Groups with few non-((locally finite)-by-Baer) subgroups

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Abstract. In this note we study groups with few non-((locally finite)-by-Baer) subgroups
and we prove that if $G$ is a locally graded group satisfying the minimal condition on non-
((locally finite)-by-Baer) subgroups or having finitely many conjugacy classes of non-((locally
finite)-by-Baer) subgroups, then $G$ is a (locally finite)-by-Baer group. We prove also that if $G$
is a minimal non-((locally finite)-by-Baer) group then $G$ is a finitely generated perfect group
which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple
group, where $Frat(G)$ stands for the Frattini subgroup of $G$.

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subgroups, minimal condition on subgroups.

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1 Introduction

The aim of this paper is to study groups that in some sense have few non-((locally finite)-by-Baer)
subgroups, namely minimal non-((locally finite)-by-Baer) groups, or groups satisfying
the minimal condition on non-((locally finite)-by-Baer) subgroups, or groups having finitely
many conjugacy classes of non-((locally finite)-by-Baer) subgroups. Recall that a group $G$ is
said to be a Baer group if all its cyclic subgroups are normal in $G$.

Let $\mathcal{X}$ be a class of groups. A group $G$ is said to be a minimal non-$\mathcal{X}$-group if it is not
an $\mathcal{X}$-group but all of whose proper subgroups are $\mathcal{X}$-groups. Many results have been obtained
on minimal non-$\mathcal{X}$-groups, for various choices of $\mathcal{X}$. In particular, in [5] (respectively, in [9]) it
is proved that if $G$ is a finitely generated minimal non-nilpotent (respectively, non-(finite-by-nilpotent)
group, then $G$ is a perfect group which has no proper subgroup of finite index and
such that $G/Frat(G)$ is an infinite simple group, where $Frat(G)$ denotes the Frattini subgroup
of $G$. Also in [7] it is proved that if $G$ is a minimal non-(periodic-by-nilpotent) (respectively,
non-((locally finite)-by-nilpotent)) group, then $G$ is a finitely generated perfect group which
has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group. We generalize this last result to minimal non-((locally finite)-by-Baer) groups. We will prove:

**Theorem 1.** If $G$ is a minimal non-((locally finite)-by-Baer) group, then $G$ is a finitely generated perfect group which has no proper subgroup of finite index and such that $G/Frat(G)$ is an infinite simple group.

Groups satisfying the minimal condition on non-$X$-subgroups or groups having finitely many conjugacy classes of non-$X$-subgroups have been studied by several authors for many choices of $X$. In particular, in [8] it is proved that if $G$ is a locally graded group satisfying the minimal condition on non-((locally finite)-by-nilpotent) subgroups or having finitely many conjugacy classes of non-((locally finite)-by-nilpotent) subgroups, then $G$ is (locally finite)-by-nilpotent. We generalize this result to (locally finite)-by-Baer groups. We will prove:

**Theorem 2.** Let $G$ be a locally graded group satisfying the minimal condition on non-((locally finite)-by-Baer) subgroups. Then $G$ is (locally finite)-by-Baer.

**Theorem 3.** Let $G$ be a locally graded group having finitely many conjugacy classes of non-((locally finite)-by-Baer) subgroups. Then $G$ is (locally finite)-by-Baer.

### 2 Proof of Theorem 1

**Lemma 1.** Let $G = \langle g, x \rangle$ be a torsion-free nilpotent group and let $n, k$ be positive integers. If $[(g)^n, (x^k)] = 1$ then $(g)^n(x^k) = 1$.

**Proof.** Put $H_1 = K_1 = \langle g \rangle$, $H_2 = \langle x \rangle$ and $K_2 = \langle x^n \rangle$. Then $H_i, K_i$ are subgroups of $G$ such that $K_i$ is a subgroup of $H_i$ of finite index. We deduce by [4, Theorem 2.3.3] that $[K_1, K_2] = [(g)^n, (x^k)]$ is of finite index in $[H_1, K_2] = [(g), (x)]$. Thus $[(g)^n, (x^k)]$ is of finite order and hence it is trivial since $G$ is torsion-free.

**Lemma 2.** Let $G$ be a torsion-free locally nilpotent group. If $G$ has a Baer subgroup of finite index, then $G$ is a Baer group.

**Proof.** Let $H$ be a normal Baer subgroup of $G$ of finite index, say $n$. So $x^n \in H$ for all $x \in G$. Therefore $(x^n)$ is normal in $G$ and hence $[G, (x^n)] = 1$ for some positive integer $k$ (see [4, p.276]). Thus $[(g)^n, (x^k)] = 1$ for all $g \in G$. By Lemma 1, we deduce that $[(g)^n, (x^k)] = 1$ for all $g \in G$, hence $[G, (x)] = 1$. It follows that $(x)$ is normal in $G$ (see [4, p.276]) which gives that $G$ is a Baer group.

**Proposition 1.** Let $G$ be a torsion-free locally graded group. If every proper subgroup of $G$ is a Baer group, then so is $G$.

**Proof.** If $G$ is finitely generated then it is nilpotent-by-finite as it is locally graded. So it satisfies the maximal condition on subgroups. Consequently, every proper subgroup of $G$ is nilpotent. Hence $G$ is nilpotent since by [5] a finitely generated minimal non-nilpotent group has no proper subgroup of finite index. Therefore $G$ is a Baer group. Assume now that $G$ is not finitely generated. Hence $G$ is locally nilpotent. Let $B$ denote the Baer radical of $G$. If $G$ is not a Baer group, then $B$ is proper in $G$. Since $B$ contains all subnormal Baer subgroups of $G$, $G/B$ must be a simple group. Hence it is cyclic of prime order since it is locally nilpotent [6, Corollary 1 of Theorem 5.27] and this is a contradiction to Lemma 2. Therefore, $G$ is a Baer group.
Proposition 2. Let $G$ be a group in which every proper subgroup is (locally finite)-by-Baer. Then $G$ is (locally finite)-by-Baer if it satisfies one of the following two conditions:

(i) $G$ is finitely generated and has a proper subgroup of finite index, or
(ii) $G$ is not finitely generated.

Proof. (i) Suppose that $G$ is finitely generated and let $H$ be a proper normal subgroup of finite index in $G$. So $H$ is finitely generated and hence it is (locally finite)-by-nilpotent. It follows that $\gamma_{k+1}(H)$ is locally finite for some integer $k \geq 0$. Clearly, $G/\gamma_{k+1}(H)$ is a finitely generated nilpotent-by-finite group. So it satisfies the maximal condition on subgroups. Consequently, every proper subgroup of $G/\gamma_{k+1}(H)$ is finite-by-nilpotent. In [2, Lemma 4] it is proved that a finitely generated locally graded group in which every proper subgroup is finite-by-nilpotent is itself finite-by-nilpotent. We deduce that $G/\gamma_{k+1}(H)$ is finite-by-nilpotent which gives that $G$ is (locally finite)-by-nilpotent, as claimed.

(ii) Suppose now that $G$ is not finitely generated and let $x, y$ be two elements of finite order in $G$. The subgroup $(x, y)$, being proper in $G$, is (locally finite)-by-Baer hence it is finite. Thus $xy^{-1}$ is of finite order, so $G$ has a torsion subgroup $T$ which is locally finite as $G$ is not finitely generated. If $G/T$ is not finitely generated, then it is a torsion-free locally nilpotent group in which every proper subgroup is a Baer group. By Proposition 1, we deduce that $G/T$ is a Baer group hence $G$ is (locally finite)-by-Baer, as desired. Now if $G/T$ is finitely generated, then there exists a finitely generated subgroup $X$ of $G$ such that $G = XT$. We deduce that $G/T$ is nilpotent since $X$ is proper in $G$. Therefore, $G$ is (locally finite)-by-nilpotent, as required.

The previous proposition admits the following immediate consequence.

Corollary 1. If $G$ is a locally graded group in which every proper subgroup is (locally finite)-by-Baer, then $G$ is (locally finite)-by-Baer.

Proof of Theorem 1. Let $G$ be a minimal non-((locally finite)-by-Baer) group. By Proposition 2, $G$ is a finitely generated group which has no proper subgroup of finite index. So $G/Frat(G)$ is infinite and $G$ has no proper locally graded factor group. In particular, $G$ is perfect. Suppose that $G/Frat(G)$ is not simple and let $N$ be a normal subgroup of $G$ such that $\mathrm{Frat}(G) \leq N \leq G$. Therefore there is a maximal subgroup $M$ of $G$ such that $N \nleq M$ hence $G = MN$. Now $G/N \simeq M/M \cap N$ is (locally finite)-by-nilpotent hence $G/N$ is locally graded which is a contradiction by the above remark. Therefore $G/Frat(G)$ is simple.

3 Proof of Theorem 2

Lemma 3. Let $G$ be a torsion-free locally graded group satisfying the minimal condition on non-Baer subgroups. Then $G$ is a Baer group.

Proof. If $G$ is not a Baer group, then it has a subgroup which is a minimal non-Baer group and this is a contradiction to Proposition 1. Therefore $G$ is a Baer group.

Proof of Theorem 2. (i) First suppose that $G$ is finitely generated. If $G$ is infinite, there is a chain of subgroups, $G > G_1 > G_2 > ...$, such that $|G_i : G_{i+1}|$ is finite for all integers $i$. Thus there exists a positive integer $k$ such that $G_k$ is (locally finite)-by-Baer and $|G : G_k|$ is finite. So $G_k$ is (locally finite)-by-nilpotent. Therefore there is an integer $c \geq 0$ such that $\gamma_{c+1}(G_k)$ is locally finite. Now $G/\gamma_{c+1}(G_k)$ is a finitely generated nilpotent-by-finite group. So it satisfies the maximal condition on subgroups. It follows that $G/\gamma_{c+1}(G_k)$ satisfies the minimal condition on non-(finite-by-nilpotent) subgroups. But Lemma 4 of [2] can easily be
extended to that a finitely generated locally graded group which satisfies the minimal condition on non-(finite-by-nilpotent) subgroups is itself finite-by-nilpotent. Consequently, $G/\gamma_{c+1}G_k$ is finite-by-nilpotent hence $G$ is (locally finite)-by-nilpotent, as required.

(ii) Now assume that $G$ is not finitely generated. So $G$ has a torsion subgroup $T$ which is locally finite. By (i) every finitely generated subgroup of $G$ is (locally finite)-by-nilpotent. So $G$ has a torsion subgroup $T$ which is locally finite and $G/T$ is a torsion-free locally nilpotent group satisfying the minimal condition on non-Baer subgroups. It follows by Lemma 3 that $G/T$ is a Baer group which gives that $G$ is a (locally finite)-by-Baer group.

\section{Proof of Theorem 3}

\textbf{Lemma 4.} Let $G$ be a torsion-free locally graded group having finitely many conjugacy classes of non-Baer subgroups. Then $G$ is a Baer group.

\textbf{Proof.} In [3, Proposition 3.3] it is proved that if $\mathcal{X}$ is a subgroup closed class of groups and $K$ is a locally graded group having finitely many conjugacy classes of non-$\mathcal{X}$-subgroups, then $K$ is locally in the class $\mathcal{X}\mathcal{F}$, where $\mathcal{F}$ denotes the class of finite groups. So every finitely generated subgroup of $G$ is a Baer-by-finite group hence it is nilpotent-by-finite. So $G$ satisfies locally the maximal condition on subgroups. Now Lemma 4.6.3 of [1] states that if $K$ is a group locally satisfying the maximal condition on subgroups and if $H$ is a subgroup of $K$ such that $H^x \leq H$ for some element $x$ of $K$, then $H^x = H$. We deduce that $G$ satisfies the minimal condition on non-Baer subgroups. Now Lemma 3 gives that $G$ is a Baer group. \hfill \qed

\textbf{Lemma 5.} Let $\mathcal{X}$ be a quotient closed class of groups and let $G$ be a group having finitely many conjugacy classes of non-$\mathcal{X}$-subgroups. If $N$ is a normal subgroup of $G$ then $G/N$ has finitely many conjugacy classes of non-$\mathcal{X}$-subgroups.

\textbf{Proof.} This follows from the fact that if $N \leq K \leq G$, then

$$\{(K/N)^xN : xN \in G/N\} \subseteq \{K^x/N : x \in G\}.$$ \hfill \qed

\textbf{Proof of Theorem 3.} (i) First assume that $G$ is finitely generated. So by [3, Proposition 3.3] $G$ is (locally finite)-by-nilpotent)-by-finite. Let $N$ be a normal subgroup of $G$ of finite index such that $N$ is (locally finite)-by-nilpotent and let $T$ its torsion subgroup. So $T$ is locally finite and $G/T$ is a finitely generated nilpotent-by-finite group having finitely many conjugacy classes of non-(locally finite)-by-Baer subgroups by Lemma 5. Hence $G/T$ has finitely many conjugacy classes of non-(finite-nilpotent) subgroups. But in [8, Proposition 1.1] it is proved that a finitely generated locally graded group which has finitely many conjugacy classes of non-(finite-nilpotent) subgroups is itself finite-nilpotent. Consequently, $G/T$ is finite-nilpotent, which gives that $G$ is (locally finite)-by-nilpotent, as claimed.

(ii) Now assume that $G$ is not finitely generated. So by (i) every finitely generated subgroup of $G$ is (locally finite)-by-nilpotent. Hence $G$ has a torsion subgroup $T$ such that $T$ is locally finite and $G/T$ is a locally nilpotent group having finitely many classes of non-Baer subgroups. We deduce by Lemma 4 that $G/T$ is a Baer group which implies that $G$ is (locally finite)-by-Baer, as claimed. \hfill \qed
References


