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Large quartic groups on translation planes, I—odd order: Characterization of the Hering planes

Mauro Biliotti
Dipartimento di Matematica, Università di Lecce,
Via per Arnesano, 73100 Lecce, Italy
biliotti@ilenic.unile.it

Vikram Jha[i]
Mathematics Dept., Caledonian University,
Cowcaddens Road, Glasgow, Scotland
v.jha@gcal.ac.uk

Norman L. Johnson[i]
Mathematics Dept., University of Iowa,
Iowa City, Iowa 52242, USA
njohnson@math.uiowa.edu

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Abstract. The Hering planes of order q^2 and the Walker planes of order 5^2 are shown to be the unique classes of planes with spreads in $PG(3,q)$ or $PG(3,5)$, respectively, admitting at least two 'large' quartic groups with distinct centers.

Keywords: quartic group, translation plane, Hering plane

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1 Introduction

In the theory of finite translation planes of order p^r, there are various important results classifying the collineation subgroups generated by collineations g of order p when the fixed point nature of the collineation is specified.

For example, if $Fixg$ is a component of the plane, the Hering-Ostrom theorem (see Hering [8], Ostrom [13]) provides a complete classification of the possible groups generated.

If $Fixg$ is a Baer subplane and p is strictly larger than 3, the Foulser theorem (see Foulser [4]) on collineation groups generated by Baer p-collineations shows

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that the generated groups fall into the same general classes as do the groups generated by elations.

When the spread for the translation plane is within $PG(3,q)$, any collineation g of order p that lies within $GL(4,q)$ (the linear translation complement) is either an elation, a Baer p-element or has minimal polynomial $(x-1)^4$ and $Fixg$ is a 1-dimensional $GF(q)$-subspace. However, there are no basic results on the possible groups generated by such 'quartic' collineations. In particular, we will be interested in groups generated by 'quartic groups,' so to be clear, we provide an explicit definition.

1 Definition. Let π be a translation plane of order $q^2, q = p^r$, p a prime, with spread in $PG(3,q)$. A 'quartic group' T is an elementary Abelian p-group all of whose non-identity elements are quartic (i.e. have minimal polynomials $(x-1)^4$) and which fix the same 1-dimensional $GF(q)$-subspace pointwise. The fixed-point space is called the 'quartic center' of the group and the unique component of π containing the center is called the 'quartic axis'.

On the other hand, if a translation plane of order q^2, $p^r = q$, p a prime, admits a collineation group in the translation complement isomorphic to $SL(2,q)$, the Foulser-Johnson Theorem (see [5]) does provide a complete classification of the possible planes and within this class are the Hering planes of odd order for which the p-elements are quartic and the Sylow p-subgroups of $SL(2,q)$ are 'quartic groups' of order q. Indeed, also the Walker planes of order 25 admit a collineation group isomorphic to $SL(2,5)$ where the 5-elements are quartic.

However, utilization of the Foulser-Johnson theorem is often obtained via an indirect approach and to resolve certain of such technicalities, the authors (see Biliotti, Jha, Johnson [1]) recently extended the Foulser-Johnson theorem to determine the possible translation planes of order q^2 that admit a collineation group G containing a normal subgroup N such that G/N is isomorphic to $PSL(2,q)$.

In previous articles (see Jha and Johnson [10],[9]) two of the authors have studied what might be called 'large Baer group' planes and asked the following question: What are the finite translation planes admitting two large Baer p-groups? In planes of order q^2, in this context, 'large' means of order $> \sqrt{q}$, for q odd and $\geq 2\sqrt{q}$ when q is even. Such planes were completely determined as either the Hall planes or the Dempwolff plane of order 16.

In this article, we ask the same sort of question regarding quartic groups: What are the finite translation planes with spreads in $PG(3,q)$ that admit two 'large' quartic groups? We are able to then attack this problem using the Foulser-Johnson Theorem and its generalization by Biliotti-Jha-Johnson.

To reiterate, a 'large' quartic group of odd order is a quartic p-group of order $> \sqrt{q}$. Our main result is as follows:

2 Theorem. *Let π be a translation plane with spread in $PG(3,q)$, where $q = p^r$, p a prime. If π admits two quartic p-groups of orders $> \sqrt{q}$ with distinct centers then*

(1) the group generated by the quartic p-elements is isomorphic to $SL(2,q)$, and

(2) the plane is one of the following:

(a) a Hering plane or

(b) $q = 5$ and the plane is one of the three exceptional Walker planes.

2 Background

In this section, we list for convenience of the reader three results, which are essentially lists of possibilities, and which are used most prominently in our proof.

The following result is a compilation of results of various work. The reader is referred to the references of Liebler and Kantor [11] and the remarks preceding the statement of Theorem (5.1)

3 Theorem. *(Liebler and Kantor [11] (5.1))*

Let H be a primitive subgroup of $\Gamma L(4,q)$ for $p^r = q$. Then one of the following holds:

(a) $H \geq SL(4,q)$,

(b) $H \leq \Gamma L(4,q')$ with $GF(q') \subset GF(q)$,

(c) $H \leq \Gamma L(2,q^2)$, with the latter group embedded naturally,

(d) $H \leq Z(H)H_1$, where H_1 is an extension of a special group of order 2^6 by S_5 or S_6. Here, q is odd, H_1 induces a monomial subgroup of $O^+(6,q)$, and H_1 is uniquely determined up to $\Gamma L(4,q)$-conjugacy,

(e) $H^{(\infty)}$ is $S_p(4,q)'$ or $SU(3,q^{1/2})$,

(f) $H \leq \Gamma O^{\pm}(4,q)$,

(g) $H^{(\infty)}$ is $PSL(2,q)$ or $SL(2,q)$ (many classes),

(h) $H^{(\infty)}$ is A_5. Here, H arises from the natural permutation representation of S_5 in $O(5,q)$ and $p \neq 2,5$,

(i) $H^{(\infty)}$ is $2 \cdot A_5$, $2 \cdot A_6$ or $2 \cdot A_7$. These arise from the natural permutation representation of S_7 in $O(7,q)$,

(j) $H^{(\infty)} = A_7$ and $p = 2$,

(k) $H^{(\infty)} = S_p(4,3)$, and $q \equiv 1 \bmod 3$. This arises from the natural representation of the Weyl group $W(E_6)$ in $O^+(6,q)$,

(l) $H^{(\infty)} = SL(2,7)$, and $q^3 \equiv 1 \bmod 7$. Here, $H^{(\infty)}$ lies in the group $2 \cdot A_7$,

(m) $H^{(\infty)} = 4 \cdot PSL(3,4)$, and q is a power of 9,

(n) $H^{(\infty)} = S_z(q)$, and $p = 2$.

4 Theorem. *(Ostrom [15] (2.17))*

Let π be a translation plane of order q^2 with spread in $PG(3,q)$. Let G be a subgroup of the linear translation complement and assume that $(p, |G|) = 1$ where $p^r = q$. Let $\overline{G} = GK/K$ where K is the kernel homology group of order $q-1$.

Then at least one of the following holds:

(a) G is cyclic.

(b) $\overline{G}$ has a normal subgroup of index 1 or 2 which is cyclic or dihedral or isomorphic to one of $PSL(2,3)$, $PGL(2,3)$ or $PSL(2,5)$.

(c) G has a cyclic normal subgroup H such that G/H is isomorphic to a subgroup of S_4.

(d) G has a normal subgroup H of index 1 or 2. H is isomorphic to a subgroup of $GL(2,q^a)$, for some a, such that the homomorphic image of H in $PSL(2,q^a)$ is one of the groups in the list given under (b).

(e) There are five pairs of points on ℓ_∞ such that if (P,Q) is any such pair, there is an involutory homology with center P (or Q) whose axis goes through Q (or P). $\overline{G}$ has a normal subgroup $\overline{E}$ which is elementary Abelian of order 16. For each pair (P,Q), each element of $\overline{E}$ either fixes P and Q or interchanges them. $\overline{G}$ induces a transitive permutation group on these ten points.

(f) $\overline{G}$ has a subgroup isomorphic to $PSL(2,9)$ and acts in the following manner: Each Sylow 3-subgroup has exactly two fixed points on ℓ_∞. If (P,Q) is such a pair, G contains (P,OQ) and (Q,OP) -homologies of order 3. There are ten such pairs and $\overline{G}$ is transitive on these ten pairs.

(g) G has a reducible normal subgroup H not faithful on its minimal subspaces and satisfying the following conditions:

Either the minimal H-spaces have dimension two and H has index 2 in G or the minimal H-spaces have dimension 1. In the latter case, if H_0 is a subgroup fixing some minimal H-space pointwise, then H/H_0 is cyclic and G/H is isomorphic to a subgroup of S_4.

5 Theorem. *(Biliotti, Jha, Johnson [1]) Let π denote a translation plane of order q^2 and assume that π admits a collineation group G containing a normal subgroup N such that G/N is isomorphic to $PSL(2,q)$.*

Then π is one of the following planes:

(1) Desarguesian, (2) Hall, (3) Hering, (4) Ott-Schaeffer, (5) one of three planes of Walker of order 25 or (6) the Dempwolff plane of order 16.

3 Order 25

There are three exceptional Walker planes of order 25 admitting $SL(2,5)$. In this situation, the group $SL(2,5)$ leaves invariant a component of the spread and induces $SL(2,5)$ faithfully on that component. The group is reducible but not completely reducible. One of these planes is the Hering plane of order 25 and admits a second group isomorphic to $SL(2,5)$ that acts irreducibly. It is noted that when one refers to the Walker planes, it is normally considered that the group acts reducibly. Reference to the Hering plane normally indicates that we are considering the irreducible $SL(2,5)$.

Furthermore, all translation planes of order 25 and their collineation groups are known. It is straightforward to check that the only translation planes of order 25 admitting two large quartic groups are the Walker planes of order 25 and the Hering plane of order 25 (see Charnes [2], Czerwinski [3], Heimbeck [6]).

Hence, in the following we may assume that q is not 5.

4 p is forced to be 5

In the following sections, we will give the proof of Theorem 2 listed in the introduction.

We shall give the proof as a series of lemmas. The hypothesis in the body of the statement shall be assumed throughout all sections.

6 Lemma. *$p > 3$.*

Proof. There are no quartic elements when $p = 3$, by Theorems (49.4), (49.5) of [12]. $\boxed{QED}$

7 Lemma. *Suppose T and S be quartic groups of orders $> \sqrt{q}$, with distinct centers. Then the axes L_T and L_S, respectively, are distinct.*

Proof. Assume that $L_T = L_S$. $FixT \neq FixS$. We may now assume that $FixT \cap FixS = 0$ and the group generated by T and S on L_T, clearly is isomorphic to $SL(2,p^t)$, where $p^t > \sqrt{q}$, since the Hering-Ostrom theorem ([8], [13]), applies to the Desarguesian spread π_{L_T} of L_T. However, the elation net within π_{L_T} defines a set of Desarguesian subplanes of π_{L_T}, each of which is left invariant by $SL(2,p^t)$. Since these Desarguesian subplanes will have orders at least $> \sqrt{q}$ and correspond to subfields of $GF(q)$, it follows that $SL(2,q)$ is generated on L_T. By Theorem 5, the plane is determined. However, the plane is either Walker of order 25, Hering, Hall or Desarguesian. Since the order is $\neq 25$ then only the Hering planes have quartic groups and in this setting no two of the quartic groups share the same axis. Hence, we have a contradiction so $L_T \neq L_S$. $\boxed{QED}$

8 Lemma. *Assume that no two quartic axes are equal. Then there are no non-trivial elations in $\langle T, S\rangle$.*

Proof. There are at least two quartic axes with quartic groups T and S. If g is an elation with axis M, then at least one of the quartic groups, say T , does not have axis M, implying that M is in an orbit of length $> \sqrt{q}$ under T. We may again use the Hering-Ostrom theorem ([8], [13]) to conclude that, since $p > 3$, then the group generated by the elations is isomorphic to $SL(2, p^t)$, and there is an orbit of length $p^t + 1$. Hence, $p^t + 1 > \sqrt{q}$, so that $p^t > \sqrt{q}$ as p is odd. The elation net $\mathcal{E}$ is a Desarguesian net of order q^2 and coordinatized by $GF(p^t)$. This implies that $GF(p^t)$ is a subfield of $GF(q^2)$. Hence, $p^t - 1$ divides $q^2 - 1$ and $p^t - 1 > \sqrt{q} - 1$. So, it follows that $p^t = q^{2/3}, q$ or q^2. In the latter two cases, $SL(2, q)$ is generated by elations so that the plane is Desarguesian by Foulser-Johnson [5]. However, there are no quartic elements in Desarguesian planes.

Hence, we may assume that we have $SL(2, q^{2/3})$ generated by elations within $\langle T, S \rangle$. Moreover, an elation axis is a quartic axis and conversely. Hence, there are exactly $q^{2/3} + 1$ elation axes permuted by a Sylow p-subgroup of order $p^a q^{2/3}$, where $p^a > \sqrt{q}$. It follows that there is a p-element subgroup of order p^a that fixes two components. Since the group is linear, it follows that there is a Baer group of order $p^a > \sqrt{q}$. However, since there are elations, this is a contradiction to the mains results of Foulser [4]. $\boxed{QED}$

9 Remark. For the rest of this article, we assume that $G = \langle T, S \rangle$,where T and S are large quartic groups with distinct centers.

10 Lemma. *There are no non-trivial Baer p-elements in G. Hence, the Sylow p -subgroups in G have orders dividing q.*

Proof. Suppose that g is a Baer p-element. Since $p > 3$, by results of Foulser [4], there is an invariant net N of degree $q + 1$ wherein lives $Fixg$ and all other Baer p-axes. Then, T must leave N invariant so that the axis L_T of T is a component of the net N. Moreover, also S must leave N invariant so that L_S is a component of N. Similarly, all quartic axes corresponding to quartic groups in G are in N.

First assume that $Fixg = \pi_o$ is invariant under all quartic groups Q in G.

Then Q induces a p-group on π_o, a Desarguesian plane of order q. This implies that Q induces a faithful elation group of order $> \sqrt{q}$ on π_o. Hence, the group induced on π_o is $SL(2, p^t)$, where $p^t > \sqrt{q}$. Thus, $SL(2, q)$ is induced on π_o. We may apply Theorem 5 to show that π is a Hering plane. But, the Hering planes do not admit Baer p-elements. Hence, π_o cannot be left invariant by G. In particular, some quartic group must move π_o.

Thus, without loss of generality, $Fixg = \pi_o$ is moved by T, implying that $FixT \cap \pi_o = 0$, since π_o is moved on the net N.

Therefore, there are at least $> \sqrt{q} + 1$ kernel subplanes of N. But, if there are at least three Desarguesian Baer subplanes in N, then there are $1 + q$, since the kernel of any one of them is $GF(q)$, by results of Foulser [4]. This then implies that the net N is a regulus net. However, this also means that

$SL(2,q)$ is generated by Baer p-elements. Thus, the plane is Hall by Theorem 5, which cannot occur due to the existence of quartic elements. This completes the proof. $\boxed{QED}$

Thus, the only p-elements of G are quartic.

11 Lemma. *There are $1+kp^a$ quartic axes if $|T|=p^a$ and G is transitive on the quartic axes. Furthermore G is irreducible on the 4-dimensional $GF(q)$-vector space π.*

Proof. Since all p-elements are quartic, there are $1+kp^a$ quartic axes, where p^a is the order of T. Since each quartic axis is fixed by a quartic element of order p that fixes no other quartic axis, we may utilize Gleason's Theorem [7], which implies that the group G is transitive on the set of quartic axes,

If the group is reducible, then the only possible irreducible submodules are 1 or 2-dimensional over $GF(q)$. Hence, there is an invariant 2 -subspace which certainly cannot be a component since there are at least two quartic groups with distinct quartic axes. Thus, by questions of order, there is an invariant Baer subplane π_o of order q. Since each quartic group induces on π_o a non-trivial p-group, the p-group must be an elation group acting on π_o.

Therefore, the quartic groups acting on π_o must be elation groups of order p^a and there are at least $1+p^a$ axes. Hence, the group acting on π_o induces $SL(2,p^a)$,where $p^a > \sqrt{q}$, so again we obtain that the induced group must be $SL(2,q)$, since the elation net is a Desarguesian subnet of π_o, and we may apply Theorem 5. $\boxed{QED}$

12 Lemma. $p=5$ *or the theorem is proved.*

Proof. If $p>5$, we may apply the main theorems of Ostrom [16] (2.6) and (2.8), which says there is an invariant Hering subplane of order p^{2t} and there is a normal subgroup isomorphic to $SL(2,p^t)$ leaving the subplane invariant. Since the quartic groups normalize $SL(2,p^t)$, they must be within $SL(2,p^t)$ so that clearly, $p^t > \sqrt{q}$. Thus, $p^{2t} > q$, so the subplane order is larger than the order of a Baer subplane and so it must be q^2. Hence, there is a group isomorphic to $SL(2,q)$ and the plane is Hering (the Walker planes of order 25 are excluded, in this argument, as $p>5$). $\boxed{QED}$

5 G is primitive

We now may assume that $p=5$.

13 Lemma. *G is primitive.*

Proof. Since $\langle T,S\rangle$ cannot permute a set of four 1 -dimensional K-subspaces nor permute a set of two 2-dimensional K -subspaces, it follows that G is primitive. $\boxed{QED}$

We may now utilize the list of primitive subgroups of $\Gamma L(4,5^r)$ of Theorem 3. We emphasize that all p-groups are quartic and a quartic p-group has order bounded by q.

14 Lemma. *G does not contain $SL(4,q)$, so case (a) of Theorem 3 cannot occur.*

Proof. If G does contain $SL(4,q)$, the Sylow p-subgroups are too large. $\boxed{QED}$

15 Lemma. *G is not in $\Gamma L(4,q')$ for $GF(q')$ a proper subfield of $GF(q)$. Hence, case (b) of Theorem 3 does not occur.*

Proof. The question is whether G may be considered a subgroup of $\Gamma L(4,5^s)$ if $s<r$. That is, if there is a $GF(5^r)$-basis over which we may consider a 4-dimensional $GF(5^s)$-subspace and the elements of G may be written over $GF(5^s)$. If so then we would be able to write T of order at least $>\sqrt{5^r}$ over a smaller field. T is faithful on L_T and with appropriate coordinates can be represented on L_T as a subgroup of the Desarguesian elation group $\langle(x,y)\longmapsto(x,x\alpha+y);\alpha\in GF(5^r)\rangle$. If T could be considered over a proper subfield $GF(5^r)$ then $s>\sqrt{5^r}$, implying that $s=r$, a contradiction. $\boxed{QED}$

16 Remark. By adjoining the kernel homology K^*, a group of order $q-1$, if G is any group of $\Gamma L(4,q)$, then we may assume without loss of generality that G is not in $\Gamma L(4,q')$ for $GF(q')$ a proper subfield of $GF(q)$.

17 Lemma. *G cannot be embedded in $\Gamma L(2,q^2)$.*

Proof. $G=\langle T,S\rangle$, where T and S are large quartic groups with distinct centers. Let K^* denote the kernel homology group of the plane. Let Z denote the center of $\Gamma L(2,q^2)$. Then $\overline{G}=GK^*Z/Z$ is a subgroup of $P\Gamma L(2,q^2)$, which is also within $PGL(4,q)$. Clearly, $\overline{G}$ is generated by two distinct p-groups of orders $>\sqrt{q}$, is $PSL(2,q)$ since $q>5$. It follows that G is isomorphic to $SL(2,q)$ and we may apply Foulser-Johnson [5]. Hence case (c) of Theorem 3 does not occur. $\boxed{QED}$

18 Lemma. *The center $Z(G)$ is a subgroup of the kernel homology group of order $q-1$.*

Hence, $G/Z(G)$ is the group induced on the line at infinity of π.

Proof. Let $h\in Z(G)$, so that h fixes each quartic axis and fixes each quartic center. Let X be a quartic center and let k be in the kernel K^* such that hk fixes X pointwise. Then hk is trivial or Baer since it fixes all of the quartic axes. If hk is Baer, and fixes π_1 pointwise, then π_1 is invariant by all quartic elements, since h and k are both central. However, this says that G is reducible, contrary to the above. Hence, $hk=1$ and h is a kernel homology. $\boxed{QED}$

19 Lemma. *GK^*, where K^* is the kernel homology group, is primitive.*

Proof. G is primitive if and only if GK^* is primitive. $\boxed{QED}$

20 Lemma. *Let M be a normal subgroup of G containing $Z(G)$.*

(1) Then M has a unique homogeneous space.

(2) If M is reducible then M fixes at least three components or fixes at least three Baer subplanes that mutually intersect in the zero vector.

Proof. (1) follows by Clifford's theorem. Note that we are saying that if S is an irreducible $GF(q)M$-module then the subspace generated by all submodules which are $GF(q)M$-isomorphic to S is π, considered as a vector space. Assume that M is reducible. Then there is an irreducible module S of dimension 1 or 2. If S has dimension 1, there is an M-invariant component. Since G is primitive, there must be at least three M -invariant components. Thus, assume that S has dimension 2. If S is a component, the previous argument applies. Hence, S becomes a Baer subplane of π. It follows that M must leave invariant at least three distinct Baer subplanes, and either there is an irreducible submodule of dimension 1 or the three subplanes have trivial intersection. $\boxed{QED}$

21 Lemma. *Assume that M is a minimal normal subgroup of G containing $Z(G)$. If $M/Z(G)$ contains a p-element then $M/Z(G)$ is a non-Abelian simple group.*

Proof. First assume that $M/Z(G)$ is Abelian. Hence, there is a unique Sylow p-subgroup S_p in M, since M is minimal. However, S_p will fix a component, implying that S_p fixes at least three components by the previous lemma. Since G is generated by quartic elements, G is a subgroup of $GL(4, q)$. Thus, S_p is a Baer group. However, all p-elements are quartic.

Hence, $M/Z(G)$ is a direct sum of isomorphic simple groups.

If M contains a p-element g and there are at least two non-Abelian simple groups, then g is a quartic element and fixes a unique component L_g. It then follows that there is a non-Abelian simple group $S/Z(G)$ of order divisible by p such that S fixes L_g and must fix the center $Fixg$ on L_g, by a previous lemma, as there cannot be two different fixed point spaces for the same quartic axis. Thus, let S^- denote the normal subgroup of S generated by the p-elements in S. Then $S^- Z(G) = S$. Since S^- must fix $Fixg$ pointwise and is generated by p-elements, it follows that S^- is a p-group, a contradiction.

Hence, if M contains a p-element, there is exactly one non-Abelian simple group. $\boxed{QED}$

We adopt the notation of the previous lemma in what follows.

22 Lemma. *Either M is transitive on the $1 + kp^a$ quartic axes of $\langle T, S\rangle$, or M fixes each quartic axis and fixes the center of each quartic group.*

Proof. Recall that $G = \langle T, S\rangle$ is transitive on the set of $1 + kp^a$ quartic axes and that any normal subgroup M of $\langle T, S\rangle$ has orbits of the same size t

dividing $1+kp^a$ acting on the quartic axes. Since a quartic group fixes an infinite point, it must fix the corresponding M-orbit containing that infinite point so that $t-1$ is divisible by p^a. If $t>1$ then G fixes an M-orbit, as G is generated by two quartic groups. This completes the proof. $\boxed{QED}$

23 Lemma. *M acts transitively on the set of quartic axes.*

Proof. If this is not true, then M fixes each quartic axis. That is, in the previous lemma, we may assume that $t=1$. In this case, M fixes and hence normalizes all quartic groups. Then M fixes all centers of the quartic groups, since there is a unique fixed point space for each quartic axis.

Let g be an element of M and first assume that g is planar. Then, g must be Baer, since there are $>1+\sqrt{q}$ quartic axes. Hence, g has order dividing $q-1$, since there are no non-quartic p-elements.

If g is not planar then g has order dividing $q-1$, since g can fix no non-zero point. However, there is a kernel element k such that gk fixes a non-zero point, implying that gk is Baer or trivial. In either case, the order of gk divides $q-1$, implying in all cases that the order of g divides $q-1$. Let X be a quartic axis and let g be a Baer element of M. Let N_g denote the net of degree $q+1$ containing $Fixg$. Since M leaves each quartic center invariant, all of these 1-dimensional K-subspaces must lie on components of N_g. Since g leaves invariant exactly two Baer subplanes of N_g, say π_o and π_1, then if $Fixg$ is π_o, there exists a kernel homology k such that $Fixgk$ is π_1. Since M leaves invariant all quartic axes, and all such axes are in either π_o or π_1, it follows that M must leave invariant two Baer subplanes. The argument shows that the order of M divides $(q-1)^2$. In any case, M leaves invariant a Baer subplane and induces a group of order dividing $q-1$ on it. Thus, $M=BW$, where B is a cyclic group of order dividing $q-1$ and W is a cyclic subgroup of the kernel homology group of order $q-1$. Since M is normal, this implies that G leaves invariant a Baer subplane, which is contrary to that fact that G is primitive. $\boxed{QED}$

In the following, we assume that M is transitive on the quartic axes.

24 Lemma. *Assume that $M/Z(G)$ is solvable.*

(1) Then $M/Z(G)$ is a direct sum of isomorphic cyclic groups of prime order u.

(2) $M/Z(G)$ is regular on the set of quartic axes.

Hence,

$$u^\alpha = 1+kp^a.$$

Proof. If $M/Z(G)$ is solvable then it is an elementary Abelian u-group. Suppose that some element g of $M-Z(G)$ fixes a quartic axis. Then g must fix a 1-dimensional $GF(q)$-subspace so that the order of g must divide $q-1$ or g is a Baer p'-element by the previous lemma so that again the order of g must divide $q-1$. In either case, there is a kernel homology k so that gk is a Baer p'-element since g is not in $Z(G)$. Moreover, since $M/Z(G)$ is Abelian and M is transitive on the quartic axes then g must fix all quartic axes. Let h be a quartic p -element such that $Fixh$ is a component of $Fixgk$. Let N_g denote the net of degree $q+1$ containing $Fixgk$. We note that all quartic axes must lie on one of the two Baer subplanes of N_g, so that $1+kp^a \leq 1+q$. Furthermore, each quartic group must permute $1+kp^a$ components of $Fixgk$. Hence, G must act on the set of Baer subplanes of N_g. But, G is irreducible, implying that $Fixgk$ cannot be G-invariant. But, this means that there is a quartic p-element that interchanges two Baer subplanes of N_g. Since $1+kp^a > 1+\sqrt{q}$, it then follows that each quartic group must leave $Fixgk$ invariant a contradiction. Hence, $M/Z(G)$ is regular on quartic axes. This proves (2). $\boxed{QED}$

25 Lemma. *If $M/Z(G)$ is elementary Abelian of order u^c, let S_u be a Sylow u-subgroup of M. Hence, $S_uZ(G)=M$. Then $u \nmid (q-1)$.*

Proof. If u divides $q-1$ then S_u is a group of order u^b that acts on a set of q^2+1 components and if S_u fixes no components, then u divides q^2+1. So, u divides $(q^2+1, q-1) = (q^2-1+2, q-1) = 2$, since q is odd. Hence, $u=2$. By Theorem 3 (see remark on 2-groups in the first part of the proof of Theorem (5.1), p. 31 of [11]), it follows that we have case (d), so that $G \leq Z(G)G_1$, where G_1 is an extension of a special group of order 2^6 by S_5 or S_6. Hence, it follows that $M/Z(G)$ has order divisible by 2^6 and the order 5^a of a quartic group is exactly 5. Since $5^a = 5 > \sqrt{5^r}$, this implies that $5^2 > 5^r$, so that $r=1$ and the order of the plane is 25, which has been previously considered. $\boxed{QED}$

26 Lemma. *If $M/Z(G)$ is elementary Abelian then M is irreducible.*

Proof. If M is not irreducible then M fixes at least three 2-dimensional $GF(q)$-subspaces. Moreover, each quartic group permutes the fixed subspaces in orbits of length 5^a. Note that M acts faithfully as a subgroup of $GL(2,q)$ acting on one of the 2-dimensional $GF(q)$-subspaces. Thus, $M/Z(G)$ is an elementary Abelian subgroup of $PGL(2,q)$ and contains no $p=5$-elements. Hence, $M/Z(G)$ is a subgroup of a dihedral group of order dividing $2(q\pm 1)$. Suppose that u is an odd divisor of $q+1$. Then M contains a prime q-primitive divisor of q^2-1 and it follows that $M=S_uZ(G)$, where S_u is cyclic. But then there is a Desarguesian plane Σ consisting of all g_u-invariant subspaces and the normalizer of S_u in G is a collineation group of Σ. Hence, G is a subgroup of $\Gamma L(2,q^2)$, which is a contradiction by a previous lemma. Thus, u must divide $q-1$, a contradiction by the previous lemma. Note that since $q>5$, there is always an odd divisor of $q+1$. $\boxed{QED}$

27 Lemma. *If $M/Z(G)$ is a u-group, there is a cyclic u-group S_u of order u of M such that $S_uZ(G)=M$.*

Proof. Since $Z(G)$ is K^* (or contained in K^*), and u does not divide $q-1$ there is a unique Sylow u-subgroup S_u of M that is necessarily disjoint from $Z(G)$. However, M is irreducible, implying that S_u is irreducible so, by Schur's lemma, it follows that S_u is contained in a field. Since S_u is elementary Abelian, it follows that S_u is cyclic of order u. $\boxed{QED}$

28 Lemma. *There is a minimal normal subgroup M of G such that $M/Z(G)$ is a direct product of isomorphic non-Abelian simple groups.*

Proof. Let $M/Z(G)$ be a minimal normal subgroup of $G/Z(G)$. First assume that $M/Z(G)$ is Abelian. Then by the previous lemma, there is a cyclic and irreducible normal subgroup S_u of G, which implies by Theorem (19.6) of Passman [17], for example, that G is a subgroup of $\Gamma L(1, q^4)$, a contradiction since 5 divides $|G|$.

Hence, $M/Z(G)$ is a direct product of isomorphic non-Abelian simple groups. $\boxed{QED}$

We have seen in Lemma 21 that when M contains a $p = 5$-element, then $M/Z(G)$ is a non-Abelian simple group. In the following sections, we consider the two cases, when M does not or does contain a 5-element.

6 When M does not contain a 5-element

29 Lemma. *Assume that M does not contain a 5-element. Then the theorem is proved.*

Proof. So, in any case, M is transitive on the set of $1 + kp^a$ quartic axes. Furthermore, since the order of M is prime to p, we may utilize Ostrom's analysis of the possible subgroups that can act on translation planes. Note that the group G is in $GL(4, q)$ and $G/Z(G)$ is the group acting on the line at infinity. We consider the cases as listed in Theorem 4. Note that $M/Z(G)$ is a direct product of non-Abelian isomorphic simple groups.

Case (a) M is cyclic—clearly cannot hold.

Case (b) $M/Z(G)$ has a normal subgroup of index 1 or 2, which is cyclic or dihedral or isomorphic to one of $PSL(2, 3)$, $PGL(2, 3)$ or $PSL(2, 5)$. The only possibility is when $M/Z(G)$ is isomorphic to $PSL(2, 5)$. Since we have quartic groups of order $5^a > \sqrt{q}$, this again implies that we have $q = 5$ and the plane has order 25.

(c) M has a cyclic normal subgroup H such that M/H is isomorphic to a subgroup of S_4. This cannot hold since $M/Z(G)$ is not solvable.

(d) M has a normal subgroup H of index 1 or 2. H is isomorphic to a subgroup of $GL(2, m)$, such that the homomorphic image of H in $PSL(2, m)$ is one of the groups in the list given under (b).

That is, M has a normal subgroup H subgroup of $GL(2,m)$ such that the index of H in G is 1 or 2 and $H \cap SL(2,m)/(Z(GL(2,m)) \cap H)$ is cyclic, dihedral or isomorphic to $PSL(2,3)$, $PGL(2,3)$ or $PSL(2,5)$.

Since $M/Z(G)$ is a minimal normal subgroup then we may assume that $M = H$ and M is a subgroup of $GL(2,m)$ so that $M/Z(G)$ is a subgroup of $PGL(2,m)$. Hence, it can only be that $M/Z(G)$ is $PSL(2,5)$, implying that $q = p = 5$.

Consider (e), where $M/Z(G)$ permutes a set of five pairs on the line at infinity such that each pair corresponds to the axis/coaxis pair of an involutory homology. However, since we have quartic groups of order $5^a > \sqrt{q}$, this implies that a quartic group must have order exactly 5. Then for $5^a > \sqrt{5^r}$, implies that $q = 5$.

Case (f). $M/Z(G)$ contains a group isomorphic to $PSL(2,9)$ such that each Sylow 3-subgroup fixes two infinite points. Moreover, $M/Z(G)$ acts transitively on 10 pairs, again implying that 5 divides the order of M, a contradiction.

Case (g). M has a reducible normal subgroup H not faithful on its minimal subspaces. The normal subgroups H are such that $H/(Z(G) \cap H)$ is a non-Abelian simple group. It follows that if H_o is the subgroup fixing a 1-dimensional K-subspace pointwise, then H/H_o is cyclic. Hence, $H = H_o$. Now the 1-dimensional K-subspace X lies on a component L, implying that H acts as a central collineation group of an associated Desarguesian affine plane of order q defined by the spread of L. This is a contradiction unless L is fixed pointwise by H. Hence, the minimal subspaces of H are dimension 2. By the restrictions stated in case (g), H has index 2 in M. Thus, since $H/Z(G)$ is a direct product of non-Abelian simple groups, it follows that $H/Z(G) = M/Z(G)$. But, then M leaves invariant a component L, a contradiction. This completes the proof of the lemma. $\boxed{QED}$

7 When M contains a 5-element

Hence, M contains an element of order $p = 5$. Furthermore, $M/Z(G)$ is a non-Abelian simple group and M is transitive on $1 + kp^a$ quartic axes and we may assume that M is generated by quartic elements, although we do not know the order of a Sylow 5-subgroup.

30 Lemma. *M is irreducible and primitive.*

Proof. Assume that M is reducible. Then there is an M-invariant Baer subplane π_o, that is necessarily Desarguesian. Furthermore, the quartic elements of M induce elations on π_o and the group generated on π_o is then $SL(3, 5^t)$ for $5^t > \sqrt{5^r}$, so that the group induced is $SL(2,q)$ and we may apply Theorem 5 to complete the proof. Thus, M is irreducible and it easily follows that M is primitive. $\boxed{QED}$

Referring to Theorem 3, we have seen that case (a) cannot occur and we may always avoid case (b). Since $M/Z(G)$ is a simple group, we will not have cases (c) or (d) (see beginning of the proof of Theorem (5.1) p. 31 of [11]). Furthermore, case (f) occurs when M is reducible. Hence, we now must have one of the cases (e), or (g) through (n) of Theorem 3 for G, which has an irreducible and primitive normal subgroup M, which we may always assume contains K^*, the kernel homology group of order $q-1$ and such that M/K^* is a non-Abelian simple group of order divisible by 5.

31 Lemma. *Case (e) cannot occur.*

Proof. Case (e) cannot occur since we would have a Sylow 5-subgroup of order 5^4 or 5^6, forcing a Baer 5-group. $\boxed{QED}$

32 Lemma. *Case (g) does not occur.*

Proof. In case (g), $M/Z(G)$ is isomorphic to $PSL(2,q)$ and we are finished by Theorem 5. $\boxed{QED}$

33 Lemma. *Cases (j) or (n) do not occur.*

Proof. Cases (j) and (n) only occur for $p=2$. $\boxed{QED}$

34 Lemma. *Cases (h) and (l) do not occur.*

Proof. Case (h) does not occur since $p=5$. Case (l) is out since $SL(2,7)$ is not divisible by 5. (m) is out since $p=5$. $\boxed{QED}$

This leaves cases (i) and (k).

35 Lemma. *Case (i) cannot occur.*

Proof. In case (i), clearly the group $M/Z(G)$ that M induces on the line at infinity is A_i, for $i=5,6,7$ and this group must act transitively on the $1+k5^a$ quartic axes. Hence,

$$1+k5^a \mid 7! \text{ so that } 1+k5^a \mid 7!/5.$$

Since $1+5^4=626>7!/5=504$, it follows that $5^a<5^4$ so that $a=1,2,3$.

First assume that $a=3$. Then $1+k5^3 \mid 504$, and $k<3$. So, $1+k5^3=126,501$, for $k=1,2$ respectively, implying that the only possibility is $k=1$.

Now assume that $a=2$ so that $1+k5^2$ divides 504. Since $5^2>\sqrt{q}$, this implies that $q=5^i$, for $i=1,2,3$. $1+k5^2=26,51,76,101,126,151,176,201$ for $k=1,2,3,4,5,6,7,8$, so that the only possible solution is $k=5$ and A_i is A_7. If $k>8$ and $1+k5^2$ divides 504 then $1+k5^2=504$, a contradiction since 504 is not congruent to 1 mod 25.

Thus, for $a = 2$ or 3, A_i can only be A_7. There are 21 subgroups of A_7 isomorphic to A_5 and since A_7 is normal in $G/Z(G)$, it follows that each Sylow 5-subgroup of G normalizes one of these A_5 subgroups. But, a Sylow 5-subgroup of order 5^2 or $5^3 > 5$ cannot be within A_5, implying that there is an element of order 5 that centralizes $SL(2,5)$. Since each subgroup of order 5 necessarily fixes a unique quartic axis, this implies that a quartic element must fix at least six quartic axes, a contradiction. Hence, A_7 cannot occur.

Let $a = 1$ so that $5 > \sqrt{q}$, implies that $q = 5$, which has been excluded previously. This completes our proof. $\boxed{QED}$

We need finally to consider case (k) of Theorem 3 when $M/Z(G)$ is isomorphic to $S_p(4,3)$ of order $3^4 \cdot 2^7 \cdot 5$ and $q \equiv 1 \bmod 3$.

36 Lemma. *Case (k) does not occur.*

Proof. First assume that $5^a > 5$. Since a Sylow 5-subgroup cannot normalize $S_p(4,3)$ without centralizing (the outer automorphism group has order 2) , it follows that there is a quartic element that centralizes $S_p(4,3)$, again implying that this element must fix more than a single component.

Thus, $5^a = 5$, implying that $q = 5$. $\boxed{QED}$

This completes the proof of our main result stated in the introduction.

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