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Recent Results on Generalized Baumslag-Solitar Groups

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Abstract. Generalized Baumslag-Solitar groups are the fundamental groups of finite graphs of groups with infinite cyclic vertex and edge groups. These groups have interesting group theoretic and algorithmic properties and they also have close connections with algebraic topology. Here we present an introduction to the theory with an account of recent results.

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1 Graphs of Groups

Let Γ be a connected graph, with loops and multiple edges allowed, and write

$$V(\Gamma) \quad \text{and} \quad E(\Gamma)$$

for the respective sets of vertices and edges of Γ. If $e \in E(\Gamma)$, we assign endpoints e^+, e^- and hence a direction to e,

$$\bullet_{e^-} \longrightarrow \bullet_{e^+}$$

To each $e \in E(\Gamma)$ and $x \in V(\Gamma)$ we assign groups H_e and G_x and we assume there are injective homomorphisms

$$\phi_{e^-} : H_e \to G_{e^-} \quad \text{and} \quad \phi_{e^+} : H_e \to G_{e^+}.$$

Then the system

$$\mathcal{G} = (\Gamma, \phi_{e^-}, \phi_{e^+}, H_e, G_x \mid e \in E(\Gamma),\ x \in V(\Gamma)),$$

ia called a *graph of groups.*

Next choose a maximal subtree T in Γ. Then the *fundamental group* of the graph of groups $\mathcal{G}$ is the group

$$G = \pi_1(\mathcal{G})$$

which is generated by the groups and elements

$$G_x \text{ and } t_e, \quad (x \in V(\Gamma),\ e \in E(\Gamma \backslash T)),$$

subject to the defining relations

$$h^{\phi_{e^-}} = h^{\phi_{e^+}}, \quad (e \in E(T)), \quad (h^{\phi_{e^+}})^{t_e} = h^{\phi_{e^-}}, \quad (e \in E(\Gamma \backslash T)),$$

for all $h \in H_e$. In the case where Γ is a tree, G is called a *tree product.*

The following result is fundamental – see [3], [5], [15].

(1.1). *Up to isomorphism the group $G = \pi_1(\mathcal{G})$ is independent of the choice of maximal subtree.*

Special cases of interest

(i) Let Γ have two vertices and a single edge e. Then G is the generalized free product

$$G = G_{e^-} *_H \ G_{e^+}$$

where the subgroup $H = H_e$ is amalgamated by means of the injective homomorphisms ϕ_{e^-} and ϕ_{e^+}.

(ii) Let Γ have one vertex x and one edge e, i.e., it is a loop. Then G is the HNN-extension

$$G =< t_e, G_x \mid (h^{\phi_{e^+}})^{t_e} = h^{\phi_{e^-}},\ h \in H_e > .$$

Here G_x is the base group, $H^{\phi_{e^-}}$ and $H^{\phi_{e^-}}$ are the associated subgroups, and t_e is the stable element.

We note an important property of graphs of groups.

(1.2). *Let $\mathcal{G} = (\Gamma, \phi_{e^-}, \phi_{e^+}, H_e, G_x \mid e \in E(\Gamma),\ x \in V(\Gamma))$ be a graph of groups and let Γ_0 be a connected subgraph of Γ. Define $G_0 = \pi_1(\mathcal{G}_0)$ where*

$$\mathcal{G}_0 = (\Gamma_0, \phi_{e^-}, \phi_{e^+}, H_e, G_x \mid e \in E(\Gamma_0), x \in V(\Gamma_0)).$$

Then the natural homomorphism from G_0 to G is injective. In particular each G_x is isomorphic with a subgroup of G.

For a detailed account of the theory of graphs of groups the reader may consult [3], [5], [15].

2 Generalized Baumslag-Solitar groups

A *Baumslag-Solitar group* is a 1-relator group with a presentation of the form

$$BS(m,n) =< t, x \mid (x^m)^t = x^n >,$$

where $m, n \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$: these groups seem to have first appeared in the literature in [1], but they may be of greater antiquity.

A similar type of 1-relator group is

$$K(m,n) =< x, y \mid x^m = y^n >,$$

where $m, n \in \mathbb{Z}^*$. When m and n are relatively prime, this is a *torus knot group*.

The groups $BS(m,n)$ and $K(m,n)$ are the fundamental groups of graphs of infinite cyclic groups where the graph is a 1-loop or a 1-edge respectively. There is a natural way to generalize these groups.

Let Γ be a finite connected graph. Associate infinite cyclic groups $< g_x >$ and $< u_e >$ to each vertex x and edge e and define injective homomorphisms

$$< u_e > \to < g_{e^-} > \text{ and } < u_e > \to < g_{e^+} >$$

by the assignments

$$u_e \ \mapsto \ (g_{e^-})^{\omega^-(e)} \text{ and } \ u_e \ \mapsto \ (g_{e^+})^{\omega^+(e)}$$

where $\omega^-(e), \omega^+(e) \in \mathbb{Z}^*$. So the edge e is assigned a *weight* $(\omega^-(e), \omega^+(e))$ and the graph of groups is determined by a *weight function*

$$\omega : E(\Gamma) \to \mathbb{Z}^* \times \mathbb{Z}^*,$$

with values

$$\omega(e) = (\omega^-(e),\ \omega^+(e)).$$

We will write the weighted graph of infinite cyclic groups in the form

$$(\Gamma, \omega)$$

and refer to it as a *generalized Baumslag-Solitar graph* or *GBS-graph*.

Definition 1. A *generalized Baumslag-Solitar group*, or *GBS-group*, is the fundamental group of a GBS-graph (Γ, ω), in symbols

$$G = \pi_1(\Gamma, \omega).$$

To obtain a presentation of G choose a maximal subtree T in Γ; then G has generators

$$t_e,\ g_x,\quad e \in E(\Gamma\backslash T),\quad x \in V(\Gamma),$$

and defining relations

$$(g_{e^-})^{\omega^-(e)} = (g_{e^+})^{\omega^+(e)},\quad e \in E(T),$$

$$(g_{e^-})^{\omega^-(e)} = ((g_{e^+})^{\omega^+(e)})^{t_e},\quad e \in E(\Gamma\backslash T).$$

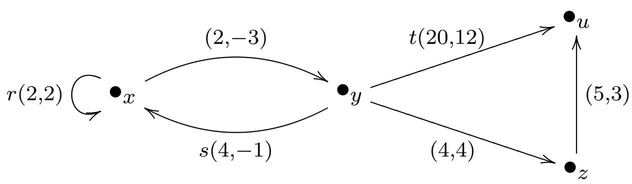
Note that up to isomorphism G does not depend on the choice of the maximal subtree.

Examples

1. If Γ is a 1-loop with weight (m,n), then $G = BS(n,m)$.

2. If Γ is a 1-edge with weight (m,n), then $G = K(m,n)$.

3. As a more complex example, consider the GBS-graph shown below.



Choose as the maximal subtree T the path $xyzu$ and let the stable letters be r, s, t as indicated. Then the corresponding GBS-group G has a presentation with generators

$$r, s, t, g_x, g_y, g_z, g_u$$

and relations

$$(g_x^2)^r = g_x^2,\ g_x^2 = g_y^{-3},\ g_y^4 = g_z^4,\ g_z^5 = g_u^3,\ (g_u^{12})^t = g_y^{20},\ (g_x^4)^s = g_y^{-1}.$$

3 Some Properties of GBS-groups

We list some known properties of GBS-groups. Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group.

(3.1). *The group G is finitely presented and torsion-free.*

For if F is a finite subgroup of G, it intersects each conjugate of a vertex group trivially, which implies that it is free and therefore trivial ([5], p.212).

(3.2). *If* Γ *is a tree, so that* G *is a GBS-tree product, then* G *is locally extended residually finite. Hence* G *is hopfian.*

Recall here that a group is *locally extended residually finite* (or LERF) if every finitely generated subgroup is closed in the profinite topology.

Proof of (3.2). Since Γ is a tree and $< g_x > \cap < g_y > \neq 1$ for all $x, y \in V(\Gamma)$, each vertex generator has a positive power lying in the centre. Hence $Z(G) =< z > \neq 1$ and $G/ < z >$ is the fundamental group of a graph of finite cyclic groups. It follows that $G/ < z >$ is virtually free (see Karrass, Pietrowski and Solitar [9]). If $n > 0$, then $G/ < z^n >$ is also virtually free. Since finitely generated free groups are LERF, (M. Hall [8]), G is LERF. □

Corollary 1. *The generalized word problem soluble in any GBS-tree product.*

GBS-tree products have another strong residual property.

(3.3). *A GBS-tree product* G *is conjugacy separable, i.e., if two elements are conjugate in every finite quotient of* G*, then they are conjugate in* G*.*

This follows from a theorem of Kim and Tang [10]: *if* G *is a (finite) tree product of groups each of which is finitely generated torsion-free nilpotent and if the amalgamations are cyclic, then* G *is conjugacy separable.*

Corollary 2. *The conjugacy problem is soluble in any GBS-tree product.*

Remark. In general $BS(m, n)$ is not hopfian, and hence is not even residually finite. For example, let $G =< t, g| \ (g^m)^t = g^n >$ where $\gcd(m, n) = 1$. Define an endomorphism θ of G by

$$t^\theta = t, \quad g^\theta = g^n.$$

Then θ is a surjective since $\mathrm{Im}(\theta)$ contains g^n and also $(g^n)^{t^{-1}} = g^m$, so $g \in \mathrm{Im}(\theta)$. But θ is not an automorphism of G if $m, n \neq \pm 1$, since $[g, g^{t^{-1}}]^\theta = 1$ and $[g, g^{t^{-1}}] \neq 1$.

The next result is an important characterization of GBS-groups due to Kropholler [11].

(3.4). *The non-cyclic GBS-groups are exactly the finitely generated groups of cohomological dimension 2 that have an infinite cyclic subgroup which is commensurable with its conjugates, i.e., intersecting each conjugate non-trivially.*

Kropholler also showed that there is a type of Tits alternative for GBS-groups, (Kropholler [11]).

(3.5). *The second derived subgroup of a GBS-group is free.*

Since free groups are residually soluble, we deduce from the last result:

Corollary 3. *Every GBS-group is residually soluble.*

The subgroups of a GBS-group are of very restricted type, as the next result shows.

(3.6). *Let H be a finitely generated subgroup of a GBS-group G. Then H is either free or a GBS-group.*

Proof. Assume that H is not free, so G is certainly non-cyclic. Now $\mathrm{cd}(H) \leq \mathrm{cd}(G) = 2$. If $\mathrm{cd}(H) = 1$, then by a result of Stallings and Swan the group H is free, since it is torsion-free: (for these results see [2], Chapter II). By this contradiction $\mathrm{cd}(H) = 2$. Now H must contain a commensurable element since otherwise it is free. Therefore by (3.4) H is a GBS-group. □

Corollary 4. *A GBS-group is coherent, i.e., all its finitely generated subgroups are finitely presented.*

Since GBS-groups have cohomological dimension 2 in general, it is natural to enquire about their (co)homology in dimensions 1 and 2. We begin with homology. Recall that

$$H_1(G) \simeq G_{ab} = G/G' \text{ and } H_2(G) \simeq M(G),$$

the Schur multiplier. We will investigate these groups in the next two sections.

4 The Abelianization of a GBS-group

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group defined with respect to a maximal subtree T of Γ. Then $G_{ab} = G/G'$ is the finitely generated abelian group with generators

$$t_e, \quad g_x \text{ where } e \in E(\Gamma \backslash T),\ x \in V(\Gamma),$$

and (abelian) defining relations

$$(g_{e^-})^{\omega^-(e)} = (g_{e^+})^{\omega^+(e)},\ e \in E(\Gamma).$$

To find the complete structure of G_{ab} the *weight matrix* W must be transformed into Smith normal form. This matrix has rows indexed by edges and columns indexed by vertices: row e has entries

$$0, \ldots, 0,\ \omega^-(e) - \omega^+(e),\ 0 \ldots, 0$$

if e is a loop, and

$$0, \dots, 0,\ \omega^-(e), 0 \dots, 0, -\omega^+(e), 0, \dots 0$$

if e is not a loop.

Let

$$r_0(G)$$

denote the *torsion-free rank* of G_{ab}, i.e., the rank of G_{ab} modulo its torsion-subgroup. A formula for $r_0(G)$ can be found without resorting to the lengthy process of determining the Smith normal form of the matrix W. Since the t_e, $(e \in E(\Gamma \backslash T))$, are linearly independent, linear algebra shows that

$$r_0(G) = |E(\Gamma)| - |E(T)| + |V(\Gamma)| - \mathrm{rank}(W) = |E(\Gamma)| + 1 - \mathrm{rank}(W).$$

Let W_0 be the submatrix of W consisting of the rows which correspond to edges of the maximal subtree T. Then W_0 gives the structure of $(G_0)_{ab}$ where $G_0 = \pi_1(T, \omega)$. Since each pair of generators of G_0 is linearly dependent, we have $r_0(G_0) = 1$ and $\mathrm{rank}(W_0) = |V(\Gamma)| - 1$. Now $\mathrm{rank}(W) = \mathrm{rank}(W_0)$ or $\mathrm{rank}(W_0) + 1$, depending on whether each non-tree row of W is linearly dependent on the rows of W_0 or not. Therefore $r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon$ where $\epsilon = 1$ if $\mathrm{rank}(W) = \mathrm{rank}(W_0)$ and otherwise $\epsilon = 0$.

Tree dependence

Let $e \in E(\Gamma \backslash T)$ and put $e^- = x$ and $e^+ = y$; then there is a unique path from x to y in T. The defining relations associated with this path lead to a relation $x^h = y^k$, $(h, k \in \mathbb{Z}^*)$. (If $x = y$, then $h = k$). Let $\omega(e) = (m, n)$, so that $x^m \equiv y^n$ mod G'. We will say that e is *T-dependent* if (m, n) is a rational multiple of (h, k), (which means that $m = n$ if $e^- = e^+$). Otherwise e is *T-independent.* If every non-tree edge of Γ is T-dependent, then (Γ, ω) is said to be *tree dependent.* By (4.1) below this property does not dependent on the tree T. If (Γ, ω) is tree dependent, then $\mathrm{rank}(W) = \mathrm{rank}(W_0)$, and otherwise $\mathrm{rank}(W) = \mathrm{rank}(W_0) + 1$. Thus we obtain:

(4.1). *Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group defined relative to a maximal subtree T. Then*

$$r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon,$$

where $\epsilon = 0$ or 1, the rule being that $\epsilon = 1$ if and only if (Γ, ω) is tree dependent.

For example, consider Example 3 above. Here the maximal subtree is the path $xyzu$. All the non-tree edges with the exception of $\langle y, x \rangle$ are T-dependent, so (Γ, ω) is not tree dependent. Therefore $\epsilon = 0$ and $r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 = 3$ by (4.1).

5 The Schur multiplier of a GBS-Group

Next we consider how to compute the Schur multiplier of a GBS-group. First recall an inequality which is valid for any finitely presented group.

(5.1). *Let G be a finitely presented group with n generators and r relations. Then*

$$n - r \leq r_0(G) - d(M(G)),$$

where $d(H)$ denotes the minimum number of generators of H.

Proof. Let $1 \to R \to F \to G \to 1$ be a presentation of G where F is free of rank n and R is the normal closure of an r-element subset of F. Then $M(G)$ is given by Hopf's formula

$$M(G) \simeq (F' \cap R)/[F, R].$$

Now $r \geq d(R/[F, R])$, and since F/F' is free abelian, we have

$$d(R/[F,R]) = d((F' \cap R)/[F,R]) + d(F'R/F')$$

$$= d(M(G)) + n - r_0(F/F'R).$$

This shows that $r \geq d(M(G)) + n - r_0(G)$, from which the result follows. □

Observe the consequence that there is a least upper bound for the integer $n - r$ over all finite presentations of G: this is the *deficiency* of G,

$$\mathrm{def}(G).$$

Now apply (5.1) to a GBS-group $G = \pi_1(\Gamma, \omega)$, using the standard presentation with respect to a maximal subtree T. Here $n = |V(\Gamma)| + |E(\Gamma \backslash T)|$ and $r = |E(\Gamma)|$, so that

$$n - r = |V(\Gamma)| - |E(T)| = 1$$

and we have $\mathrm{def}(G) \geq 1$. Then $d(M(G)) \leq r_0(G) - (n - r) = r_0(G) - 1$ by (5.1). Therefore we have:

(5.2). *If G is a GBS-group, then $d(M(G)) \leq r_0(G) - 1$. Thus $M(G) = 0$ if $r_0(G) = 1$.*

Corollary 5. *If G is a GBS-tree product, then $M(G) = 0$.*

On the other hand, a Baumslag-Solitar group can have non-zero Schur multiplier.

(5.3). *Let $G = BS(m, n)$. Then $M(G) = 0$ if $m \neq n$ and $M(G) \simeq \mathbb{Z}$ if $m = n$.*

Proof. Suppose that $m \neq n$. Then $G_{ab} \simeq \mathbb{Z} \oplus \mathbb{Z}_{|m-n|}$, so that $r_0(G) = 1$ and $M(G) = 0$ by (5.2). Now assume that $m = n$. Note that $r_0(G) = 2$ in this case and hence $d(M(G)) \leq 2 - 1 = 1$; thus it is enough to show that $r_0(M(G)) = 1$. From the exact sequence $1 \to G' \to G \to G_{ab} \to 1$ we obtain the 5-term exact homology sequence

$$M(G) \to M(G_{ab}) \to G'/[G', G] \to G_{ab} \to G_{ab} \to 1.$$

Now $G'/[G', G]$ is finite since

$$[x, t]^m \equiv [x^m, t] \equiv x^{-m}(x^m)^t \equiv 1 \bmod [G', G].$$

Also $r_0(M(G_{ab})) = 1$, because $r_0(G) = 2$. Hence $\mathrm{Im}(M(G) \to M(G_{ab}))$ is infinite. Thus we have $1 \geq d(M(G)) \geq r_0(M(G)) \geq 1$, so that $r_0(M(G)) = 1$ and $M(G) \simeq \mathbb{Z}$. □

In fact there is a remarkably simple formula for the Schur multiplier of an arbitrary GBS-group ([14]).

(5.4). *If G is an arbitrary GBS-group, then $M(G)$ is free abelian of rank $r_0(G) - 1$.*

The proof of this result uses the 5-term homology sequence and the Mayer-Vietoris sequence for the homology of a generalized free product: for details see [14].

Corollary 6. *If G is any GBS-group, then* $\mathrm{def}(G) = 1$.

For by (5.1) and (5.2) we have

$$1 \leq \mathrm{def}(G) \leq r_0(G) - d(M(G)) = r_0(G) - r_0(G) + 1 = 1.$$

Corollary 7. *Let Γ be a bouquet of k loops. Then $M(G) \simeq \mathbb{Z}^\ell$ where $\ell = k + 1$ if each loop has equal weight values and otherwise $\ell = k$.*

The underlying reason here is that a bouquet of loops is tree dependent if and only if each loop has equal weight values.

For example, consider the *GBS*-group G in Example 3. Here $r_0(G) = 3$ and thus $M(G) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Central extensions

Knowledge of the Schur multiplier of a GBS-group G allows one to study central extensions of an arbitrary abelian group C by G. By the Universal Coefficients Theorem we have

$$H^2(G, C) \simeq \mathrm{Ext}(G_{ab}, C) \oplus \mathrm{Hom}(M(G), C).$$

Now(5.4) shows that $\mathrm{Hom}(M(G), C) \simeq \bigoplus C^{r_0(G)-1}$, while $G_{ab} \simeq \mathbb{Z}^{r_0(G)} \oplus F$ with F finite. Hence $\mathrm{Ext}(G_{ab}, C) \simeq \mathrm{Ext}(F, C)$, which can be computed if the structure of F is known. On the basis of these remarks we can characterize those GBS-groups G for which every central extension by G splits.

(5.5). *Every central extension by a generalized Baumslag-Solitar group G splits, i.e., is a direct product, if and only if G_{ab} is infinite cyclic.*

Proof. Let C be a trivial G-module and denote the periodic subgroup of G_{ab} by F; thus $G_{ab} \simeq \mathbb{Z}^{r_0(G)} \oplus F$ where F is finite. Since $H^2(G, C) \simeq \mathrm{Ext}(F, C) \oplus C^{r_0(G)-1}$, we have $H^2(G, C) = 0$ for every C if and only if $r_0(G) = 1$ and $\mathrm{Ext}(F, C) = 0$ for all C. By taking C to be $\mathbb{Z}$, we see that this happens precisely when $r_0(G) = 1$ and $F = 1$, i.e., $G_{ab} \simeq \mathbb{Z}$. □

6 Nilpotent quotients of GBS-Tree Products

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group where Γ is a tree and let $\bar{G}$ be a nilpotent quotient of G. Then $\bar{G}$ has a central cyclic subgroup $\bar{Z}$ which contains a positive power of every generator. Thus $\bar{G}/\bar{Z}$ is a finitely generated periodic nilpotent group, so it is finite. Clearly $r_0(G) = 1$, which implies that all lower central factors of $\bar{G}$ after the first are finite (by the usual tensor product argument for lower central factors). Hence $r_0(\bar{Z}) = 1$, which shows that $\bar{G}$ is central cyclic-by-finite, and hence finite-by-cyclic. Thus we have:

(6.1). *A nilpotent quotient of a GBS-tree product is finite-by-cyclic.*

Information about the second derived quotient group is also available.

(6.2). *If G is a GBS-tree product, then G/G'' is virtually abelian.*

Proof. Write $\bar{G} = G/G''$ and note that there exists an element $u \in G$ such that $G/ < u > G'$ is finite. Next let x, y, z be generators of G; since G is a tree product, $< x > \cap < y > \cap < z > \neq 1$. Hence $([x, y]^{<z>})G''/G''$ is finitely generated and it follows that $[x, y]^G G''/G''$ is finitely generated, as is G'/G'' since G/G'' satisfies max-n. Also z^m centralizes G'/G'' for some $m > 0$, from which it follows that $< z^m > G'/G''$ is abelian and clearly it has finite index in G. □

Note that G'' is a free group by (3.5), so further derived factors may be complex. Furthermore the next result shows that one cannot expect to be able to say anything about finite factors of a GBS-group.

(6.3). *Every finite group is a quotient of a GBS-tree product.*

Proof. Let $F = \{f_1, \ldots, f_n\}$ be an arbitrary finite group with $m_i = |f_i|$. Let T be the line graph with edges $< f_1, f_2 >, < f_2, f_3 >, \ldots, < f_{n-1}, f_n >$, the weight of edge $< f_i, f_{i+1} >$ being (m_i, m_{i+1}). By Von Dyck's theorem there is a surjective homomorphism from $\pi_1(T,\omega)$ to F such that $g_{f_i} \mapsto f_i$ since $f_i^{m_i} = 1 = f_{i+1}^{m_{i+1}}$. $\square$

7 Geometric quotients of GBS-groups

We now restrict attention to quotients of a GBS-group which arise in a natural way from the underlying GBS-graph. Let (Γ,ω) and $(\bar{\Gamma},\bar{\omega})$ be GBS-graphs and let $G, \bar{G}$ be the corresponding GBS-groups defined with respect to the maximal subtrees $T, \bar{T}$. A pair of functions (γ, δ),

$$\gamma : V(\Gamma) \to V(\bar{\Gamma}), \quad \delta : E(\Gamma \backslash T) \to E(\bar{\Gamma} \backslash \bar{T})$$

is called a *vertex-edge pair* for $(\Gamma, \omega, T), (\bar{\Gamma}, \bar{\omega}, \bar{T})$ if
(i) $(\delta(e))^- = \gamma(e^-)$ and $(\delta(e))^+ = \gamma(e^+)$, $e \in E(\Gamma \backslash T)$;
(ii) if $< x, y > \in E(T)$ and $\gamma(x) \neq \gamma(y)$, then $< \gamma(x), \gamma(y) > \in E(\bar{T})$.

Thus non-tree edges of Γ are mapped to non-tree edges of $\bar{\Gamma}$ and an edge in T is mapped to an edge in $\bar{T}$ provided that γ has distinct values at the endpoints.

Definition 2. A homomorphism between the GBS-groups above

$$\theta : \pi_1(\Gamma,\omega) \to \pi_1(\bar{\Gamma},\bar{\omega})$$

is called *geometric* if there is a vertex-edge pair (γ, δ) such that

$$g_x^\theta = g_{\gamma(x)}^{r(x)}, \ x \in V(\Gamma)$$
$$t_e^\theta = t_{\delta(e)}^{s(e)}, \ e \in E(\Gamma \backslash T),$$

where $r(x), s(e) \in \mathbb{Z}$. Thus θ is determined by the parameters

$$(\gamma, \delta, r(x), s(e) \mid x \in V(\Gamma), e \in E(\Gamma \backslash T)),$$

which are of course subject to certain restrictions.

A quotient group G/K of a GBS-group G is called a *geometric quotient* if $K = \mathrm{Ker}(\theta)$ where θ is a *surjective* geometric homomorphism from G to some GBS-group. (Note that in general the image of a geometric homomorphism need not be a GBS-group).

Some natural examples of geometric homomorphisms

1. *Loop deletion*

Suppose that the graph Γ has two loops e, e' through the same vertex and that e has weight (1,1). Then deleting e and mapping the associated generator t_e to 1 gives rise to a geometric homomorphism $\theta : \pi_1(\Gamma, \omega) \to \pi_1(\bar{\Gamma}, \bar{\omega})$ where $\bar{\Gamma}$ is Γ with e removed and $\bar{\omega}$ is the restriction of ω. Here the vertex pair fixes vertices and maps e and e' to e'.

2. *Loop identification*

Suppose that the graph Γ has two loops e, e' through a vertex and that they have the same weight. Identify the two loops to form a new graph $\bar{\Gamma}$, which is Γ with the loop e' removed. Map t_e and $t_{e'}$ to t_e: here the vertex pair fixes vertices and maps e and e' to e, with other edges fixed.

3. *Pinch maps*

Let $G = \pi_1(\Gamma, \omega)$ and let T be a maximal subtree in Γ. Choose any $e \in E(\Gamma)$ and write $m = \omega^-(e),\ n = \omega^+(e)$. Let d be a common divisor of m and n. Define a new weight function $\bar{\omega}$ on Γ by replacing the weight (m, n) by $(m/d, n/d)$, with all other weights unchanged. Write $\bar{G} = \pi_1(\Gamma, \bar{\omega})$. Then there is a surjective homomorphism

$$\theta : G \to \bar{G}$$

in which

$$x \mapsto \bar{x}, \quad y \mapsto \bar{y}.$$

Indeed $\bar{x}^{m/d} = \bar{y}^{n/d}$ implies that $\bar{x}^m = \bar{y}^n$, while $(\bar{x}^{m/d})^t = \bar{y}^{n/d}$ implies that $(\bar{x}^m)^t = \bar{y}^n$. Note that θ is a geometric homomorphism induced by the vertex-edge pair of identity functions. Also, if $e \in E(T)$, then

$$[x^{m/d}, y^{n/d}]^\theta = 1$$

and $[x^{m/d}, y^{n/d}] \neq 1$ if $d \neq \pm 1$. There is a similar discussion if $e \notin E(T)$. Hence θ is not an isomorphism if $d \neq \pm 1$. Call θ a *pinch map on e.*

4. *Edge contractions*

Let $G = \pi_1(\Gamma, \omega)$ and let T be a maximal subtree of Γ. Suppose that $e = < y, z > \in E(T)$ has relatively prime weights $m = \omega^-(e),\ n = \omega^+(e)$. We aim to define a *contraction along the edge* $e = < y, z >$. The diagram which follows exhibits a part of the graph Γ.

$$\bullet_x \xrightarrow{(p,q)} \bullet_y \xrightarrow{(m,n)} \bullet_z \xrightarrow{(r,s)} \bullet_u$$

Form a new graph $\bar{\Gamma}$ by deleting the edge e and adjusting the weights of adjacent edges appropriately: the relevant segment of $\bar{\Gamma}$ is

$$\bullet_a \xrightarrow{(p,qn)} \bullet_b \xrightarrow{(rm,s)} \bullet_c$$

Now define a vertex pair (γ, δ) by

$$\gamma(x) = a,\ \gamma(y) = b,\ \gamma(z) = b,\ \gamma(u) = c,$$

with other vertices fixed and δ preserving non-tree edges. A homomorphism

$$\theta : G \to \bar{G}$$

is defined by the rules

$${g_x}^\theta = g_a,\ {g_y}^\theta = {g_b}^n,\ {g_z}^\theta = {g_b}^m,\ {g_u}^\theta = g_c,\ \ldots.$$

Then θ is a geometric homomorphism induced by (γ, δ): for example, the relation ${g_z}^r = {g_u}^s$ in G becomes ${g_b}^{rm} = {g_c}^s$ in $\bar{G}$. Since $\gcd(m, n) = 1$, we have $g_b \in \mathrm{Im}(\theta)$, so θ is surjective. Finally, note that

$$[g_y, g_z]^\theta = [{g_b}^n, {g_b}^m] = 1,$$

while if $|m| \neq 1$ and $|n| \neq 1$, then $[g_y, g_z] \neq 1$ and θ is not an isomorphism. Notice that edge contraction does not decrease weights in absolute value. (In a similar way it is possible to define a contraction along a loop.)

It is an important property of geometric homomorphisms that their composites are also geometric.

(7.1) *Let $G_i = \pi_1(\Gamma_i, \omega_i)$, $i = 1, 2, 3$, be GBS-groups with associated maximal subtrees T_i, and let $\phi_i : G_i \to G_{i+1}$, $i = 1, 2$, be geometric homomorphisms with parameters $(\gamma_i, \delta_i, r_i(x), s_i(e))$ relative to the T_i. Then the composite $\phi_1\phi_2$ is a geometric homomorphism from G_1 to G_3 with parameters*

$$(\gamma_2\gamma_1,\ \delta_2\delta_1,\ r_1(x)r_2(\gamma_1(x)),\ s_1(e)s_2(\delta_1(e)).$$

Proof. First observe that $(\gamma_2\gamma_1, \delta_2\delta_1)$ is a vertex-edge pair. For, if $e \in E(\Gamma_1 \backslash T_1)$, then $(\delta_2\delta_1(e))^\pm = \gamma_2(\delta_1(e)^\pm) = \gamma_2\gamma_1(e^\pm)$. Also, if $\langle x, y\rangle \in E(T_1)$ and $\gamma_2\gamma_1(x) \neq \gamma_2\gamma_1(y)$, then $\gamma_1(x) \neq \gamma_1(y)$, so $\langle\gamma_1(x), \gamma_1(y)\rangle \in E(T_2)$. Thus we have $\langle\gamma_2\gamma_1(x), \gamma_2\gamma_1(y)\rangle \in E(T_3)$. Next, if $x \in V(\Gamma_1)$, then

$$(g_x)^{\phi_1\phi_2} = (g_{\gamma_1(x)}^{r_1(x)})^{\phi_2} = (g_{\gamma_2\gamma_1(x)})^{r_1(x)r_2(\gamma_1(x))},$$

and there is a similar calculation for $(t_e)^{\phi_1\phi_2}$. $\square$

8 GBS-Simple Groups and GBS-Free groups

Every GBS-group has $\mathbb{Z}$ as a quotient, although not necessarily as a geometric quotient. We will say that a GBS-group is *GBS-simple* relative to a maximal subtree T if there are no surjective, geometric homomorphisms relative to T, with non-trivial kernel, from G to any non-cyclic GBS-group. Equivalently G has no proper, non-cyclic, geometric GBS-quotients. (Here a quotient is called proper if the associated normal subgroup is non-trivial). If a GBS-group has no proper, non-cyclic GBS-quotients at all, whether geometric or not, it is called *GBS-free*.

The following result, which is proved in [6], provides a complete classification of the GBS-groups which are GBS-simple: it also shows that the properties "GBS-free" and "GBS-simple" are identical.

(8.1). *Let (Γ, ω) be a GBS-graph and let $G = \pi_1(\Gamma, \omega)$ be the GBS-group defined with respect to a maximal subtree T. Then the following statements are equivalent:*

(a) *G is GBS-free;*
(b) *G is GBS-simple;*
(c) *there is a geometric isomorphism from G to one of the groups $BS(1,n), K(1,1), K(p,q), K(p,p^d)$, where $n \in \mathbb{Z}^*$, p, q are distinct primes and $d > 0$.*

Thus, for example, $K(2,4)$, $K(2,3)$, $BS(1,3)$ are GBS-free, but $K(4,9)$ and $BS(2,3)$ are not GBS-free. Notice that the theorem also shows that the property GBS-simple is independent of the maximal subtree T.

Sketch of proof of (8.1).
Assume G is GBS-simple, but not cyclic. The idea of the proof is to show there is a surjective, geometric homomorphism from G to a non-cyclic GBS-group whose underlying graph is either a 1-edge or a 1-loop. This will show that there is no loss in assuming the original graph to have one of these forms. Then these special cases can be dealt with. The geometric homomorphisms used will be composites of the special types (1)–(4) listed above: thus (7.1) is relevant.

Suppose first that Γ is a tree with more than one edge. Contract all edges with a weight vaue ± 1, which does not change G up to isomorphism. Thus we can assume that there are no such edges. There must be some edges left, otherwise the graph consists of a single vertex and G is infinite cyclic. If two or more edges are left, pinch and contract all edges but one, noting that after a pinch-contraction there are still no ± 1 labels. The resulting graph has a single edge and the group is non-cyclic, so we have reduced to the case of a 1-edge.

Now suppose Γ is not a tree and let T be a maximal subtree. Pinch and contract edges in T to a single vertex to get a bouquet of loops. Note that the group is non-cyclic.

From now on assume that Γ is a bouquet of $k \geq 2$ loops. Moreover, by pinching we can also assume that all the weights are relatively prime. The next step is to establish

(8.2). *If not all weights have absolute value 1, then G has a proper, non-cyclic geometric quotient and hence is not GBS-simple.*
Proof. We have

$$G =< t_1, \ldots, t_k, x | \ (x^{m_i})^{t_i} = x^{n_i}, i = 1, \ldots, k >$$

where $\gcd(m_i, n_i) = 1$. We can assume that $|m_i| \leq |n_i|$. Define

$$\ell = \ell\mathrm{cm}(n_1, \ldots, n_k);$$

then the assignments

$$x \mapsto x^{\ell}, \ t_i \mapsto t_i$$

determine a geometric endomorphism θ of G, where the vertex pair consists of identity functions. We have to prove that θ is surjective. First G^{θ} contains t_i and $x^{\ell} = x^{(\ell/n_i)n_i}$, and hence $x^{(\ell/n_i)m_i}$. Since m_i, n_i are relatively prime, $x^{\ell/n_i} \in G^{\theta}$. Also the ℓ/n_i are relatively prime, so $x \in G^{\theta}$ and $\mathrm{Im}(\theta) = G$. Notice in addition that

$$[x, x^{t_i}]^{\theta} = [x^{\ell}, (x^{\ell})^{t_i}] = [x^{\ell}, ((x^{n_i})^{t_i})^{\ell/n_i}] = [x^{\ell}, x^{m_i \ell/n_i}] = 1$$

and $[x, x^{t_i}] \neq 1$ if $|m_i| \neq 1$. On the other hand, if all the $|m_i| = 1$, then in a similar way $[x^{t_i^{-1}}, x^{t_i^{-1} t_j}] \in \mathrm{Ker}(\theta)$ and this is non-trivial if $j \neq i$. □

The discussion so far shows that we can assume that Γ is a bouquet of $k \geq 2$ loops where $|m_i| = 1 = |n_i|$ for all i. We can delete any loop with label (1,1). Then, if there are multiple loops with label $(1, -1)$, pass to a 1-loop quotient with $G = BS(1, -1)$ by identifying loops. The effect of the above analysis is to reduce to the case of a 1-loop. Thus it remains to deal with the cases of a 1-loop and a 1-edge. In these cases a complete description of all GBS-quotients is possible.

(8.3). *There is a surjective homomorphism from $G = K(m, n)$ to $\bar{G} = K(m', n')$, where $\bar{G}$ non-cyclic, if and only if there exist integers k, r, s such that either*

$$(i) \ m' = m/ks, \ n' = n/kr \ \textit{and} \ \gcd(r, m/k) = 1 = \gcd(s, n/k),$$

or

$$(ii) \ m' = n/kr, \ n' = m/ks \ \textit{and} \ \gcd(r, m/k) = 1 = \gcd(s, n/k).$$

The sufficiency of the conditions in the theorem is proved by observing that if m, n are relatively prime and p divides m, then there is a surjective geometric homomorphism

$$\theta : K(m,n) \longrightarrow K(\frac{m}{p}, n)$$

in which $x \mapsto \bar{x},\ y \mapsto \bar{y}^p$, where x, y and $\bar{x}, \bar{y}$ are the respective generators of the groups $G, \bar{G}$.

(8.4). *There is a surjective homomorphism θ from $BS(m,n)$ to $BS(\bar{m},\bar{n})$ if and only if $\bar{m} = m/q$ and $\bar{n} = n/q$ or $\bar{m} = n/q$ and $\bar{n} = m/q$ for some integer q dividing m and n.*

Sketch of proof
Let $G =< t, x >$ and $\bar{G} =< \bar{t}, \bar{x} >$ be the two groups and assume there is a surjective homomorphism from G to $\bar{G}$. To prove the result we will produce invariants of the groups. An obvious one is obtained from

$$G_{ab} \simeq \mathbb{Z} \times \mathbb{Z}_{|m-n|}.$$

Since θ maps G_{ab} onto $\bar{G}_{ab}$, we see that $\bar{m} - \bar{n}$ divides $m - n$. Assume that $m \neq n$: the case where $m = n$ requires a special argument.

Next we analyze the structure of G/T where $T/(x^G)'$ is the torsion-subgroup of $x^G/(x^G)'$. In fact

$$G/T \simeq < t > \ltimes A$$

where $A = \mathbb{Q}_\pi$ is the additive group of rational numbers with π-adic denominators, π being the set of primes involved in $\frac{n}{m}$ (after cancellation). Here t acts on A by multiplication by $\frac{n}{m}$, this being the additive version of the relation $(x^m)^t = x^n$.

Note that x^G is generated by all the elements commensurable with their conjugates, (i.e., elements g such that $< g > \cap < g >^h \neq 1$, for all $h \in G$). Therefore x^G is characteristic in G. Hence θ maps $< t > \ltimes A$ onto $< \bar{t} > \ltimes \bar{A}$, where $\bar{A} = \mathbb{Q}_{\bar{\pi}}$ is the additive group of rational numbers with $\bar{\pi}$-adic denominators, with $\bar{\pi}$ the set of primes involved in $\frac{\bar{n}}{\bar{m}}$. It follows that $\frac{n}{m} = \frac{\bar{n}}{\bar{m}}$ (or $\frac{\bar{m}}{\bar{n}}$, in which case a similar argument applies).

Let $d = \gcd(m,n)$ and write $m' = \frac{m}{d}$ and $n' = \frac{n}{d}$: similarly define $\bar{d}, \bar{m}', \bar{n}'$. Then $\frac{m'}{n'} = \frac{\bar{m}'}{\bar{n}'}$ and hence $m' = \bar{m}'$ and $n' = \bar{n}'$. Therefore $\bar{m} = \bar{d}m/d$ and $\bar{n} = \bar{d}n/d$, so that

$$\frac{m-n}{\bar{m}-\bar{n}} = \frac{d}{\bar{d}}$$

is an integer and $\bar{d}$ divides d. Writing $q = d/\bar{d}$, we have $\bar{m} = \frac{m}{q}$ and $\bar{n} = \frac{n}{q}$. Conversely, if $m, n, \bar{m}, \bar{n}$ satisfy the conditions, then by pinching we get a surjective homomorphism $G \to \overline{G}$. $\square$

The proof of (8.1) is now essentially complete: for fuller details see [6].

Remark. It follows from the discussions of (8.3) and (8.4) that (m, n) is an invariant of the groups $BS(m, n)$ and $K(m, n)$ up to multiplication by -1 (of either component in the second case) and interchange of components. It is more challenging to find invariants of arbitrary GBS-groups, although one example is the number of non-tree edges in the graph when the group is not $BS(1, -1)$.

We end with what is probably a hard question. *Is the isomorphism problem soluble for GBS-groups*, i.e., is there an algorithm which, when two GBS-graphs (Γ, ω) and $(\bar{\Gamma}, \bar{\omega})$ are given, decides if $\pi_1(\Gamma, \omega) \simeq \pi_1(\bar{\Gamma}, \bar{\omega})$? A positive answer is known in various special cases, particularly in the case of GBS-trees – for details see [4], [7], [12], [13].

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