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Existence and uniqueness of solutions of certain second order nonlinear equations

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Abstract. The existence, uniqueness and other properties of solutions of nonlinear second order Volterra integrodifferential equation in a general Banach space are studied. The main tools in our analysis is based on the theory of the strongly continuous cosine family, a modified version of contraction mapping principle and the integral inequality established by B. G. Pachpatte.

Keywords: Volterra integrodifferential equation, uniqueness, continuous dependence, cosine and sine family, integral inequality

MSC 2000 classification: 34G20, 47D09, 47H09, 35A23

1 Introduction

The problems of existence, uniqueness and other properties of solutions for the second order systems have much attention in the recent years. It is advantageous to treat second order abstract differential equations directly rather than to convert into first order systems (refer, Fitzgibbon [8]). Fitzgibbon [8] used the second order abstract system for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine family of operators. We will make use of some of the basic ideas from cosine family theory [7, 9, 19, 21, 22]. Motivation for second order systems can be found in [7, 12, 16, 19, 20].

We consider the abstract nonlinear integrodifferential equation of the type:

$$x''(t) = Ax(t) + f(t, x(t), \int_{t_0}^{t} k(t, s, x(s))ds), \quad 0 \le t_0 \le T \tag{1}$$

$$x(t_0) = x_0, \quad x^{'}(t_0) = y_0, \tag{2}$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space X, $f : [t_0, T] \times X \times X \to X$, $k : [t_0, T] \times [t_0, T] \times X \to X$ are continuous functions and x_0, y_0 are given elements of X.

The equations of these types or their special forms commonly come across in almost all phases of physics and other areas of applied mathematics, see, for example [1, 2, 3, 5] and the references given therein. The problems of existence, uniqueness, continuation and other properties of solutions of various special forms (1)–(2) have been extensively studied by using different techniques during last few years see, [4, 6, 10, 13, 15, 19, 22] and the reference listed therein. The theorems proved in this paper generalize some results obtained by A. Pazy [17], and C. C. Travis and G. F. Webb [21]. The abstract results in this work are applicable to "partial" second order integrodifferential equation, see Section 4.

The paper is organized as follows: In Section 2, we present the preliminaries and hypotheses. Section 3 deals with the our main results. In section 4, we give an example to illustrate the applications of some our results established in Section 3.

2 Preliminaries and Hypotheses

We introduce notations, definitions and preliminary facts that will be used throughout the paper.

Let X be a Banach space with norm $\|\cdot\|$. Let $B = C([t_0, T], X)$ be the Banach space of all continuous functions from $[t_0, T]$ into X endowed with supremum norm

$$\|x\|_B := \sup\{\|x(t)\| : t \in [t_0, T]\}.$$

Definition 1. A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in the Banach space X is called a strongly continuous cosine family if and only if

(a) $C(0) = I$ (I is the identity operator);

(b) $C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;

(c) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t,\ s \in \mathbb{R}$.

If $\{C(t) : t \in \mathbb{R}\}$ is a strongly continuous cosine family in X, then we define the associated sine family $\{S(t) : t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \tag{3}$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}$. Let $M \geq 1$ and N be two positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for all $t \in [0, T]$.

Definition 2. Let $f \in L^1(t_0, T; X)$. The function $x \in B$ defined by

$$\begin{aligned} x(t) &= C(t - t_0)x_0 + S(t - t_0)y_0 \\ &+ \int_{t_0}^t S(t - s)f(s, x(s), \int_{t_0}^s k(s, \tau, x(\tau))d\tau)ds, \quad t \in [t_0, T] \end{aligned} \tag{4}$$

is called mild solution of the initial value problem (1)–(2).

We list the following hypotheses for our convenience.

(H_1) For $t, s \in [t_0, T]$ and $x_i, y_i \in X$, $i = 1, 2$, there exist nonnegative constants L, K such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L[\|x_1 - x_2\| + \|y_1 - y_2\|],$$

and

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq K\|x_1 - x_2\|.$$

(H_2) There exist two continuous functions $p, q : [t_0, T] \to \mathbb{R}^+$ such that

$$\|f(t, x, y)\| \leq p(t)[\|x\| + \|y\|],$$

and

$$\|k(t, s, x)\| \leq q(t)\|x\|,$$

for all $x, y \in X$ and $t, s \in [t_0, T]$.

We require the following Lemmas in our further discussion.

Lemma 1 ([18], p. 196)**.** *Let X be a Banach space. Let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is positive integer. Then D has a unique fixed point in X.*

Lemma 2 ([14], p. 758)**.** *Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}^+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t p(s)[u(s) + \int_0^s q(\tau)u(\tau)d\tau]ds,$$

holds for all $t \in \mathbb{R}^+$, where u_0 is a nonnegative constant, then

$$u(t) \leq u_0[1 + \int_0^t p(s)\exp\left(\int_0^s (p(\tau) + q(\tau))d\tau\right)ds],$$

holds for all $t \in \mathbb{R}^+$.

3 Existence and Uniqueness of Mild Solution

Theorem 1. *Let the hypothesis (H_1) be satisfied. Then for each $x_0, y_0 \in X$, the initial value problem (1)–(2) has a unique mild solution $x \in B$ on $[t_0, T]$. Moreover, the mapping $(x_0, y_0) \to x$ is Lipschitz continuous from $X \times X$ into B.*

Proof. Define a mapping $F : B \to B$ by

$$\begin{aligned}(Fx)(t) &= C(t - t_0)x_0 + S(t - t_0)y_0 \\ &\quad + \int_{t_0}^t S(t - s)f(s, x(s), \int_{t_0}^s k(s, \tau, x(\tau))d\tau)ds, \quad t \in [t_0, T]. \end{aligned} \tag{5}$$

We observe that the mild solution of the equations (1)–(2) is a fixed point of the operator equation $Fx = x$. Let $x,\ y \in B$ and using equation (5), and the hypothesis, we obtain

$$\begin{aligned}\|(Fx)(t) - (Fy)(t)\| &\leq \int_{t_0}^{t} \|S(t-s)\| \|f(s, x(s), \int_{t_0}^{s} k(s,\tau,x(\tau))d\tau) \\ &\quad - f(s, y(s), \int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)\|ds \\ &\leq N \int_{t_0}^{t} L[\|x-y\|_B + K\|x-y\|_B \int_{t_0}^{s} d\tau]ds \\ &\leq N \int_{t_0}^{t} L[\|x-y\|_B + K\|x-y\|_B (s-t_0)]ds \\ &\leq N(t-t_0)[L + LK\frac{(t-t_0)}{2}]\|x-y\|_B. \end{aligned} \tag{6}$$

Similarly by using the equations (5), (6) and the hypothesis, we get

$$\begin{aligned}&\|(F^2x)(t) - (F^2y)(t)\| \\ &= \|(F(Fx))(t) - (F(Fy))(t)\| \\ &= \|(Fx_1)(t) - (Fy_1)(t)\| \\ &\leq \int_{t_0}^{t} \|S(t-s)\| \|f(s, x_1(s), \int_{t_0}^{s} k(s,\tau,x_1(\tau))d\tau) \\ &\quad - f(s, y_1(s), \int_{t_0}^{s} k(s,\tau,y_1(\tau))d\tau)\|ds \\ &\leq NL \int_{t_0}^{t} \|x_1(s) - y_1(s)\|ds + NL \int_{t_0}^{t} K \int_{t_0}^{s} \|x_1(\tau) - y_1(\tau)\|d\tau ds \\ &= NL \int_{t_0}^{t} \|(Fx)(s) - (Fy)(s)\|ds + NL \int_{t_0}^{t} K \int_{t_0}^{s} \|(Fx)(s) - (Fy)(s)\|d\tau ds \\ &\leq NL[NL\frac{(t-t_0)^2}{2!} + NLK\frac{(t-t_0)^3}{3!}]\|x-y\|_B \\ &\quad + NLK[NL\int_{t_0}^{t} \frac{(s-t_0)^2}{2!}ds + NLK\int_{t_0}^{t} \frac{(s-t_0)^3}{3!}ds]\|x-y\|_B \\ &\leq N^2\frac{(t-t_0)^2}{2!}[L^2 + 2L^2K\frac{(t-t_0)}{3} + L^2K^2\frac{(t-t_0)^2}{4 \times 3}]\|x-y\|_B \\ &\leq N^2\frac{(t-t_0)^2}{2!}[L^2 + 2L^2K\frac{(t-t_0)}{2!} + L^2K^2\frac{(t-t_0)^2}{4}]\|x-y\|_B \\ &\leq \frac{(t-t_0)^2}{2!}[N(L + LK\frac{(t-t_0)}{2})]^2\|x-y\|_B. \end{aligned} \tag{7}$$

By making use of the equations (5), (7) and iteration it follows that

$$\begin{aligned}\|(F^nx)(t) - (F^ny)(t)\| &\leq \frac{(t-t_0)^n}{n!}[N(L + LK\frac{(t-t_0)}{2!})]^n\|x-y\|_B \\ &\leq \frac{1}{n!}[TN(L + \frac{LKT}{2})]^n\|x-y\|_B,\end{aligned}$$

which yields

$$\|F^nx - F^ny\|_B \leq \frac{1}{n!}[TN(L + \frac{LKT}{2})]^n\|x-y\|_B. \tag{8}$$

For n large enough, $\frac{1}{n!}[TN(L+\frac{LKT}{2})]^n < 1$. Thus, there exists a positive integer n such that F^n is a contraction in B. From Lemma 1, it follows that F has a unique fixed point, say $x \in B$. This fixed point x is the required mild solution of (1)–(2).

Suppose that y is another mild solution of the initial value problem (1) with $y(t_0) = {x^*}_0, y^{'}(t_0) = {y^*}_0$ on $[t_0, T]$. Using the equation (4) and the hypothesis (H_1), we have

$$\begin{aligned} \|x(t)-y(t)\| &\le \|C(t-t_0)\|\|x_0 - {x^*}_0\| + \|S(t-t_0)\|\|y_0 - {y^*}_0\| \\ &\quad + \int_{t_0}^{t} \|S(t-s)\|\|f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau) \\ &\quad - f(s,y(s),\int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)\|ds \\ &\le [M\|x_0 - {x^*}_0\| + N\|y_0 - {y^*}_0\|] \\ &\quad + \int_{t_0}^{t} NL[\|x(s)-y(s)\| + \int_{t_0}^{s} K\|x(\tau)-y(\tau)\|d\tau]ds. \end{aligned} \tag{9}$$

By applying Lemma 2 known as the Pachpatte's inequality with $u(t) = \|x(t) - y(t)\|$ and $u_0 = 0$ to the inequality (9), we get

$$\begin{aligned} \|x(t)-y(t)\| &\le [M\|x_0 - {x^*}_0\| + N\|y_0 - {y^*}_0\|] \\ &\quad \times [1 + \int_{t_0}^{t} NL\exp(\int_{t_0}^{s}(NL+K)d\tau)ds], \end{aligned}$$

which yields

$$\begin{aligned} \|x-y\|_B &\le [M\|x_0 - {x^*}_0\| + N\|y_0 - {y^*}_0\|] \\ &\quad \times [1 + \int_{t_0}^{t} NL\exp(\int_{t_0}^{s}(NL+K)d\tau)ds]. \end{aligned} \tag{10}$$

This proves that the uniqueness of x, i. e. for $x_0, y_0 \in X$, the initial value problem (1)–(2) has a unique mild solution $x \in B$ on $t_0 \le t \le T$ and also Lipschitz continuity of the mapping $(x_0, y_0) \to x$. This completes the proof of the Theorem 1. QED

Theorem 2. *Let the hypothesis* (H_2) *be satisfied. Then all solutions of* (1)–(2) *are bounded on* $[0, T]$

Proof. Let

$$\begin{aligned} x(t) &= C(t-t_0)x_0 + S(t-t_0)y_0 \\ &\quad + \int_{t_0}^{t} S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \quad t \in [t_0, T] \end{aligned} \tag{11}$$

be a solution of (1)–(2). Using hypothesis (H_2), we have

$$\begin{aligned} \|x(t)\| &\le \|C(t-t_0)\|\|x_0\| + \|S(t-t_0)\|\|y_0\| \\ &\quad + \int_{t_0}^{t} \|S(t-s)\|\|f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)\|ds \\ &\le M\|x_0\| + N\|y_0\| + \int_{t_0}^{t} Np(s)[\|x(s)\| + \int_{t_0}^{s} q(\tau)\|x(\tau)\|d\tau]ds. \end{aligned} \tag{12}$$

Applying Lemma 2, with $u(t) = \|x(t)\|$, we get

$$\begin{aligned}\|x(t)\| &\leq [M\|x_0\| + N\|y_0\|][1 + \int_{t_0}^{t} Np(s)\exp\big(\int_{t_0}^{s}[Np(\tau) + q(\tau)]d\tau\big)ds] \\ &\leq [M\|x_0\| + N\|y_0\|][1 + \int_{0}^{t} NP\exp\big(T[NP + Q]\big)ds] \\ &\leq [M\|x_0\| + N\|y_0\|][1 + TNP\exp\big(T[NP + Q]\big)], \qquad (13)\end{aligned}$$

where

$$P = \max_{t\in[0,T]} p(t) \quad \text{and} \quad Q = \max_{t\in[0,T]} q(t).$$

Thus, the boundedness of $x(t)$ follows from inequality (13). This completes the proof of the Theorem 2. $\boxed{QED}$

Remark 1. It is important to note that Theorem 2 proves not only the boundedness, but also the stability of $x(t)$, if $\|x_0\|, \|y_0\|$ are small enough.

Theorem 3. *Let the hypothesis* (H_1) *be satisfied and* $x_0, y_0 \in X$. *Suppose that the functions* $x_1(t)$ *and* $x_2(t)$ *satisfy the equation* (1) *for* $t_0 \leq t \leq T$ *with* $x_1(t_0) = x_0{}^*$, $x'_1(t_0) = y_0{}^*$ *and* $x_2(t_0) = x_0{}^{**}$, $x'_2(t_0) = y_0{}^{**}$, *respectively and* $x_1(t), x_2(t) \in B$, *then*

$$\|x_1 - x_2\|_B \leq [M\|x_0{}^* - y_0{}^*\| + N\|x_0{}^{**} - y_0{}^{**}\|][1 + TNL\exp\big((NL + K)T\big)].$$

Proof. The continuous dependence of solutions depends upon the initial data can be proved as in the last part of the proof of Theorem 1. Hence, we omit the details. This completes the proof of the Theorem 3. $\boxed{QED}$

Remark 2. In general, cosine family $C(t)$ and sine family $S(t)$ are not bounded in $\mathbb{R}$. They are bounded only in a finite interval, and may be exponential growth in $\mathbb{R}^+$. Therefore, the all solutions (1)-(2) are need not bounded on $\mathbb{R}^+$.

4 Example

In order to illustrate the applications of some of our result established in previous section, we consider the following partial nonlinear differential equation of the form:

$$\begin{aligned}\frac{\partial^2 w(t,u)}{\partial t^2} &= \frac{\partial^2 w(t,u)}{\partial u^2} + \frac{w(t,u)\sin(w(t,u))}{(1+t)(1+t^2)} \\ &\quad + \int_0^t \frac{sw(s,u)}{(1+t)}ds, \quad t \in [0,1], \quad u \in I = [0,\pi], \qquad (14)\end{aligned}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in [0,1], \qquad (15)$$

$$w(0,u) = x_0(u), \quad u \in I, \qquad (16)$$

$$\frac{\partial w(t,u)}{\partial t}\Big|_{t=0} = y_0(u), \quad u \in I, \qquad (17)$$

Let us take $X = L^2([0,\pi])$ and w(t,u)=x(t)(u). Since

$$f(t, x(t), \int_{t_0}^{t} k(t,s,x(s))ds) = \frac{x(t)\sin(x(t))}{(1+t)(1+t^2)} + \int_0^t \frac{sx(s)}{(1+t)}ds$$

and

$$k(t,s,x(s)) = \frac{sx(s)}{(1+t)},$$

we have

$$\begin{aligned} &\|f(t,x_1,Kx_1) - f(t,x_2,Kx_2)\| \\ &\qquad \leq \frac{2}{(1+t)(1+t^2)}\|x_1 - x_2\| + \int_0^t \frac{s\|x_1(s) - x_2(s)\|}{(1+t)}ds \\ &\qquad \leq \frac{2}{(1+t)(1+t^2)}\|x_1 - x_2\|_B + \frac{t^2}{2(1+t)}\|x_1 - x_2\|_B \\ &\qquad \leq L\|x_1 - x_2\|_B, \end{aligned}$$

where

$$L = \max_{t\in[0,1]}\{\frac{2}{(1+t)(1+t^2)}, \frac{t^2}{2(1+t)}\} \quad \text{and} \quad Kx := \int_{t_0}^t k(t,s,x(s))ds.$$

Also, we obtain

$$\begin{aligned} \|f(t,x,Kx)\| &\leq \frac{1}{(1+t)(1+t^2)}\|x\|_B + \int_0^t \frac{s\|x\|_B}{(1+t)}ds \\ &\leq [\frac{1}{(1+t)(1+t^2)} + \frac{t^2}{2(1+t)}]\|x\|_B \\ &\leq p(t)\|x\|_B, \end{aligned}$$

where

$$p(t) = [\frac{1}{(1+t)(1+t^2)} + \frac{t^2}{2(1+t)}].$$

Similarly, we can estimate for the function k:

$$\begin{aligned} \|k(t,s,x_1) - k(t,s,x_2)\| &\leq \frac{s}{(1+t)}\|x_1 - x_2\| \\ &\leq K\|x_1 - x_2\|_B, \end{aligned}$$

where

$$K = \sup_{0\leq s\leq t\leq 1}\{\frac{s}{(1+t)}\},$$

and for $0 \leq s \leq t \leq 1$

$$\|k(t,s,x)\| \leq \frac{s}{(1+t)}\|x\|_B \leq q(t)\|x\|_B,$$

where

$$q(t) = \frac{t}{(1+t)}.$$

We define the operator $A : D(A) \subset X \to X$ by $Aw = w_{uu}$, where $D(A) = \{w(\cdot) \in X : w(0) = w(\pi) = 0\}$. It is well known that A is the generator of strongly continuous cosine function $\{C(t) : t \in \mathbb{R}\}$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with corresponding normalized characteristics vectors $w_n(u) := \sqrt{\frac{2}{\pi}}\sin(nu),\ n = 1,2,3...$, and the following conditions hold :

(1) $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(2) If $w \in D(A)$ then $Aw = -\sum_{n=1}^{\infty} n^2 < w, w_n > w_n$.

(3) For $w \in X$, $C(t)w = \sum_{n=1}^{\infty} \cos(nt) < w, w_n > w_n$. Moreover, from these expression, it follows that $S(t)w = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < w, w_n > w_n$, that $S(t)$ is compact for every $t > 0$ and that $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in [0, 1]$.

(4) If H denotes the group of translations on X defined by $H(t)x(u) = \tilde{x}(u + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}\Big(H(t) + H(-t)\Big)$. If $G : X \to X$ is defined by $Gx = x^{'}$, $D(G) = \{x \in X : x^{'} \in X\}$, then it follows that $A = G^2$, where G is the infinitesimal generator of the group H , see [7, 11].

With this choice of A, f and k, we observe that the equations (1)–(2) is an abstract formulation of (14)–(17) and the reported Theorems, therefore, can be applied to guarantee the existence, uniqueness and other properties of solutions of the nonlinear partial integrodifferential equation (14)–(17).

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