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Multiple nests

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Abstract. A description is given of all spreads in $PG(3,q)$, $q = p^r$, p odd, whose associated translation planes admit linear Desarguesian collineation groups of order $q(q+1)$

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1 Introduction

In the study of the translation planes and spreads associated with flocks of quadratic cones in $PG(3,q)$, the 'conical planes' or 'conical spreads' in $PG(3,q)$ that correspond to q-nests are an important class of spreads that may be constructed from an associated Desarguesian spread by the replacement of a set of q reguli that cover a net of degree $q(q+1)/2$. The flocks are said to be of 'Fisher type' and hence we call the associated translation planes 'Fisher planes'. In this setting, every line of the net is incident with exactly two of the q reguli.

More generally, a t-nest is a set of t reguli defining a net of degree $t(q+1)/2$ such that each line of the net is incident with exactly two of the q-reguli. This concept actually originated with A. Bruen in [6] and generalized by Baker and Ebert in [1] and considered in a variety of articles (see, e.g. [2],[3],[4]). In particular, Bruen termed what are now known to be $(q+3)/2$-nests 'chains' and these have the property that each pair of distinct reguli within the set share exactly two lines.

In each of the known cases, a t-nest is 'nest replaceable' in the sense that the associated net may be replaced by a net of the same degree and consisting of exactly $(q+1)/2$ Baer subplanes incident with the zero vector of each of the t regulus nets (i.e. lines of the opposite regulus). There are infinite classes of $q-1, q, q+1, 2(q-1)$-nests thus producing a variety of new and interesting translation planes with spreads in $PG(3,q)$.

In this article, the concept of t-nest is generalized in two steps. First, we consider sets of t reguli that define a net of degree $t(q+1)/2k$ such that each line of the net is incident with exactly $2k$ of the reguli. This is called a '$(k;t)$-nest' of reguli. Although we postpone a formal definition to the next section, we next generalize these notions to allow a set of t reguli to be partitioned into subsets that are to some degree $(k_i;t_i)$-nests, that we generally call 'multiple-nests'.

As noted, the use of t-nests has been important in the construction of translation planes and it would be expected that this more general construction technique based on multiple-nests would also find wide application.

The Fisher planes may be constructed using a 'regulus-inducing' elation group E and the kernel subgroup of squares H of an associated Desarguesian affine plane. A regulus-inducing elation group E is an elation group each of whose component orbits union the axis is a regulus. If Σ is the associated Desarguesian plane and $\mathcal{Q}$ is a q -nest of reguli then there is a Baer subplane π_o of Σ disjoint from the axis of E such that $\pi_o EH$ is a replacement for $\mathcal{Q}$. Actually, it was noted by Baker and Ebert [2] for odd prime square and more generally by Payne [16] for odd square order that such q-nests may be replaced in a Desarguesian plane to produce the Fisher planes.

In this article, we consider a new construction technique but based on this idea: Since the group H may be replaced by the subgroup of order $(q+1)$, we now assume that H has order $q+1$ and consider possible subgroups of index 2 and 4. Use a subgroup H_1 of H and the appropriate number of Baer subplanes $\pi_i, i=1,2,\ldots,k$ so that $\{\pi_i; i=1,2,\ldots,k; \pi_i EH_1\}$ is a replacement net for a set of t reguli in a Desarguesian affine plane. Although this could conceivably be accomplished in a very general manner for subgroups H^{2^k} of H, we consider in this article the sets of reguli and replacements that may be obtained for groups H^2 of order $(q^2-1)/4$ and H^4 of order $(q^2-1)/8$ where $(q-1)/2$ is odd.

As an example, the Desarguesian plane of order 9 admits such multiple-nests and re-constructs itself using the nest replacements.

Furthermore, we show that the Thas-Herssen-De Clerck conical flock of order 11^2 and associated conical plane may be constructed from the Desarguesian affine plane using EH^2 and hence are double-nest planes.

Generally, assume that we have a translation plane π with spread in $PG(3,q)$ that admits a linear collineation group EH of order $q(q+1)$ such that there is an associated Desarguesian affine plane Σ where EH is a collineation group of Σ and E is a normal regulus-inducing elation group of order q. Then, it must be the case that H contains a kernel homology subgroup of order either $(q+1)$, $(q+1)/2$ or $(q+1)/4$. Note that such a translation plane π admits E as a collineation group and the axis of E is either a component of π or a Baer subplane. By the work of Gevaert and Johnson [8], Johnson [11] and Payne and

Thas [17], it will follow that π is either a conical flock plane or a derived conical flock plane.

In this article, we give a complete classification of the translation planes that have such a group EH^2 or EH^4 where H^2 and H^4 are kernel homology groups of orders $(q+1)/2$ and $(q+1)/4$ respectively of an associated Desarguesian plane Σ as above.

Thus, in particular, we have a complete classification of the translation planes of odd order q^2 with spread in $PG(3,q)$ that are so constructed.

In a related article, the authors show that whenever there is a translation plane of odd order with spread in $PG(3,q)$ that admits a linear group G of order $q(q+1)$ either the plane is Desarguesian or Hall or there is, in fact, an associated Desarguesian affine plane Σ admitting the group G as a collineation group and $G = EH$. Hence, our analysis of multiple-nests in this setting provides an integral step in the classification of all such translation planes admitting linear groups of order $q(q+1)$.

2 (k,t)-nests

1 Definition. Let Σ be a Desarguesian spread of order q^2 in $PG(3,q)$. A '(k,t)-multiple nest' of reguli is a set of t reguli of Σ such that for each line L of the union, there are exactly $2k$ reguli of Σ that contain L and the cardinality of the union is $t(q+1)/2k$, (where $2k$ divides $q+1$). When $k = 1$, 2, 3 we use the terms 't -nest', 'double t-nest' and 'triple t-nest' of reguli, respectively.

We note that a reference to a 'Baer' subplane shall always mean a Baer subplane that is a line in $PG(3,q)$.

If $\mathcal{P}$ is a (k,t)-multiple nest of reguli assume that for each regulus R of $\mathcal{P}$, there are exactly $(q+1)/2k$ Baer subplanes of the associated affine plane incident with the zero vector such that the union of these $t(q+1)/2k$ subplanes covers $\mathcal{P}$. We call such a set $\mathcal{P}^*$ of Baer subplanes a '(k,t)-nest replacement' for $\mathcal{P}$ and say that $\mathcal{P}$ is '(k,t)-nest replaceable'.

2 Remark. Let $\mathcal{P}$ be a (k,t)-nest replaceable multiple nest of reguli in a Desarguesian spread Σ, with (k,t)-nest replacement $\mathcal{P}^*$, then $(\Sigma - \mathcal{P}) \cup \mathcal{P}^*$ is a spread in $PG(3,q)$.

2.1 $(k_1, k_2, \ldots, k_z, ; t)$-nests

3 Definition. A set $\mathcal{P}$ of t reguli in a Desarguesian affine plane of order q^2 with spread in $PG(3,q)$ is said to be a 'multiple-nest of type $(k_1, \ldots, k_z; t)$' or a '$(k_1, k_2, \ldots, k_z; t)$-nest', where the k_j are distinct integers, $k_i \leq k_{i+1}$, if the t reguli can be partitioned into z sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_z$ with the following properties:

(i) $\cup_{j=1}^{z}\mathcal{S}_j = \mathcal{P}$,

(ii) lines of $\mathcal{S}_1$ are lines of exactly $2k_1$ reguli of $\mathcal{P}$, lines of $\mathcal{S}_2 - \mathcal{S}_1$ are lines of exactly $2k_2$ reguli of $\mathcal{P}$, lines of $\mathcal{S}_3 - \cup_{j=1}^{2}\mathcal{S}_j$ are lines of exactly $2k_3$ reguli of $\mathcal{P}$ ..., lines of $\mathcal{S}_w - \cup_{j=1}^{w-1}\mathcal{S}_j$ are lines of exactly $2k_w$ reguli of $\mathcal{P}$, for $w = 2, 3, \ldots, z$.

Note that if $k_i = k_{i+1}$, we may take $\mathcal{S}_i \cup \mathcal{S}_{i+1}$ as one of the sets of the partition. However, when we consider replacements for this set of reguli, we might lose information as to the replacement set (see below)..

If $\mathcal{P}$ is a $(k_1, k_2, \ldots, k_z; t)$-nest of reguli, assume that for each regulus R of $\mathcal{S}_j$, there are exactly $w_j(q+1)/2k_z$ Baer subplanes of the associated affine plane incident with the zero vector such that the union of these subplanes covers $\mathcal{P}$. We call such a set $\mathcal{P}^*$ of Baer subplanes a '$(k_1, k_2, \ldots, k_z; t)$-nest replacement' for $\mathcal{P}$ and say that $\mathcal{P}$ is '$(k_1, k_2, \ldots, k_z; t)$-nest replaceable'.

Note that two reguli in different sets can nevertheless share lines.

Let $|\mathcal{S}_j| = t_j$ so that $\sum_{j=1}^{z} t_j = t$ then the cardinality of the set of lines of the union is

$$\sum_{j=1}^{z} t_j w_j (q+1)/2k_z.$$

We shall also use the notation $(k_1, k_2, \ldots, k_z; \{t_1, w_1\}, \{t_2, w_2\}, \ldots, \{t_z, w_z\})$ for a $(k_1, k_2, \ldots, k_z; t)$ nest where $\sum_{j=1}^{z} t_w = t$ and $\sum_{j=1}^{z} t_j w_j (q+1)/2k_z$ is the degree of the net in the Desarguesian plane defined by the set of reguli.

4 Remark. Let $\mathcal{P}$ be a $(k_1, k_2, \ldots, k_z; t)$-nest replaceable multiple nest of reguli in a Desarguesian spread Σ, with $(k_1, k_2, \ldots, k_z; t)$-nest replacement $\mathcal{P}^*$, then $(\Sigma - \mathcal{P}) \cup \mathcal{P}^*$ is a spread in $PG(3, q)$.

3 Fundamental Double and Triple Nests

In the q-nest setting, there is a q-nest of reguli in a Desarguesian affine plane of odd order and a Baer subplane π_o such that if H is the kernel homology subgroup of squares and E is a regulus-inducing elation group of order q then $\pi_o E H^2$ is a replaceable partial spread of $q(q+1)/2$ 2-dimensional K-subspaces. Replacement of the corresponding q-nest of reguli in the Desarguesian plane Σ, produces the Fisher planes. Note that we merely consider that H has order $(q+1)$. Then, since 2 divides $(q+1)$, there is always a $GF(q)$-linear group of order 2 fixing all Baer subplanes (that are $GF(q)$-subspaces).

We consider a more general construction. Now assume that 4 divides $q+1$, H is the kernel homology group of order $(q+1)$ of a Desarguesian affine plane Σ and we take the subgroup H^2 of order $(q+1)/2$ and consider three different q-reguli as follows:

Consider three distinct Baer subplanes of Σ, π_1, π_2, π_3 such that the reguli R_i of Σ containing π_i, $i = 1, 2, 3$ respectively, share exactly two components of $(q + 1)/2$ base reguli (E-orbits of components). Let the set of such base reguli be denoted by B_i, $i = 1, 2, 3$. Furthermore, assume that B_i and B_j share components of exactly $(q + 1)/4$ base reguli for all $i \neq j$, $i, j = 1, 2, 3$ and the union of the three sets of base reguli is $3(q + 1)/4$ and $B_1 \cap B_2 \cap B_3 = \phi$.

Then $\pi_i EH^2$ is a partial spread of $3q(q + 1)/4$ 2-dimensional K -subspaces involving q reguli and each 'line' of this set is in exactly four of the q-reguli.

5 Definition. If H^2 is the homology group of order $(q+1)/2$, for $(q+1)/2$ even, of the Desarguesian affine plane Σ then $\cup_{i=1}^{3} \pi_i EH^2$ is a set of $3(q(q+1)/4)$-2-dimensional K-subspaces, and there are exactly $3q$-reguli and each 'line' lies on exactly 4 of the reguli. We shall call this a 'double-nest' of $3q$-reguli.

If $\cup_{i=1}^{3} \pi_i EH^2$ is a partial spread, we shall call the double-nest of $3q$-reguli, 'double-nest replaceable'.

Hence, in this case, there is a corresponding translation plane admitting the collineation group EH^2 of order $q(q + 1)/2$ constructed from a Desarguesian affine plane. Since E is regulus-inducing, it follows that the constructed translation plane is a conical flock plane.

6 Definition. Suppose we have a set of Nq reguli of a Desarguesian affine plane of order q^2 covering exactly $N(q + 1)/8$ lines and such that each line lies on exactly 8-reguli. We shall call such a set of reguli a 'triple- nest of Nq reguli.

Assume that there is a regulus-inducing group E of order q and a kernel homology group of H^4, of order $(q+1)/4$ of the kernel homology group. Assume that there are N subplanes π_i such that $\pi_i EH^4$ are partial spreads and that the associated sets of base reguli B_i and B_j corresponding to π_i and π_j share exactly $(q + 1)/8$ base reguli, the intersection of any three distinct sets of B_k is empty and $\cup_{i=1}^{N} \pi_i EH^4$ involves $Nq(q+1)/8$ lines such that each line is incident with exactly 8 reguli. If this set is a partial spread, we call the triple-nest of Nq-reguli, 'triple-nest replaceable.'

Hence, we obtain a translation plane admitting a collineation group EH^4 of order $q(q + 1)/4$. Since E is regulus-inducing, the plane is a conical flock plane.

We will show that every EH^2 group leads to a double-nest replacement and there are some more general nest structures corresponding to EH^4 groups.

We finally list the following result showing that all translation planes in question are either conical flock planes or derived conical flock planes.

7 Theorem (Gevaert and Johnson, [8], Johnson [11] and Payne and Thas [17]). *Let ρ be a translation plane of order q^2 with spread in $PG(3, q)$ that admits a linear collineation group E of order q that is either regulus-inducing or is a Baer group of order q. Then ρ is a conical flock plane or derived conical* *flock plane.*

4 Preliminary Lemmas

Assume that ρ is any translation plane that admits a group E as above. We assume that Σ is a Desarguesian affine plane of order q^2, q odd, where 4 divides $q+1$, which is coordinatized by a field F isomorphic to $GF(q^2)$ containing a field K isomorphic to $GF(q)$ such that K is contained in the kernel of ρ.

8 Lemma. *Let Σ be a Desarguesian affine plane of order q^2, q odd, and 4 divides $q+1$ and let $x=0$ be a component of Σ. Let E denote the regulus-inducing elation group of Σ with axis $x=0$, call the set of reguli containing $x=0$ defined by the orbits of E the 'base reguli' and let π_o be a 2-dimensional $GF(q)$-subspace which is not a component and which is disjoint from $x=0$. Then, one of the following situations occur:*

(1) π_o intersects exactly $(q+1)/2$ base reguli in two components each, or

(2) π_o intersects at least two base reguli in one component each and if π_o is $y=x^q s+xt$, then s^{q+1} is a square in $GF(q)$. We call the base reguli that intersect in exactly one components, '1-cuts'.

Proof. Assume that π_o intersects the standard base regulus in exactly one component. Represent π_o by $y=x^q s+xt$ where s is non-zero, $s,t\in GF(q^2)$. Without loss of generality, we may assume that the one component of intersection is $y=0$. Hence, it follows that $x_o^q s=-x_o t$, for some non-zero element x_o of $GF(q^2)$. Hence, $s^{q+1}=t^{q+1}$. Moreover, since the other components of the standard regulus net are of the form $y=x\alpha$ for all $\alpha\in GF(q)-\{0\}$, there is not a solution to the equation:

$$s^{q+1}=(\alpha-t)^{q+1}=\alpha^2+t^{q+1}-\alpha(t+t^q).$$

Now assume that

$$t+t^q=\beta\neq 0.$$

Then,

$$\beta(\beta-(t+t^q))=0,$$

and since $s^{q+1}=t^{q+1}$ we have

$$s^{q+1}=(\beta-t)^{q+1}=\beta^2+t^{q+1}-\beta(t+t^q).$$

But, this implies that

$$((\beta-t)/s)^{q+1}=1$$

and by Hilbert's Theorem 90, there exists non-zero element x_o of F, isomorphic to $GF(q^2)$, so that

$$x_o^{q-1} = (\beta - t)/s.$$

However, this is implies that

$$x_o^q s = x_o(\beta - t) \iff x_o^q s + x_o t = x_o \beta,$$

which, in turn, implies that $\beta = 0$. Hence, $t + t^q = 0$. Let $\{1, e\}$ be a basis for $GF(q^2)$ over $GF(q)$ and let $e^2 = -1$, since we may choose -1 as a non-square in $GF(q)$. Then, if $t^q = -t$, it must be that $t^{q+1} = -t^2$, so that $t^2 = (et_1 + t_2)^2 = e(2t_1t_2) + t_2^2 - t_1^2$, implying that $t_1t_2 = 0$. If $t_2 = 0$, then $-t^2 = t_1^2$ is a square, implying that $t^{q+1} = s^{q+1}$ is a square. If $t_1 = 0$, again $t^2 = t_2^2$ implies that $t \in GF(q)$ but then $t = t^q$ so that $t = 0$, a contradiction. This proves the lemma. $\boxed{QED}$

9 Lemma. *Assume that* 4 *divides* $q + 1$. *Let* $b \in F$ *be any element of order dividing* $(q + 1)$. *Then* $2 + b^{q-1} + b^{1-q}$ *is a square in* K *and is* 0 *if and only if* $b^{q-1} = -1$.

Proof. We note that $b^{(q+1)/2} = \pm 1$ implying $b^{(q+1)(q-1)/2} = 1$. Hence, $b^{q(q-1)/2+(q-1)/2} = 1$ or rather $b^{q(q-1)/2} = b^{(1-q)/2}$and raising to the q^{th}-power, $b^{q(1-q)/2} = b^{(q-1)/2}$. This implies that $(b^{(q-1)/2} + b^{(1-q)/2})^q = (b^{(q-1)/2} + b^{(1-q)/2})$ is in K isomorphic to $GF(q)$. Then $(b^{(q-1)/2} + b^{(1-q)/2})^2$ is a square in K. But, $(b^{(q-1)/2} + b^{(1-q)/2})^2 = b^{(q-1)} + 2b^{(q-1)/2}b^{(1-q)/2} + b^{(1-q)} = 2 + b^{q-1} + b^{1-q}$.

Now assume that $2 + b^{q-1} + b^{1-q} = 0$, then $b^{(q-1)/2} = -b^{(1-q)/2}$, implying that $b^{(q-1)/2+(q-1)/2} = b^{q-1} = -1$. $\boxed{QED}$

10 Lemma. *If the order of* b *divides* $(q + 1)$ *and* 4 *divides* $q + 1$ *then* $2 - (b^{1-q} + b^{q-1})$ *is a square in* K *and is* 0 *if and only if* $b^{q-1} = 1$.

Proof. $(b^{q-1} - 1)^{q+1} = 2 - (b^{1-q} + b^{q-1}) = -(b^{(1-q)/2} - b^{(q-1)/2})^2$.

Then $(b^{q-1} - 1)^{(q+1)(q-1)/2} = (-1)^{(q-1)/2}(b^{(1-q)/2} - b^{(q-1)/2})^{(q-1)}$ and $(b^{(1-q)/2} - b^{(q-1)/2})^{(q-1)} = (b^{(1-q)/2} - b^{(q-1)/2})^q/(b^{(1-q)/2} - b^{(q-1)/2}) =$

$-(b^{(1-q)/2} - b^{(q-1)/2})/(b^{(1-q)/2} - b^{(q-1)/2}) = -1$ using the arguments of the previous lemma. Hence, $(b^{q-1} - 1)^{(q^2-1)/2} = 1$, implying that $(b^{q-1} - 1)^{q+1}$ is a square in $GF(q)$. $\boxed{QED}$

11 Lemma. *Let* Z *be any kernel homology group of* Σ *of order dividing* $q + 1$ *but not dividing* $q - 1$. *Assume that* 4 *divides* $q + 1$ *and consider*

$$EZ(y = x^q s + xt) = \{y = x^q s b^{1-q} + x(t + \beta) \ \forall\ b \in Z, \forall \beta \in GF(q)\}.$$

Then this is an orbit under EZ *and is a partial spread if and only if* s^{q+1} *is non-square.*

Proof. We need to check the intersections with $y = x^q s + xt$, which exist if and only if $(s(b^{1-q} - 1)^{q+1}) = \beta^2$, implying that s^{q+1} is non-square by the previous lemma since b^{1-q} is not always 1. Hence, we must have $\beta = 0$ and $b^{1-q} = 1$. QED

12 Lemma. *If 4 divides $q + 1$ and if we have a partial spread then there are no 1-cuts.*

Proof. Suppose so. Then, assume that the standard regulus is a 1-cut for $y = x^q s + xt$. Then s^{q+1} is a square by lemma 8 and also a non-square by lemma 11. QED

Since there are no 1-cuts, we obtain:

13 Lemma. *Let ρ be a translation plane with spread in $PG(3, q)$ that admits a collineation group EZ where EZ is a collineation group of an associated Desarguesian affine plane Σ. Assume that E is a regulus inducing elation group acting on Σ and Z is a kernel homology group of order dividing $q + 1$ but not dividing $q - 1$.*

(1) Then any component of ρ which is not in Σ is a Baer subplane π_i in Σ.

(2) If π_i is disjoint from the axis of E then π_i intersects $(q + 1)/2$ base reguli in 2 components each. (The base reguli are the reguli in Σ defined by the component orbits of E union the axis).

5 Groups of Order $q(q + 1)/2$

14 Theorem. *Let π be a translation plane of order q^2 with spread in $PG(3, q)$ that admits a linear group G with the following properties:*

(i) G has order $q(q + 1)/2$,

(ii) there is an associated Desarguesian affine plane Σ of order q^2 such that $G = EZ$ where E is a normal, regulus-inducing elation group of Σ and Z is a kernel homology group of order $(q + 1)/2$ of Σ.

Then

(1) π is either a conical flock plane or a derived conical flock plane.

(2) π is either Desarguesian or Hall or

(3) if 4 does not divide $(q + 1)$ then π is Fisher or derived Fisher.

(4) If π is of odd order q^2, $4 | (q + 1)$ then either π is one of planes of part (2) or (3) or π may be constructed from a Desarguesian plane by either

(a) double-nest replacement of a 3q-double-nest or

(b) derived from a plane which may be so constructed, by a base regulus net fixed by the group of order $q(q + 1)/2$.

(5) If π is constructed by 3q-double-nest replacement, the replacement net consists of a set of exactly $3(q + 1)/4$ base reguli (E -orbits of components of

Σ). This set is replaced by $\{\pi_i EZ; i = 1, 2, 3\}$ where π_i are Baer subplanes of Σ that intersect exactly $(q + 1)/2$ base reguli in two components each.

The sets $\mathcal{B}_i$ of $(q+1)/2$ base reguli of intersection pairwise have the property that $|\mathcal{B}_i \cap \mathcal{B}_j| = (q + 1)/4$ for $i \neq j$, $i, j = 1, 2, 3$ and $\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 = \phi$.

Proof. We now assume that EZ is a group of order $q(q + 1)/2$ as in the previous setting. By theorem 7, we may, without loss of generality, assume that π is a conical flock plane.

Assume that 4 does not divide $(q + 1)$. Then the plane π must be Fisher or Desarguesian since if not Desarguesian, there is a Baer subplane π_1 of Σ that is a component of π and we obtain an orbit of length $q(q + 1)/2$ under EZ. But, there cannot be any other components of π that are not components of Σ since we would then be forced to have at least $q(q + 1)$ components. Hence, there is a linear subflock of $q - (q + 1)/2 = (q - 1)/2$ so that the plane must be Fisher by Johnson [13].

Hence 4 divides $(q + 1)$ and we may apply the previous lemmas.

It follows that the components of $\pi - \Sigma$ are in orbits of lengths $(q + 1)/4$ under Z since the involutory kernel homology is in Z. Furthermore, these orbits are in reguli of Σ disjoint from the elation axis. Since E acts as a collineation group of π, it follows that we have orbits of length $q(q + 1)/4$ under EZ.

If π_o is a component of $\pi - \Sigma$ then π_o intersects exactly $(q+1)/2$ base reguli each in two components. Now $\pi - \Sigma$ has either $1, 2$ or 3 orbits of length $q(q+1)/4$ under EZ.

$\pi \cap \Sigma$ consists of a set of E-orbits; base reguli. If this set has cardinality at least $(q - 1)/2$, then π is Fisher or Desarguesian by [13]. Hence, there are exactly three orbits of length $q(q + 1)/4$ under EZ in $\pi - \Sigma$.

Hence, $\pi - \Sigma$ has exactly $3q(q + 1)/4$ components. Furthermore, by lemma 7, each orbit corresponds to a subplane π_i of Σ (a component of π) and a set B_i of $(q + 1)/2$ base reguli, $i = 1, 2, 3$ such that π_i shares two components with each regulus (E-orbit) of B_i, $i = 1, 2, 3$.

We consider how the lines of B_1 must be covered. First of all, each line of each regulus of B_1 is intersected by two sets of $(q + 1)/4$ subplanes that form a partial spread so each line is what might be called 'half' covered from $\pi_1 EZ$. Hence, in order that a given line of a regulus of B_1 is covered, it must be the case that the line is covered either by orbits of π_2 or orbits of π_3 but not both. Let k_{1j} denote the number of common base reguli between B_1 and B_j for $j = 2$ or 3. We note that each line of each base regulus of B_1 is initially 'half' covered using $\pi_1 EZ$.

Note that if B_1 and B_2 are equal then $\pi_1 EZ \cup \pi_2 EZ$ would cover the lines on the common base reguli. But, then the lines of B_3 could be covered only by $\pi_3 EZ$, which cannot be the case. Thus, in this case, no two base regulus sets can be equal. Hence, we obtain

$$k_{12} + k_{13} = (q+1)/2.$$

Similarly,

$$k_{21} + k_{23} = (q+1)/2$$

and

$$k_{31} + k_{32} = (q+1)/2,$$

where k_{ij} is the number of common base subplanes of B_i and B_j. Hence, $k_{13} = k_{23} = k_{12} = (q+1)/4$. So, we have three distinct sets B_i, $i = 1, 2, 3$ of $(q+1)/2$ base reguli any two of which share a subset of exactly $(q+1)/4$ base reguli. Each Baer subplane $\pi_i e$, for $e \in E$ defines a unique regulus, which we denoted by $R_{\pi_{je}}$. So, corresponding to $\pi_i EZ$ are three sets of q reguli.

In general, a regulus of $R_{\pi_i}E$ is a regulus of $R_{\pi_j}E$ only if the base reguli of intersection B_i and B_j are equal, for any Baer subplanes π_i and π_j of Σ.

To see this note that R_{π_i} is an image of R_{π_j} under an element of E only if π_i and π_j share their base reguli of intersections.

Again, take a common base regulus R_{12} of B_1 and B_2. If a subplane ρ_1 of $\pi_1 EZ$ shares a regulus with a subplane ρ_2 of $\pi_2 EZ$ then $\{\rho_1, \rho_2\}Z$ has $(q+1)/2$ Baer subplanes of the regulus defined by ρ_1 or ρ_2. Then $\{\rho_1, \rho_2\}EZ$ is a q-nest replaceable net of $q(q+1)/2$ components. But, this means that B_1 and B_2 share all of their base reguli, a contradiction.

Hence, there are exactly $3q$ reguli and it follows that each line of the set of $3q(q+1)/4$ 'lines' are covered by exactly four reguli.

So, if we initially have a conical flock plane, we have shown that we have either a plane constructed by q-nest replacement (i.e. a Fisher plane) or a plane constructed by double $3q$-nest replacement. This proves our theorem when $|Z| = (q+1)/2$. QED

6 Groups of Order $q(q+1)/4$

15 Theorem. *Let π be a translation plane of order q^2 with spread in $PG(3,q)$ that admits a linear group G with the following properties:*

(i) G has order $q(q+1)/4$,

(ii) there is an associated Desarguesian affine plane Σ of order q^2 such that $G = EZ$ where E is a normal, regulus-inducing elation group of Σ and Z is a kernel homology group of order $(q+1)/4$ of Σ.

Then

(1) π is either a conical flock plane or a derived conical flock plane.

(2) If 8 *does not divide* $q+1$ *then* π *is either*

(a) Desarguesian,

(b) Hall,

(c) Fisher,

(d) derived Fisher,

(e) constructed by double-nest replacement of a $3q$*-double nest or*

(f) derived from a plane which may be so constructed, by a base regulus net fixed by the group of order $q(q+1)/4$.

(3) If 8 *divides* $q+1$ *and is not one of the planes of part (2) then* π *may be constructed from the Desarguesian plane* Σ *by*

(a) triple-nest replacement of a Nq*-triple nest where* $N=5,6$ *or* 7 *or*

(b) derived from a plane which may be so constructed as in (a), by a base regulus net fixed by the group of order $q(q+1)$,

(c) is constructed from a Desarguesian plane by replacing a $(2,4;4q)$ *or* $(2,4;5q)$*-nest of degree* $Nq(q+1)/8$ *for* $N=6$ *or* 7 *or*

(d) derived from a plane which may be so constructed as in (c), by a base regulus net fixed by the group of order $q(q+1)$,

(e) is constructed from a Desarguesian plane by replacing a $(3,4;6q)$ *-nest of degree* $7q(q+1)/8$ *or*

(f) derived from a plane which may be so constructed as in (e), by a base regulus net fixed by the group of order $q(q+1)$.

Proof. If $(q+1)/4$ is even then it is possible to have merely a kernel subgroup H^4 of order $(q+1)/4$ producing in $\pi-\Sigma$ a set of orbits under EH^4 of length $q(q+1)/8$. Hence, there are i such orbits for $i=1,2,3,4,5,6,7$. Suppose that $i=1,2,3$ or 4, Then, the plane is Fisher or Desarguesian by the cardinality of the linear subset of base reguli by Johnson [13]. Hence, $i=5,6$ or 7. Furthermore, EH^4 is normal in EH as H^- is characteristic in $H\cap GL(2,q^2)$ which is normal in H.

Hence, it is at now possible in this case that $R_{\pi_k}=R_{\pi_j}$ for $k\neq j$ and/or, more generally, two base regulus sets of $(q+1)/2$ base reguli of intersection could be equal. In any case, there are i Baer subplanes π_j, $j=1,2,\dots,i$ and $i^*\leq i$ sets of base reguli B_j each of $(q+1)/2$ reguli each $i^*<i$ if and only if $R_{\pi_k}=R_{\pi_j}$ for $k\neq j$, or if two base regulus sets are equal.

Any given line of a base regulus of one of the B_j is '1/4'-covered by π_jEH^4. Furthermore, the line is 'half' covered by images from π_kEH^4 for $k\neq j$, '3/4' covered by images from a third π_sEH^4 and then covered using four such image sets.

If z base regulus sets of $(q+1)/2$ base reguli originating from the various R_{π_k} are equal, we shall say that we have a 'z-cover'. Since we have $(q+1)/2$ base reguli of intersection corresponding to each R_{π_j}, it follows that we have a design of base reguli and sets B_j of $(q+1)/2$ base reguli and each such base reguli within any such set is incident with exactly four such sets. Hence, it follows by a standard 'flag counting' that there are exactly $N(q+1)/8$ base reguli of intersection where $N = 5, 6, 7$. QED

First assume that there are only $z = 1$-covers.

6.1 1-covers

PROOF. Let k_{ijkl} denote the number of base reguli in $\cap_{z \in \{i,j,k,l\}} B_z$ for $z = i, j, k, l$, where the i, j, k, l are distinct even if the base regulus sets are not. We note that the intersection of any five base regulus sets is empty. So,

$$\sum_{i \text{ fixed}} k_{ijkl} = (q+1)/2.$$

Hence,

$$\sum_{i=1}^{N} \sum_{i \text{ fixed}} k_{ijkl} = N(q+1)/2.$$

Notice that each term k_{ijkl} appears exactly four times in the double sum. Hence, we obtain

$$4 \sum_{(ijkl) \in \binom{N}{4}} k_{ijkl} = N(q+1)/2,$$

where the subscript notation is intended to indicate the sum over an intersection of four of the N possible base regulus sets B_z.

Hence, we obtain

$$\sum_{(ijkl) \in \binom{N}{4}} k_{ijkl} = N(q+1)/8.$$

Now if N=5, then we have

$$\sum_{i \text{ fixed}} k_{ijkl} = (q+1)/2,$$

and this sum involves all but one of the terms. Hence, it follows that any missing term $k_{jklm} = 5(q+1)/8 - (q+1)/2 = (q+1)/8$.

So, when $N = 5$, all $k_{ijkl} = (q+1)/8$.

Actually, even if the k_{ijkl} are not completely determined, it follows directly that we have a triple-nest of Nq reguli which is replaceable. QED

Now we consider the possibility that there might be a z-cover for $z = 2, 3, 4$
.

6.2 4-covers

Proof. First of all if $z = 4$, then this means that there are four of the subplanes, say π_j for $j = 1, 2, 3, 4$ such that the base regulus sets for R_{π_j} and R_{π_k} for $j, k = 1, 2, 3, 4$ are all equal. This means that $\{\pi_1, \pi_2, \pi_3, \pi_4\}EH^-$ is a partial spread of cardinality $q(q+1)/2$ completely covering ER_{π_1}. However, since $N = 5, 6, 7$, it follows that R_{π_5} cannot cannot be covered, since the base reguli of intersection must be covered by exactly four intersection sets and there are at most three remaining. Hence, there cannot be 4 -covers. QED

6.3 3-covers

Proof. Next assume that $z = 3$, and assume that the base regulus sets for R_{π_j} for $j = 1, 2, 3$ are identical. Then to cover the lines of $(q+1)/2$ base reguli of intersection, for each base regulus of intersection, we would require a contribution of one orbit $\pi_k EH^-$ for $k = 4, \ldots, N$. Since there are no 4-covers, any extra orbit defines base reguli of intersection that are among the base reguli of intersection of the 3-cover. However, this means that we would require a contribution of four additional orbits, implying that $N = 7$. In this case, then the base reguli of intersection of $R_{\pi_4}$and not in the base reguli of intersection of R_{π_1} must be exactly those for R_{π_k} for $k = 4, 5, 6, 7$. Hence, if R_{π_4} has $(q+1)/2-z$ base reguli outside of that of the base reguli of intersection of R_{π_1}, then there are exactly $(q+1)/2 + (q+1)/2 - z = q+1-z$ base reguli. However, there are exactly $7(q+1)/8$ base reguli, implying that $z = (q+1)/8$. Moreover, there are z base reguli within the base reguli of intersection of R_{π_1} required to complete each base regulus set of $(q+1)/2$ and each such base regulus requires a unique contribution since there are already three intersections. But, there are exactly four remaining sets that can contribute. This again implies that $4z = (q+1)/2$, as noted above.

We now consider the possible types of s-covers possible. If the reguli R_{π_j} and R_{π_k} define the set base regulus set $\mathcal{B}_j$ of $(q+1)/2$ base reguli that share two components each of the reguli in question, it is possible that none of the q-reguli in $\pi_j EH^-$, defined by images of R_{π_j}, are in the q-reguli in $\pi_k EH^-$, defined by images of R_{π_k} even though there are but $q(q+1)/2$ components of Σ that the subplanes in $\pi_j EH^-$ or $\pi_k EH^-$ lay across. Clearly, either all of the two sets of q -reguli are equal or the sets are disjoint; there are a total of $2q$ image reguli within EH^- and each component in Σ of intersection is in exactly two components of each set of q-reguli.

Hence, we have that a 2-cover produces two possible types: 2(1), the reguli $R_{\pi_k} = R_{\pi_j}$ and 2(2), the R_{π_j} is not a regulus corresponding to E-images of R_{π_k}. In the first type, 2(1), each component lies on exactly two reguli and in the second type 2(2), each component lies on exactly four reguli.

To consider this with a 3-cover, it is possible to have type 3(1), where all three reguli R_{π_i} are equal, $i = 1, 2, 3$, a type 3(2), where no regulus R_{π_k} is an E-image of R_{π_j}, and a type 3(3), where two of the reguli are equal but the third is not an E-image of the previous. QED

We consider the previous possibilities in turn.

6.3.1 Case 3(1)

Proof. The three Baer subplanes that produce the same regulus provide q reguli in $\pi_i EH^-$. The remaining four Baer subplanes must define distinct reguli, in sets of q each. Note that this means that we have a total of $5q$ reguli partitioned into a set of q reguli such that each line of the union of $q(q+1)/2$ lines of this set shares lines of exactly 4 reguli (a contribution of 2 lines from the q 3-cover reguli and a contribution of 2 lines from the remaining $4q$ reguli) and there are $4q$ reguli such that the lines of the union of this set that are not in the previous set are lines of exactly 8 reguli. Hence, we have what we have termed a $(2, 4; 5q)$ -nest of reguli, $5q$ reguli in a Desarguesian spread that has the required partition where $\mathcal{S}_1$ consists of q reguli, $\mathcal{S}_2$ consists of $4q$ reguli, lines of $\mathcal{S}_1$ are lines of exactly 4 reguli and lines of $\mathcal{S}_2 - \mathcal{S}_1$ are lines of exactly 8 reguli.

Note that there are 7 orbits $\pi_i EH^-$ whose union covers a line set of $7q(q+1)/2 \cdot 4$. To see how this number is calculated in the manner presented in the background section, we note that there are q reguli that have $3(q+1)/8$ Baer subplanes and $4q$ reguli that share $(q+1)/8$ so that

$$q3(q+1)/8 + 4q(q+1)/8 = 7q(q+1)/8.$$

Then, we obtain a partial spread of $7(q+1)/8$ defining a $(2, 4; 5q)$-nest or also a $(2, 4; \{q, 3\}, \{4q, 1\})$-nest. QED

6.3.2 Case 3(2)

Proof. We have a 3-cover where the reguli involved define the same base regulus set of $(q+1)/2$ base reguli but there is a total of q reguli for each of three subplanes. Hence, we have $3q$ reguli such that the lines of the union are in six of these reguli. Furthermore, there are $4q$ additional reguli such that lines of original set are in an additional 2 of these reguli. Then, the lines of the $4q$ reguli that are not within the original set of reguli are incident with 8 reguli. Hence, in both cases, each line is incident with exactly 8 reguli of the set of $7q$ reguli. So, we have a triple nest of $7q$ reguli, with a corresponding 3-cover and a corresponding partition of set of $3q$ and $4q$ reguli. So, we obtain a type $(4, 4; 7q)$-nest, also a triple-nest of $7q$ reguli.

Furthermore, there are $3q$ reguli each of which has $(q+1)/8$ Baer subplanes and $4q$ reguli each of which also have $(q+1)/8$ reguli so

$$3q(q+1)/8 + 4q(q+1)/8 = 7q(q+1)/8.$$

In summary, we obtain a triple-nest of $7q$ reguli. QED

6.3.3 Case 3(3)

PROOF. We now have two of the R_{π_j} reguli equal and a third within the E -image set of the first two. Hence, we have a set of $6q$ reguli partitioned into three sets of q, q and $4q$ reguli. Lines of the first two sets are incident with 6 lines (four from the first set and two from the second set) and lines of the second set are incident with 8 lines. So, we have a type $(3,3,4;6q)$-nest. In the first set of q reguli, there are exactly $2(q+1)/8$ subplanes, in the second set of q reguli, there are exactly $(q+1)/8$ subplanes each and in the third set of $4q$ reguli, there are also exactly $(q+1)/8$ Baer subplanes each.

Hence, we obtain:

$$q2(q+1)/8 + q(q+1)/8 + 4q(q+1)/8 = 7q(q+1)/8.$$

Thus, we obtain a $(3,3,4;6q)$-nest, or more specifically a

$$(3,3,4;\{q,2\},\{q,1\},\{4q,1\})-\textbf{nest}.$$

Note we may also describe this as a $(3,4;6q)$-nest, although this notation will give less information.

Hence, in summary, for 3-covers, we obtain a net as follows:

degree $7q(q+1)/8$ and
(i) $(2,4;5q)$-nest, a
(ii) triple-nest of $7q$ reguli, or a
(iii) $(3,4;6q)$-nest.

QED

6.4 2-Covers, $N = 6$

PROOF. Now we turn to the possibility that there is a 2-cover.

First assume that N is 5. If there is a 2-cover, and we have seen that there is not a 3-cover in this case, the remaining three EH^{-}-orbits define sets of $(q+1)/2$ base reguli such each base regulus outside of the 2-cover requires four orbits for a cover. Hence, N is at least 6. Assume that $N = 6$. This implies that any base regulus of intersection outside of the initial 2-cover set of base reguli requires all four of the orbits for the cover. This implies that the number of base reguli of intersection outside of the 2-cover set is $(q+1)/2 - z$, where z in the number of the given base regulus set within the 2-cover base regulus set. Furthermore, the total number of base reguli is

$$\begin{aligned}(q+1)/2 + (q+1)/2 - z &= 6(q+1)/8,\\ \text{so that } z &= (q+1)/4.\end{aligned}$$

In order to complete the $(q+1)/2$ base reguli in R_{π_j} for $j = 3, 4, 5, 6$ (assume that R_{π_1} and R_{π_2} share their base reguli of intersection), there must be exactly $(q+1)/4$ base reguli in R_{π_1} in common with R_{π_j} for $j = 3, 4, 5, 6$. Hence, there are exactly $(q+1)/4$ base reguli outside of the initial 2-cover.

In these situations, the number of reguli and how their lines share reguli depends on the nature and number of 2-covers. *QED*

Three 2-covers.

Proof. The three 2-covers could be of the following:

All of type 1. This leads directly to a **double-nest of $3q$ reguli** or more accurately type $(2,2,2;3q)$-nest or rather a

$$(2,2,2;\{q,2\},\{q,2\},\{q,2\}) - nest.$$

One of type 1 and two of type 2.

This leads to a type $(2,4,4;5q)$-nest, or rather a

$$(2,4,4;\{q,2\},\{2q,1\},\{2q,1\}) - nest.$$

We may also refer to this as a $(2,4;5q)$-nest.

Two of type 1 and one of type 2.

It follows that this produces a type $(2,2,4;4q)$-nest, or rather as a

$$(2,2,4;\{q,2\},\{q,2\},\{2q,1\}) - nest.$$

More simply, we may consider this as a $(2,4;4q)$-nest.

All of type 2. This gives rise to a triple-nest of $6q$ reguli, which is also described as a $(4,4,4;6q)$-nest.

In summary, for three 2-covers we have:

a net of degree $6(q+1)/8$ so $3(q+1)/4$ and

(i) a double-nest of $3q$ reguli,

(ii) a $(2,4;5q)$-nest or

(iii) a $(2,4;4q)$-nest.

QED

Two 2-covers.

PROOF. The possibilities are as follows.
Both of type 1, leading to a type

$$(2,2,4,4;4q) = (2,2,4,4;\{q,2\},\{q,2\},\{q,1\},\{q,1\}) - nest.$$

We may also consider this as a $(2,4;4q)$**-nest.**
Both of type 2**,** giving rise to a type a **double-nest** or rather a

$$(2,2,2,2;6q) = (2,2,2,2;\{q,2\},\{q,2\},\{2q,1\},\{2q,1\}) - nest.$$

Mixed type, producing a $(2,4,4,4;5q)$-nest, or rather a

$$(2,4,4,4;\{q,2\},\{2q,1\},\{q,1\},\{q,1\}) - nest.$$

We may also consider this simply as a $(2,4;5q)$**-nest.**
In summary for two 2**-covers, we obtain:**

a net of degree $3(q+1)/4$ and
(i) a $(2,4;4q)$-nest,
(ii) a double-nest of $3q$ reguli or
(iii) a $(2,4;5q)$-nest.

QED

Exactly one 2**-cover.**

PROOF. **Type** 1, producing a $(2,4;5q)$**-nest**, or rather $(2,4;\{q,2\},\{4q,1\})$.
Type 2, giving rise to a **triple-nest** or rather of type

$$(4,4;6q) = (4,4,\{2q,1\},\{4q,1\}) - nest.$$

In summary, we obtain:

a net of degree $6q(q+1)/8$ that is
(i) a $(2,4;5q)$-nest or a
(ii) a triple-nest of $6q$ reguli.

QED

6.5 2-Covers, $N = 7$

PROOF. We first assert that there cannot be three 2-covers. If so, there since each base regulus of one of the 2-covers must be a base regulus of another 2-cover in order that a line of the base regulus can be covered. However, since this is true for all 2-covers, this implies that the lines of any base regulus set which is not involved in a 2-cover cannot be covered. QED

6.5.1 Two 2-covers

PROOF. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote the set of base reguli corresponding to the two 2-covers. We note any line of a base regulus in a base regulus set not in the union of these two sets can be covered by at most $3(q+1)/4$ 1-dimensional K-subspaces. Hence, $\mathcal{B}_1 \cup \mathcal{B}_2$ has cardinality $7(q+1)/8$, implying the intersection has cardinality $(q+1)/8$. Let w_{ij} denote the intersection with the base reguli $\mathcal{B}_k$ for $k = i$ or j and $\mathcal{B}_1 \cup \mathcal{B}_2 - \mathcal{B}_1 \cap \mathcal{B}_2$. Note that

$$\begin{aligned} w_{56} + w_{57} + w_{67} &= 3(q+1)/4, \text{ and} \\ w_{56} + w_{57} &= (q+1)/2 = w_{57} + w_{67} = w_{56} + w_{67}. \end{aligned}$$

Hence,

$$w_{ij} = (q+1)/4.$$

QED

Both Type 1.

PROOF. Then, we obtain q reguli from each of the 2-covers and q reguli each from the remaining three R_{π_k}'s. Hence, we obtain a $(2,2,4,4,4;5q)$-nest or rather a

$$(2,2,4,4,4;\{q,2\},\{q,2\},\{q,1\},\{q,1\},\{q,1\}) - nest.$$

We may also consider this a $(2,4;5q)$-nest. QED

Both Type 2.

PROOF. Then, we obtain $2q$ reguli from each 2-covers, implying we obtain a **triple-nest** or a $(4,4,4,4,4;7q)$-nest, or rather a

$$(4,4,4,4,4;\{q,1\},\{q,1\},\{q,1\},\{2q,1\},\{2q,1\}) - nest.$$

QED

Mixed Type.

PROOF. We obtain q reguli from a type 1 2-cover and $2q$ reguli from a type 2 2-cover, producing a $(2,4,4,4,4;6q)$-nest or rather a

$$(2,4,4,4,4;\{q,2\},\{q,1\},\{q,1\},\{q,1\},\{2q,1\})-nest.$$

We may regard this more simply as a $(2,4;6q)$-nest.

In summary, for two 2-covers, we obtain:

a net of degree $7q(q+1)/8$ and
(i) a $(2,4;5q)$-nest, a
(ii) a triple-nest of $7q$ reguli or
(iii) a $(2,4;6q)$-nest.

QED

6.6 One 2-cover

Type 1.

PROOF. We obtain a $(2,4,4,4,4,4;6q)$-nest or rather a

$$(2,4,4,4,4,4;\{q,2\},\{q,1\},\{q,1\},\{q,1\},\{q,1\},\{q,1\})-nest$$

or **more simply a $(2,4;6q)$-nest.** QED

Type 2.

PROOF. Here, we obtain a **triple-nest** or rather a $(4,4,4,4,4,4;7q)$-nest or rather a

$$(4,4,4,4,4,4;\{q,1\},\{q,1\},\{q,1\},\{q,1\},\{2q,1\})-nest.$$

In summary, for one 2-cover, we obtain:

a net of degree $7q(q+1)/8$ and
(i) a $(2,4;6q)$-nest or
(ii) a triple-nest of $7q$ reguli.

QED

7 Groups of Order $q(q+1)$

16 Theorem. *Let π be a translation plane of order q^2 with spread in $PG(3,q)$ that admits a linear group G with the following properties:*

(i) G has order $q(q+1)$,

(ii) there is an associated Desarguesian affine plane Σ of order q^2 such that $G = EH$ where E is a normal, regulus-inducing elation group of Σ.

Then

(1) π is either a conical flock plane or a derived conical flock plane.

(2) If 8 does not divide $q+1$ then π is either

(a) Desarguesian,

(b) Hall,

(c) Fisher,

(d) derived Fisher,

(e) constructed by double-nest replacement of a $3q$-double nest or

(f) derived from a plane which may be so constructed, by a base regulus net fixed by the group of order $q(q+1)/4$.

(3) If 8 divides $q+1$ and is not one of the planes of part (2) then
π may be constructed from the Desarguesian plane $\Sigma$by

(a) by triple-nest replacement of a Nq-triple nest where $N = 5, 6$ or 7 or

(b) derived from a plane which may be so constructed as in (a), by a base regulus net fixed by the group of order $q(q+1)$ or

(c) is constructed from a Desarguesian plane by replacing a $(2,4;4q)$ or $(2,4;5q)$-nest of degree $Nq(q+1)/8$ for $N = 6$ or 7 or

(d) derived from a plane which may be so constructed as in (c), by a base regulus net fixed by the group of order $q(q+1)$, or

(e) is constructed from a Desarguesian plane by replacing a $(3,4;6q)$ -nest of degree $7q(q+1)/8$ or

(f) derived from a plane which may be so constructed as in (e), by a base regulus net fixed by the group of order $q(q+1)$.

Proof. Since we have a group H of order $q+1$, it remains to show that we have a kernel subgroup of order $(q+1)$, $(q+1)/2$ or $(q+1)/4$. However, H is a linear subgroup so in $GL(4,q)$ and also in $\Gamma L(2,q^2)$. Hence, $H \cap GL(2,q^2)$ has order at least $(q+1)/2$. Since E is normal, we may assume that H fixes two components. Let H^- denote the subgroup of H consisting of kernel homologies of Σ. It then follows that if h is an element of H that is not a kernel homology group of Σ, then the order of $h\, H^-$must divide $q-1$. Hence, h^2 is in H^-, implying that H^- has order at least $(q+1)/4$. This completes the proof. QED

8 The Structure of $3q$-Double-Nest Planes admitting groups of order $q(q+1)$

In this section, we consider a possible construction of a $3q$-double-nest plane when there is a group EH of the associated Desarguesian affine plane Σ such that E is a regulus-inducing elation group of order q and H is a subgroup of order $q+1$ of $GL(4,q)$ acting in Σ as a subgroup of $\Gamma L(2,q^2)$. When we do not have a kernel homology group of order $q+1$, but have a group of order $2(q+1)$ where there is a linear part H^- of order $(q+1)$, we either have a Fisher plane or 4 divides $q+1$. So, we have a unique non-kernel element σ such that σ^2 is in the kernel subgroup K^+ of order $(q+1)/2$. We have a $3q$ -double-nest so σ must fix at least one of the three EK^+-orbits of length $q(q+1)/4$. Note that σ acting on the associated Desarguesian affine plane Σ must fix exactly two components, since it fixes one.

We have a set of $3q(q+1)/4$ components of Σ. Moreover, our above arguments for σ and τ (the non-linear element) as listed above in a different context, show that $\tau\sigma$ fixes all base regulus nets. Furthermore, σ must one of the EK^+ orbits, say $\Gamma_{(q+1)/4}$. So, there is an element in $EK^+\langle\sigma\rangle$ that fixes a component L of π. L is a Baer subplane of Σ.

Hence, we have three orbits of length $q(q+1)/4$, implying that H must fix one of these orbits. We note that there are $3(q+1)/4$ base reguli involved in the union, leaving $q-3(q+1)/4=(q-3)/4$ left over reguli in $\Sigma-\pi$. Now the stabilizer of a component in the fixed orbit has order 8 (if the group has order $2q(q+1)$).

Since $(q-3)/4=(q+1)/4-1$, we see that H must fix a base regulus. Since H has order 8 on the line at infinity of Σ, it follows that H must fix a component of Σ in the fixed base regulus. Choose the fixed base regulus to be the standard regulus.

Represent H^{lin} in the form $\sigma:(x,y)\longmapsto(xc,yc\alpha)$. Since H fixes an EH^- orbit, it follows that in this orbit, a Sylow 2 -subgroup of H fixes a Baer subplane of that orbit. Since σ^2 must be in H^-, it follows that $\alpha=-1$, since σ can't be in the kernel of Σ or the plane is Fisher.

Now σ maps $y=x^q m+xn$ onto $y=x^q(-mc^{1-q})-xn$. Hence, it can only be that a fixed Baer subplane has $n=0$ and $c^{1-q}=-1$. But, then σ maps $y=x^q s+xt$ onto $y=x^q s-xt$.

We note that H^{lin} cannot fix two base reguli without H being in the kernel homology group of Σ. Hence, H^{lin} fixes a unique base regulus R_1.

Recall that we must have m^{q+1} and s^{q+1} both non-square in order we obtain the EH^- orbits as partial spreads. Now ρ must fix $x=0$ and $y=0$ as H^{lin} is normal in H and fixes exactly two components of Σ. Hence, $\rho:(x,y)\longmapsto$

$(x^q a, y^q a\beta)$. Note that there is a collineation in H that fixes all three orbits and the stabilizer of a component (Baer subplane) in one of them is S_2. This group acts on the remaining orbits implying that a subgroup, namely generated by ρ, of order 4 fixes components in both of these orbits. $\rho^2 : (x, y) \longmapsto (xa^{q+1}, a^{q+1}\beta^2)$, implying that $\beta^2 = 1$.

Now ρ fixes Baer subplanes in each of the orbits and ρ fixes $y = x^q r + xd$, implies that

$$a^q r = ar^q \beta.$$

If $y = x^q s^* + xt^*$ and $y = x^q m$ is fixed then

$$a^q s^* = as^{*q}\beta, \text{ and } a^q m = am^q \beta,$$

implying that

$$(m/s^*)^{q-1} = 1.$$

Thus, $m/s^* = \alpha_{s^*}$ for some α_{s^*} in K.

Let $y = x^q m$, $y = x^q s + xt$ and $y = x^q s^* + xt^*$ be fixed by ρ. Note that these define the orbits of EH^-. Since σ maps $y = x^q s + xt$ onto $y = x^q s - xt$, we may assume that $s^* = s$ and $t^* = -t + \delta$ for some δ in K.

To get a partial spread among $x^q m$ and $y = x^q s + xt$, we see that we need that $(s(b^{q-1} - \alpha_s)^{q+1}) = (t + \alpha)^{q+1}$, cannot occur for any b and α.

To get a partial spread among $x^q s + xt$ and $y = x^q s - xt$ we need that $(s(b^{q-1} - 1))^{q+1} = (-2t + \delta)^{q+1}$ cannot occur.

To get a partial spread among $x^q m$ and $y = x^q s - xt$, we need to show that $(s(b^{q-1} - \alpha_s)^{q+1}) = (-t + \alpha)^{q+1}$ cannot occur.

Hence, we have the following:

17 Theorem. *$EH^-\{y = x^q s \pm t, y = x^q \alpha_s s\}$ is a partial spread of cardinality $3q(q + 1)/4$ if and only if s^{q+1} is non-square and*

$$\begin{aligned} (s(b^{q-1} - \alpha_s)^{q+1}) &\neq (t + \alpha)^{q+1} \text{ and} \\ (s(b^{q-1} - 1))^{q+1} &\neq (-2t + \delta)^{q+1} \forall \alpha, \delta \in GF(q), \\ & \forall b \text{ of order dividing } (q + 1)/2. \end{aligned}$$

18 Theorem. *If $p = 3$ then $EH^-\{y = x^q s \pm t, y = x^q s\}$ is a partial spread of cardinality $3q(q + 1)/4$ if $s^{q+1} = -t_1^2$, for $t^{q+1} = t_1^2 + t_2^2$ and*

$$b^2 + b^{-2} \text{ is non-square for all } b \text{ or order dividing } (q + 1)/2.$$

PROOF. $(s(b^{q-1} - 1))^{q+1} \neq (-2t + \delta)^{q+1}$ and $(s(b^{q-1} - \alpha_s)^{q+1}) \neq (t + \alpha)^{q+1}$, for $p = 3$ are equivalent for $\alpha_s = 1$.

If $s^{1+1} = -t_1^2$ then $(et_1 + t_2 + \delta) = (et_1 + \beta)^{q+1} = \beta^2 + t_1^2$. Hence, assume that $(s(b^{q-1} - 1))^{q+1} = (-2t + \delta)^{q+1}$,then

$$\begin{aligned} -t_1^2(2-(b^{q-1}+b^{1-q}) &= t_1^2+\beta^2, \text{ so that} \\ t_1^2(1+b^{q-1}+b^{1-q}) &= t_1^2+\beta^2. \end{aligned}$$

Hence, we have that

$$(b^{q-1}+b^{1-q})=\rho^2,$$

for some ρ in $GF(q)$. Note that $b^{q-1=(q+1)-2}=b^{-2}$ and similarly, $b^{1-q}=b^2$. Hence, the condition for a partial spread is that

$$(b^2+b^{-2}) \text{ is nonsquare in } GF(q).$$

QED

8.1 3/4-Theory

19 Theorem. *If the above structure is a partial spread and can be embedded into a spread π then the spread is a conical flock spread, and there is a unique embedding.*

PROOF. Apply Johnson and Storme [15], as we have at least $3(q+1)/4$ reguli sharing a component, we have q reguli that share a component, implying that the spread is a conical flock spread.

If we have a second conical flock spread containing the partial spread then, there are at least $3(q+1)/4>(q+1)/2$ common reguli, implying the the conical flock spreads are identical. QED

20 Theorem. *Let $EH^-\{\pi_i; i=1,2,3\}$ define a partial spread of degree $3q(q+1)/4$. Let $(q-3)/4-d$ denote the number of base reguli of the associated Desarguesian spread Σ that that are not intersected by base of reguli of intersection of the π_i, for $i=1,2,3$.*

Then, there is a maximal partial spread in $PG(3,q)$ of cardinality $3q(q+1)/4+q(q-3)/4-d)$ and a corresponding maximal partial flock of a quadratic cone of deficiency $\leq d$.

21 Theorem. *Every double-nest plane of order q^2 admitting a collineation group of order $2q(q+1)$ has the form listed above: Let $\mathcal{B}$ denote the set of exactly $(q-3)/4$ Desarguesian base reguli that are not intersected non-trivially by one of the three subplanes. Then, the double-nest spread is as follows:*

$$EH^-\{y=x^qs\pm t, y=x^q\alpha_s s\}\cup\mathcal{B},$$

for some elements s and t of $GF(q^2)$ such that s^{q+1} is a nonsquare in $GF(q)$, and for some element α_s of $GF(q)$.

8.2 Double Nest Planes Produce Nest planes

We now assume that 4 divides $q+1$ and we have a double nest plane. This means by our previous analysis that we never have a 1-cut and thus we obtain, by the previous section, a Fisher plane by extending the group H^- of order $(q+1)/2$ to the kernel group, say $\mathcal{K}$ of order $q+1$ of the associated Desarguesian affine plane Σ. We have three Baer subplanes π_i, $i = 1,2,3$ and a double nest spread denoted by $EH^-\{\pi_i; i = 1,2,3\}$. But, each Baer subplane π_i determines a Fisher plane, denoted by $E\mathcal{K}\{\pi_i\}$. Furthermore, each such Fisher plane shares with the double nest plane an EH^--orbit of length $q(q+1)/4$ and the remaining $(q-3)/4$ reguli; a set of $1+q((q+1)+(q-3))/4 = q(2(q-1))/4 = 1+q(q-1)/2$ components.

We may always start with a Fisher plane of the form $E\mathcal{K}\{y = x^q s; s^{q+1}$ is non-square$\}$.

Since -1 is non-square, we may assume that $s^{q+1} = -1$, without loss of generality. The other two orbits under EH^- arise from mappings that normalize EH^- and may be taken in the form $\tau_{b,\beta} : (x, y) \longmapsto (x, xb + y\beta)$ for $b \in GF(q^2)$ and $\beta \in GF(q)$.

Hence, the three Fisher planes arise from the Baer subplanes $y = x^q s$, $\tau_{b_1,\beta_1}\{y = x^q s\}$ and $\tau_{b_2,\beta_2}\{y = x^q s\}$.

22 Theorem. *Given a $3q$-double nest plane π of order q^2, there are three q -nest planes ρ_i, $i = 1,2,3$ such that the $3q$-double net is obtained from the three q-nests using the group EH^-.*

8.3 Characteristic 3 and the 3/4-Problem

Recall if $p = 3$ then $EH^-\{y = x^q s \pm t, y = x^q s\}$ is a partial spread of cardinality $q(q+1)/4$ if $s^{q+1} = -t_1^2$, for $t^{q+1} = t_1^2 + t_2^2$ and

$$b^2 + b^{-2} \text{ is non-square for all } b \text{ or order dividing } (q+1)/2.$$

Hence, we see that if $q = 3$, then $(q+1)/4 = 1$ and $b^2 = b^{-2} = 1$ and $b^2+b^{-2} = 2$ is non-square in $GF(3)$. Choose $t = e$, where $e^2 = -1$ then we obtain a partial spread of cardinality $3q = 3^2$. This partial spread covers the Desarguesian affine spread Σ except for the component $x = 0$ and hence defines an associated conical flock spread, obviously Desarguesian, from Σ but obtained using double-nests. One may verify that this cannot work when $q = 3^3$, but we leave the general question as an open problem.

9 The Thas, Herssens, De Clerck Conical Flock Plane

It is known that the Thas, Herssens, De Clerck conical flock of order 11^2 admits a group acting on the flock isomorphic to $Z_2 \times S_3$. If $\mathcal{G}$ is the full group of the conical flock and $\mathcal{F}$ is the full group of the associated translation plane π, let EK^* denote the subgroup of $\mathcal{F}$ where K^* is the kernel homology group of order 10 and E is the regulus-inducing elation group of order 11. Then, $\mathcal{F}/EK^*$ is isomorphic to $\mathcal{G}$. Hence, there exists a preimage group G of order $11 \cdot 12 \cdot 10$ corresponding to $Z_2 \times S_3$ containing EK^* and note that E is normal within G.

23 Lemma. *(1) There is a normal subgroup $\langle\theta\rangle$ of G of order 3.*

(2) There is a Desarguesian plane Σ of order 11^2 whose spread is defined as the set of $GF(11)$-subspaces of dimension 2 that are left invariant by $\langle\theta\rangle$.

(3) The normalizer of $\langle\theta\rangle$ acts as a collineation group of Σ so G is a subgroup of $\Gamma L(4,q)$ may be considered a subgroup of $\Gamma L(2,11^2)$ acting on Σ.

(3) $G \cap GL(4,11)$ contains a subgroup of order $11(12)$.

Proof. Since there is a normal subgroup of order 3 in $Z_2 \times S_3 \simeq G/EK^*$, there is an element θ of order 3 in G,such that $\langle\theta\rangle\, EK^* \simeq \langle\theta\rangle \,/\, \langle\theta\rangle \cap EK^* \simeq \langle\theta\rangle$, it follows that there is a normal subgroup of order 3 in G. Hence, we have the proof to (1). Now θ permutes a set of 11 reguli in π so must fix at least two reguli sharing the axis of E. Since 3 does not divide 10, it follows that θ fixes at least three components. Hence, by Johnson [12], we have the proof to part (2).

$G \cap GL(4,11)$ has order $11 \cdot 12 \cdot 5$ and contains a kernel homology group of order 10. Since the group is solvable, there is a subgroup of order $11 \cdot 12$.

We may now apply our main theorem to show that π may be constructed from a Desarguesian affine plane of order 11^2 by $3 \cdot 11$ -double-nest replacement.

QED

24 Theorem. *The Thas, Herssens, De Clerck conical flock plane of order 11^2 may be constructed from a Desarguesian plane of order 11^2 by double-nest replacement of a $3 \cdot 11$-double nest.*

We shall illustrate the T-H-DC plane of order 11^2 in the form

$$EH^-\{y = x^q s \pm xt, y = x^q \alpha_s s\} \cup \mathcal{B},$$

and provide the explicit elements s and t and scalar element α_s. These matrices were found by analyzing the group EH isomorphic to $Z_2 \times S_3$ on the line at infinity of the associated translation plane.

We define the field coordinatizing the associated Desarguesian affine plane Σ as

$$\left\langle \begin{bmatrix} u & -t \\ t & u \end{bmatrix}; u,t \in GF(11) \right\rangle.$$

We let

$$s=\begin{bmatrix}3 & -2\\ 2 & 3\end{bmatrix},\ t=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}, \alpha_s = 4.$$

Let

$$\theta=\begin{bmatrix}5 & -3 & 0 & 0\\ 3 & 5 & 0 & 0\\ 0 & 0 & 5 & -3\\ 0 & 0 & 3 & 5\end{bmatrix}, \text{ and note that } \theta^3 = I.$$

We also obtain the following characterization theorem when $p = 11$.

25 Theorem. *Let π be a translation plane of order 11^2 that admits a linear collineation group of order $11(12)$ then π is one of the following six planes:*

(1) Desarguesian,
(2) Hall,
(3) Fisher,
(4) Derived Fisher,
(5) Thas-Herssens-De Clerck,
(6) derived Thas-Herssens-De Clerck.

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