_Q(P_G) \cap Q = C_Q(P_G)$. Since $[Q]H = N_G(Q) \simeq \frac{G}{P_G}$, it is $\nu_c(QH) = 1$ and for every $b \in Q$ one has

$h^{-1}bh = b^s$ for some $s \in \mathbb{N}$; as $[h, Q] \neq \langle 1 \rangle$ it is also $[h, \Omega_1(Q)] \neq \langle 1 \rangle$, so $s \not\equiv 1 (\bmod\, q)$. Then $Q = [h, Q] \leq C_Q(P_G)$, against $C_P(Q) = \langle 1 \rangle$. QED

Theorem 4. *Let G be a non-nilpotent group which is not a direct product of proper subgroups. Then $\nu_c(G) = 2$ if and only if G is isomorphic to one of the following groups:*

1) *the minimal non-abelian group $[Q]C_p$ where Q is elementary abelian of order q^n, $n \geq 2$ and $|C_p| = p$;*
2) *$\langle x, A | x^{p^n} = 1_G, a^x = a^r$ for any $a \in A\rangle$, p prime, $n \geq 2$, A abelian, $(|A|, p) = 1$, $(r(r^p - 1), |A|) = 1$, $r^{p^2} \equiv 1 \pmod{\exp A}$;*
3) *$[A]D_n = \langle x, y, A | x^{2^{n-1}} = y^2 = 1_G, x^y = x^{-1}, [a, x] = 1_G, a^y = a^{-1}$ for any $a \in A\rangle$, where A is abelian, $|A|$ is odd and $n \geq 3$;*
4) *$[A]S_n = \langle x, y, A | x^{2^{n-1}} = y^2 = 1_G, x^y = x^{-1+2^{n-2}}, [a, x] = 1_G, a^y = a^{-1}$ for any $a \in A\rangle$, where A is abelian, $|A|$ is odd and $n \geq 4$;*
5) *$[A]Q_n = \langle x, y, A | x^{2^{n-1}} = 1_G, y^2 = x^{2^{n-2}}, x^y = x^{-1}, [a, x] = 1_G, a^y = a^{-1}$ for any $a \in A\rangle$, where A is abelian, $|A|$ is odd and $n \geq 4$;*
6) *$\langle x, y, A \mid x^{2^{n-2}} = y^4 = 1_G, x^y = x^{-1}, [a, x] = 1_G, a^y = a^{-1}$ for any $a \in A\rangle$, where A is abelian, $|A|$ is odd and $n \geq 4$;*
7) *$\langle x, y, A \mid x^{2^{n-2}} = y^4 = 1_G, x^y = x^{2^{n-3}-1}, [a, x] = 1_G, a^y = a^{-1}$ for any $a \in A\rangle$, where A is abelian, $|A|$ is odd and $n \geq 5$.*

Proof. Let $\nu_c(G) = 2$ and $P \in \mathrm{Syl}_p(G)$ with $P \ntriangleleft G$.

Suppose G is not the minimal non-abelian group 1); there exist two non-normal non-conjugate cyclic subgroups $H = \langle h \rangle$ and $K = \langle k \rangle$ contained in P.

By Proposition 9 it is $G = [A]P$, where A is abelian and P induces on A a group of power-automorphisms; furthermore $C_A(H) = C_A(K) = \langle 1 \rangle$, which implies $N_A(H) = N_A(K) = \langle 1 \rangle$.

If P is cyclic, then without loss of generality $K = P$, $H = \Phi(P)$ and G is isomorphic to 2).

Suppose P non-cyclic.

If $a \in A$ is such that $a^{-1}Ha \leq P$ or $a^{-1}Ka \leq P$, then $a^{-1}Ha \leq \langle a \rangle H \cap P = H$ (respectively $a^{-1}Ka \leq \langle a \rangle K \cap P = K$), so $a = 1$. It follows that every subgroup of P_G is normal in G and for $g \in G$ it is $g^{-1}Hg \subseteq P$ (respectively $g^{-1}Kg \subseteq P$) if and only if $g \in P$; this implies $\nu_c(P) \leq 2$

If $S = \langle s \rangle \leq P_G$ and $S \ntriangleleft G$, then $S \sim_G H$ or $S \sim_G K$, so $H \leq P_G$ or $K \leq P_G$; since $[A, P_G] = \langle 1 \rangle$, one would have $A = C_A(H)$ or $A = C_A(K)$, a contradiction. So a subgroup T of P_G is normal in G if and only if $T \leq P_G$.

Suppose $H \ntriangleleft P$ and $K \ntriangleleft P$. Then $\nu_c(P) = 2$ and by Proposition 7 it is $p = 2$.

If $P = M_{n-1}(2) \times C_2$, then $G = AM_{n-1}(2) \times C_2$ and G would be a direct product of proper subgroups, against the hypothesis.

If $\exp P = 2^{n-1} = |x|$ with $x \in P$, it is $\langle x \rangle \triangleleft G$; P is isomorphic to D_n, S_n or Q_n with $n \geq 4$ and G is isomorphic to 3), 4) or 5) with $n \geq 4$.

Otherwise $P = [\langle x \rangle]\langle y \rangle \simeq [C_{2^{n-2}}]C_4$ with $x^y \in \{x^{-1}, x^{-1+2^{n-3}}, x^{1+2^{n-3}}\}$. If $x^y = x^{1+2^{n-3}}$ with $n \geq 5$ it is $\langle xy \rangle \triangleleft P$, then $P = \langle x, xy \rangle \triangleleft G$, a contradiction. In the other two cases G is isomorphic to 6) or 7).

Now we prove that $\frac{P}{P_G}$ is cyclic.

It is $N_G(P) = P \times C_A(P)$; since $C_A(P) \leq C_A(H) = \langle 1 \rangle$, then $N_G(P) = P$.

Let $g \in G \setminus P$ and $T = g^{-1}Pg \cap P$. If $t \in T$ and $\langle t \rangle \ntriangleleft G$, then $\langle t \rangle = a^{-1}Ha$ or $\langle t \rangle = a^{-1}Ka$ for some $a \in P$. Since $g\langle t \rangle g^{-1} \leq P$, then $ga^{-1}Hag^{-1} \leq P$ or $ga^{-1}Kag^{-1} \leq P$, so $ag^{-1} \in P$ and $g \in P$, a contradiction. Then for any $g \in G \setminus P$ one has $P \cap g^{-1}Pg \triangleleft G$, so $P \cap g^{-1}Pg = P_G$ and $\frac{G}{P_G}$ is a Frobenius group with complement $\frac{P}{P_G}$ (see [8], 10.5). It follows $\frac{P}{P_G}$ cyclic or $\frac{P}{P_G} \simeq Q_r$.

If $\frac{P}{P_G} = \langle \overline{x}, \overline{y} | \overline{x}^{2^{r-1}} = 1, \overline{x}^{2^{r-2}} = \overline{y}^2, [\overline{y}, \overline{x}] = \overline{x}^2 \rangle \simeq Q_r$, the subgroups $\langle x \rangle$, $\langle x^2 \rangle$ and $\langle y \rangle$ would be non-normal in G (because they are not contained in P_G) and pairwise non-conjugate in G, but then $\nu_c(G) \geq 3$.

So $\frac{P}{P_G}$ is cyclic and $P_G \nleq \Phi(P)$.

Without loss of generality we may distinguish two cases: either $H \leq K$, or neither H nor K contains properly a non-normal subgroup of G (equivalently, $\Phi(H) \triangleleft G$ and $\Phi(K) \triangleleft G$).

Case 1 : $H \leq K$.

In this case one has $H = \Phi(K)$, $\Phi(H) \triangleleft G$ and $\Phi(H) = H \cap P_G = K \cap P_G$.

Since $\frac{P}{P_G}$ is cyclic, it is $P = KP_G$. Since P is not cyclic, $\Phi(H) \neq P_G$. Let $\Phi(H) \leq L \leq P_G$ with $[P_G : L] = p$; then $\frac{G}{L} = \frac{AKL}{L} \times \frac{P_G}{L}$. By Proposition 1 it follows $\nu_c\left(\frac{AK}{\Phi(H)}\right) = \nu_c\left(\frac{AKL}{L}\right) = 1$.

If $\frac{K}{\Phi(H)} \triangleleft \frac{AK}{\Phi(H)}$, then $[A, K] \leq A \cap \Phi(H) = \langle 1 \rangle$, a contradiction to $C_A(K) = \langle 1 \rangle$. So $\frac{K}{\Phi(H)} \ntriangleleft \frac{AK}{\Phi(H)}$ and analogously $\frac{H}{\Phi(H)} \ntriangleleft \frac{AK}{\Phi(H)}$, so $\nu_c\left(\frac{AK}{\Phi(H)}\right) \geq 2$, a contradiction.

Case 2 : $\Phi(H) \triangleleft G$ and $\Phi(K) \triangleleft G$.

It is $\Phi(H) = H \cap P_G$ and $\Phi(K) = K \cap P_G$, so $\left|\frac{HP_G}{P_G}\right| = \left|\frac{KP_G}{P_G}\right| = p$. For $\frac{P}{P_G} = \langle sP_G \rangle$, one has $\langle s \rangle \sim_P H$ or $\langle s \rangle \sim_P K$, so $P = HP_G$ or $P = KP_G$. It follows $\left|\frac{P}{P_G}\right| = p$, so $P = HP_G = KP_G$.

Without loss of generality we may suppose $H \triangleleft P$, so $\frac{P}{\Phi(H)} = \frac{H}{\Phi(H)} \times \frac{P_G}{\Phi(H)}$ and $\frac{G}{\Phi(H)} = \frac{AH}{\Phi(H)} \times \frac{P_G}{\Phi(H)}$; it follows $\Phi(H) \neq \langle 1 \rangle$.

It is $\frac{H}{\Phi(H)} \ntriangleleft \frac{AH}{\Phi(H)}$, so $\nu_c\left(\frac{AH}{\Phi(H)}\right) \geq 1$. It must be $\mu_c\left(\frac{P_G}{\Phi(H)}\right) = 2$ and $\nu_c\left(\frac{P_G}{\Phi(H)}\right) = 0$, so $\left|\frac{P_G}{\Phi(H)}\right| = p$, $\left|\frac{P_G}{\Phi(H)}\right| = p^2$ and $\Phi(P) = \Phi(H) \triangleleft G$.

Let $x \in P \setminus \Phi(H)$ be such that $\langle x\Phi(H) \rangle \neq \frac{P_G}{\Phi(H)}$ and $\langle x\Phi(H) \rangle \neq \frac{H}{\Phi(H)}$; then $\langle x \rangle \ntriangleleft G$ and $\langle x \rangle \nsim_G H$ and so $\langle x \rangle \sim_G K$ and $\langle x\Phi(H) \rangle = \frac{K\Phi(H)}{\Phi(H)}$. Therefore $\frac{P}{\Phi(H)}$ has only three proper subgroups and $p = 2$.

Suppose $\Phi(K) \neq \Phi(H)$; let $\Phi(K) \leq J \leq \Phi(H)$ with $[\Phi(H) : J] = 2$. Then $\left|\frac{KJ}{J}\right| = 2$ and $\frac{P}{J} = \left[\frac{H}{J}\right] \frac{KJ}{J}$ would be either abelian or isomorphic to D_3.

Since $\left|\frac{P_G}{J}\right| = 4$ and every subgroup of $\frac{P_G}{J}$ is normal in $\frac{G}{J}$, $\frac{P}{J}$ cannot be isomorphic to D_3. If $\frac{P}{P_J}$ were abelian, then $\frac{G}{J} = \frac{AKJ}{J} \times \frac{P_G}{J}$ with $\mu_c\left(\frac{P_G}{J}\right) \geq 3$, a contradiction by Proposition 1.

We conclude that $\Phi(H) = \Phi(K) = \Phi(P)$, $K \triangleleft P$ and P is a Dedekind group with $|H| = |K| \geq 4$.

If there exists $C \leq P$ with $|C| = 2$ and $C \cap H = \langle 1 \rangle$, then $P = H \times C$ with $C \triangleleft G$, so $G = AH \times C$, a contradiction. Then P is isomorphic to Q_3 and G is as in 5) with $n = 3$.

QED

Remark 3. Theorem I in [6] shows that among the groups presented in Theorem 4 only the alternating group A_4 and the groups of type 2) with $|A| = q$ (q prime) have just two conjugacy classes of non-normal subgroups.

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