Note di Matematica **23,** n. 1, 2004, 167–171.

Banach–Steinhaus type theorems in locally convex spaces for linear bounded operators

Lahrech Samir[i]
Department of mathematics, Mohammed first university,
Faculty of sciences, Oujda, Morocco
lahrech@sciences.univ-oujda.ac.ma

Received: 16/3/2003; accepted: 9/6/2004.

Abstract. Banach–Steinhaus type results are established for linear bounded operators between locally convex spaces without barrelledness.

Keywords: Barrelledness, sequential continuity, boundedness, locally convex spaces.

MSC 2000 classification: 47B37 (primary), 46A45.

Introduction

In the past, all of Banach-Steinhaus type results have been established only for some special classes of locally convex spaces, e.g.,barrelled spaces ([2],[3],[4]), s-barrelled spaces ([5]), strictly s-barrelled spaces ([6]), etc. Recently, Cui Chengri and Songho Han ([1]) have obtained a Banach-Steinhaus type result which is valid for every locally convex space as follows

1 Theorem. *Let $(X,\lambda),(Y,\mu)$ be locally convex spaces and $T_n : X \to Y$ bounded linear operators, $n \in \mathbf{N}$. If weak-$\lim_n T_n y = Ty$ exists at each $y \in X$, then the limit operator T send $\eta(X,X^b)-$bounded sets into bounded sets.*

In this paper we would like to obtain the same result by taking the topology λ in place of $\eta(X,X^b)$.

Let (X,λ) and (Y,μ) be locally convex spaces. Assume that the locally convex topology μ is generated by the family $(q_\beta)_{\beta\in I}$ of semi-norms on Y.

An operator $T : X \to Y$ is said to be sequentially continuous if $\{x_n\}$ is a sequence in X such that $x_n \to x$ then $Tx_n \to Tx$; T is said to be bounded if T sends bounded sets into bounded sets. Clearly, continuous operators are sequentially continuous, and sequentially continuous operators are bounded but in general, converse implications fail. Let X', X^s and X^b denote the families

[i]This work is partially supported by Department of mathematics of Mohammed first university in Oujda.

of continuous linear functionals, sequentially continuous linear functionals and bounded linear functionals on X, respectively. In general, the inclusions $X' \subset X^s \subset X^b$ are strict.

For a linear dual pair (E, F) let $\beta(E, F)$ denote the strongest (E, F) polar topology on E which is just the topology of uniform convergence on $\sigma(F, E)$-bounded subsets of F.

Let $\mathcal{C}(X_\lambda)$, $\mathcal{B}(X_\lambda)$ and $\mathcal{C}_0(X_\lambda)$ denote the families of conditionally $\lambda-$ sequentially compact sets, bounded sets in (X, λ) and convergent sequences in (X, λ) to 0, respectively.

Let $\sigma \subset \mathcal{B}(X_\lambda)$ such that $\bigcup\limits_{C \in \sigma} C = X$.

Let ζ be the topology on X^b generated by the family of semi-norms

$$P_C(f) = \sup_{y \in C} \mid f(y) \mid, \quad C \in \sigma.$$

Let $\eta(X, X^s_\zeta)$ denote the topology of uniform convergence on conditionally $(X^s, \zeta)-$ sequentially compacts sets of X^s.

Remark that $\eta(X, X^s_\zeta)$ is coarser than $\eta(X, X^b)$. It follows immediately

2 Proposition. *For every locally convex space X the following conditions are equivalent.*
(1) For every locally convex space Y and for every sequence $\{T_n\}_n$ of bounded linear operators from X into Y such that for every $C \in \sigma$ $\mu - \lim\limits_n T_n x = Tx$ uniformly in $x \in C$, the limit operator T is also (λ, μ)-bounded.
(2) (X^b, ζ) is sequentially complete.

Proof. (1)$\Rightarrow$ (2). Let $\{f_n\}$ be a $X^b_\zeta-$ Cauchy sequence in X^b. then, there exists a linear functional f such that for every $C \in \sigma$ $\lim\limits_n f_n(x) = f(x)$ uniformly in $x \in C$. Consequently, $f \in X^b$ by (1).

(2)$\Rightarrow$ (1). Let Y be a locally convex space and $\{T_n\}$ a sequence of bounded linear operators from X into Y such that for every $C \in \sigma$ $\mu - \lim\limits_n T_n x = Tx$ uniformly in $x \in C$. Suppose that B is a bounded subset of X_λ and $y' \in Y'$. Then there exists $\beta_0 \in I$ and $c_1 > 0$ such that for every $z \in Y$ $\mid y'(z) \mid \leq c_1 q_{\beta_0}(z)$. Therefore, for every $C \in \sigma$ $\lim\limits_n y'(T_n x) = y'(Tx)$ uniformly in $x \in C$. Since $y' \circ T_n \in X^b$ for all $n \in \mathbf{N}$, $y' \circ T \in X^b$ by (2). Therefore, $\{y'(Tx) : \ x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is $\mu-$bounded by the classical Mackey theorem. QED

The proof of proposition 1 gives the following

3 Proposition. *For every locally convex space X the following conditions are equivalent.*
(1) For every locally convex space Y and for every sequence T_n of sequentially

continuous linear operators from X into Y such that for every $C \in \sigma$ $\mu-\lim_n T_n y = Ty$ uniformly in $y \in C$, the limit operator T is also (λ, μ)-bounded.
(2) X^s_ζ is sequentially complete.

Assume now that σ satisfies also the condition $\mathcal{C}(X_\lambda) \subset \sigma$. It follows immediately

4 Proposition. *X^s_ζ is sequentially complete.*

Proof. Suppose that $\{A_n\}_n$ is Cauchy sequence in X^s_ζ. Then,

$$\forall \varepsilon > 0 \;\; \forall C \in \sigma \;\; \exists n_0 \in \mathbf{N}$$

such that

$$\forall n, m \geq n_0 \;\; \forall y \in C \quad | A_n y - A_m y | < \varepsilon. \tag{1}$$

On the other hand, $\forall y \in X$ $\{A_n y\}_n$ is Cauchy sequence in $\mathbf{R}$. Consequently, $A_n y \to Ay$ in $\mathbf{R}$, as $n \to \infty$. Letting $m \to \infty$ in (1), it follows that

$$\forall \varepsilon > 0 \;\; \forall C \in \sigma \;\; \exists n_0 \in \mathbf{N}$$

such that

$$\forall n \geq n_0 \;\; \forall y \in C \quad | A_n y - Ay | \leq \varepsilon. \tag{2}$$

We will show now that $A \in X^s$.

Let $\{x_n\}_n \in \mathcal{C}_0(X_\lambda)$. Pick any $\varepsilon > 0$. As $\{x_n\} \in \mathcal{C}(X_\mu) \subset \sigma$, then there exists $n_0 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ and forall $y \in C$

$$| A_{n_0} x_n - A x_n | \leq \frac{\varepsilon}{2}.$$

On the other hand, there exists $n_1 \in \mathbf{N}$ such that $\forall n > n_1$ $| A_{n_0} x_n | \leq \frac{\varepsilon}{2}$.

In this case

$$\forall n > n_1 \;\; | A x_n | \leq \varepsilon.$$

Consequently, $A \in X^s$. Thus, X^s_ζ is sequentially complete. QED

5 Proposition. *Let $(X, \lambda), (Y, \mu)$ be locally convex spaces and $T_n : X \to Y$ sequentially continuous linear operators, $n \in \mathbf{N}$. If for every $C \in \sigma$ $\mu-\lim_n T_n z = Tz$ uniformly in $z \in C$, then the limit operator T send $\eta(X, X^s_\zeta)-$bounded sets into bounded sets.*

Proof. Let $y' \in Y'$, $C \in \sigma$. Then there exists $\beta_0 \in I$ and $c_1 > 0$ such that for every $z \in Y$ $| y'(z) | \leq c_1 q_{\beta_0}(z)$. Consequently, $\sup_{z \in C} | y'(T_n z) - y'(Tz) | \to 0$.

Since X^s_ζ is sequentially complete, then $\{y' \circ T_n :\ n \in \mathbf{N}\}$ is conditionally $(X^s, \zeta)-$sequentially compact.

Suppose that B is a $\eta(X, X^s_\zeta)-$bounded subset of X and $\{x_k\} \subset B$. Then $\exists c > 0\ \forall k \in \mathbf{N}\ \forall n \in \mathbf{N} \mid y'(T_n x_k) \mid \leq c$. Fix a $k \geq k_0$ and $\varepsilon > 0$. Since $\lim_n y'(T_n x_k) = y'(T x_k)$ there is an $n_0 \in \mathbf{N}$ such that $\mid y'(T_{n_0} x_k) - y'(T x_k) \mid < \frac{\varepsilon}{2}$. Therefore,

$$\mid y'(Tx_k) \mid \leq \mid y'(Tx_k) - y'(T_{n_0} x_k) \mid + \mid y'(T_{n_0} x_k) \mid < \frac{\varepsilon}{2} + c.$$

This shows that $\{y'(Tx) : x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is $\mu-$bounded by the classical Mackey theorem. Thus, we achieve the proof. *QED*

Let us denote by $\theta(X, X^s_\zeta)$ the topology of uniform convergence on $(X^s, \zeta)-$Cauchy sequences. A subset B of X is said to be $\theta(X, X^s_\zeta)-$bounded if for every $X^s_\zeta-$Cauchy sequence $\{f_n\}$ there exists $c > 0$ such that for every sequence $\{x_k\}$ in $B \mid f_n(x_k) \mid \leq c\ \ \forall n \in \mathbf{N}\ \ \forall k \in \mathbf{N}$.

Then the proof of proposition 4 gives the following.

6 Proposition. *Let $(X, \lambda), (Y, \mu)$ be locally convex spaces and $T_n : X \to Y$ bounded linear operators, $n \in \mathbf{N}$. If for every $C \in \sigma$ $\mu-\lim_n T_n y = Ty$ uniformly in $y \in C$, then the limit operator T send $\theta(X, X^s_\zeta)$bounded sets into bounded sets.*

Now we have a useful proposition as follows.

7 Proposition. *Let $(X, \lambda), (Y, \mu)$ be locally convex spaces and $T_n : X \to Y$ sequentially continuous linear operators, $n \in \mathbf{N}$. If for every $C \in \sigma$ $\mu-\lim_n T_n y = Ty$ uniformly in $y \in C$, then the limit operator T send $\lambda-$bounded sets into bounded sets.*

Proof. By propositions 2 and 3, we deduce the result. *QED*

References

[1] Cui Chengri, Songho Han: *Banach-Steinhaus properties of locally convex spaces*, Kangweon-Kyungki Math. Jour. **5**, no. 2 (1997), 227–232.

[2] J. Horvath: Topological vector spaces and distributions, Addison-Wesley, 1966.

[3] A. Wilansky: Modern Methods in TVS, McGraw-Hill, 1978.

[4] G. Kothe: Topological vector spaces, I, Springer-Verlag, 1983.

[5] R. Snipes: *S-barrelled topological vector spaces*, Canad. Math. Bull. **21**, no. 2 (1978), 221–227.

[6] W. H. Hsiang: *Banach-Steinhaus theorems of locally convex spaces based on sequential equicontinuity and essentially uniform boundedness*, Acta Sci.Math. **52** (1988), 415–435.

[7] Ronglu Li, Min-Hyung Cho: *Banach-Steinhaus type theorem which is valid for every locally convex space*, Appl. Funct. Anal. **1** (1993), 146–147.