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On spatial theta-curves with the same $(Z_2 \oplus Z_2)$-fold and 2-fold branched covering

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Abstract. In this note, we study two types of spatial theta-curves having two $(1,1)$-knots whose each has two $(1,1)$-knots and a trivial knot or two trivial knots and a 2-bridge knot as constituent knots. We show that there is a 3-manifold M such that M is the $(Z_2 \oplus Z_2)$-fold and 2-fold covering of S^3 branched over each type of spatial theta-curve. Furthermore, we investigate certain relations between the spatial theta-curves and between the closed 3-manifolds which are coverings of S^3 branched over them.

Keywords: Spatial theta-curve, Constituent knot, $(1,1)$-knot.

MSC 2000 classification: 57M12 (primary), 57M25.

1 Introduction

A theta-curve G is a graph formed by two vertices and three edges(joining the vertices). Let $f : G \to S^3$ be an embedding of G into the 3-sphere S^3. Then $f(G)$ is called a spatial theta-curve. We denote the three edges of G by x, y, and z. Then $f(x \cup y)$, $f(y \cup z)$, and $f(z \cup x)$ are called the constituent knots of $f(G)$. From now on, we use G, θ, or $\theta(\cdot)$, instead of $f(G)$, as a spatial theta-curve.

The concept of spatial theta-curve is important both in knot theory and in the theory of branched coverings. Knot theory of graphs is to seek knots or links associated with a graph so that questions about the graph can be translated into questions about knots and links. A spatial theta-curve contains three knots and, of course, if two theta-curves contain different knots, then they are different. Branched covering theory of theta-curve is related to the following open question: how many different knots may have the same 2-fold branched covering? For example, in [16], Zimmermann constructed infinitely many triples of different knots such that the three knots of each triple have the same 2-fold branched covering. Here the three knots are the preimages of the third edge in the 2-fold branched coverings of these three trivial knots, that is, if the three knots defined by a spatial theta-curve are trivial we get three knots with the same 2-fold branched covering. The case of three(resp. two) different knots was considered in [16](resp. [7]). For the literature on open question written above, we refer to [1] and [12]-[14].

We are interested in two spatial theta-curves having the following restricted forms: (1) For a knot K with a strong inversion i, we has a double covering projection $\pi : S^3 \to S^3/i$ branched over a trivial knot $\pi(fix(i))$, where $fix(i)$ is the axis of the strong inversion i. Then the set $\pi(fix(i) \cup K)$ is called the spatial theta-curve associated with (K, i), denoted by $\theta(K, i)$; (2) Let $\bar{K}$ be a tunnel number one knot and τ an unknotting tunnel for $\bar{K}$ in S^3. Then $\bar{K} \cup \tau$ is called the spatial theta-curve associated with $(\bar{K}, \tau)$, denote $\theta(\bar{K}, \tau)$.

Since tunnel number one knot is a strongly invertible knot, for any tunnel number one knot $\bar{K}$, there are two spatial theta-curves $\theta(\bar{K}, i)$ and $\theta(\bar{K}, \tau)$ such that i and τ are a strong inversion and an unknotting tunnel for $\bar{K}$, respectively. Thus at least one constituent knot of $\theta(\bar{K}, i)$ is trivial, and at least one constituent knot of $\theta(\bar{K}, \tau)$ is trivial if $\bar{K}$ is a $(1,1)$-knot([8]).

In this paper, we consider the class of covering spaces having the spatial theta-curves $\theta(K, i)$ and $\theta(\bar{K}, \tau)$ as branching sets in S^3. Furthermore, we show that $\theta(K, i)$ and $\theta(\bar{K}, \tau)$ have the same $(Z_2 \oplus Z_2)$-fold and 2-fold branched covering, and investigate relations between these theta-curves.

In section 2, we construct a spatial theta-curve $\theta(K, i)$ obtained from a genus two Heegaard splitting with three involutions. Furthermore, we show that a closed 3-manifold admitting such Heegaard splittings is the $(Z_2 \oplus Z_2)$-fold covering of S^3 branched over $\theta(K, i)$.

In section 3, we construct a spatial theta-curve $\theta(\bar{K}, \tau)$ obtained from a $(1,1)$-decomposition and an unknotting tunnel. Note that there is a closed 3-manifold M admitting genus two Heegaard splittings which is homeomorphic to the 2-fold covering of S^3 branched over $\theta(\bar{K}, \tau)$ and to the $(Z_2 \oplus Z_2)$-fold covering of S^3 branched over $\theta(K, i)$. Further, we show that given a $(1,1)$-knot $\bar{K}$ and a strongly invertible knot K there exist 3-manifold $M(\theta)$ such that (i) $M(\theta)$ is the $(Z_2 \oplus Z_2)$-fold covering of S^3 branched over $\theta(\bar{K}, \tau)$ and (ii) $M(\theta)$ is the 2-fold branched covering of the $(Z_2 \oplus Z_2)$-fold covering of S^3 branched over $\theta(K, i)$.

2 The construction of spatial theta-curve derived by genus two Heegaard splitting

We introduce a class of genus two Heegaard splittings determined by 6-tuples of integers (d, a, b, c, r, s). In connection with such a class, Neuwirth([9]) describes an algorithm (called the Neuwirth algorithm) for deciding if a group presentation with n generators and n relations corresponds to the spine (or equivalently the Heegaard diagram) of a closed compact 3-manifold. Based on the Neuwirth algorithm, Dunwoody([3]) has introduced 6-tuples (d, a, b, c, r, s), where $d = 2a + b + c$, yielding a family of genus n Heegaard splittings (or, combinatorial, Heegaard diagram) of closed orientable 3-manifolds, denoted $D_n(d, a, b, c, r, s)$ for $n \geq 1$.

1 Theorem. *([3]) Let $d = 2a + b + c$ be odd. The 6-tuple (d, a, b, c, r, s) represents the Heegaard diagram of a closed orientable 3-manifold if and only if (i) $\alpha\beta$ has two cycles of length d, and (ii) $ps + q \equiv 0 (mod\ n)$, where p is the number of arrows pointing down the page minus the number of arrows pointing up whereas q is the number of arrows pointing from left to right minus the number of arrows pointing from right to left in the oriented path determined by $\alpha\beta$ (for the definition of the permutations α and β we refer to [3]).*

We are interested in certain classes of genus two Heegaard splittings of the closed orientable 3-manifolds $D_2(d, a, b, c, r, s)$, or D_2 in short, which are obtained by truncating mod 2 the infinite Heegaard diagram with $b + c \equiv 1(mod\ 2)$. For example, $D_\infty(9, 3, 1, 2, 2, -1)$ is the Heegaard diagram which is obtained from $p = 2 - 1 = 1$, $q = 4 - 3 = 1$, and $s = -1$. Thus it can be reduced mod 2 to give the genus two Heegaard diagram $D_2(9, 3, 1, 2, 2, -1)$. See Figure 1.

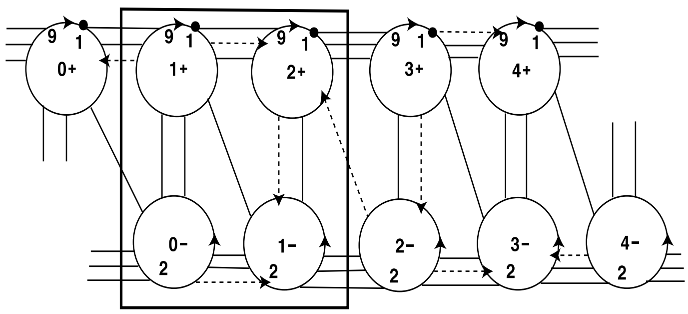

Figure 1. $D_\infty(9, 3, 1, 2, 2, -1)$ and $D_2(9, 3, 1, 2, 2, -1)$

Due to cyclic symmetry of the infinite Heegaard diagram, $D_2(d, a, b, c, r, s)$ is independent of method used for truncating mod 2. For more details on the structure of the infinite Heegaard diagram see [3], [5], and [7].

In [2], Birman and Hilden proved that any 3-manifolds with genus two Heegaard splittings admits an orientation preserving involution, called the standard involution in [11] and denoted ϵ, whose quotient space is S^3 and whose branching locus is a 3-bridge knot. Thus $D_2(d, a, b, c, r, s)$ represent closed orientable 3-manifolds which are homeomorphic to 2-fold cyclic coverings of S^3 branched over 3-bridge knots.

Consider other involution, denote σ, that rotates a genus two handlebody around the diameter of a separating disk exchanging the two solid tori. This will be called the minor involution on genus two handlebody. See Figure 2. The diameter is the set of fixed points, and its quotient space by the diameter is a solid torus. Thus $D_2(d, a, b, c, r, s)$ is homeomorphic to the 2-fold cyclic covering of a lens space branched over the set of fixed points which is a $(1, 1)$-knot, as we see later.

The final involution, denote ρ, is the product of two involutions introduced above, i.e., $\rho = \epsilon \circ \sigma = \sigma \circ \epsilon$. Note that it has a different axis of symmetry, and the quotient space by the axis is a solid torus. Thus $D_2(d, a, b, c, r, s)$ is homeomorphic to the 2-fold cyclic covering of a lens space branched over a $(1, 1)$-knot. As result, we have three involutions ϵ, σ and ρ on D_2, and denote the set of such involutions by $I = \{\epsilon, \sigma, \rho\}$. See Figure 2 below.

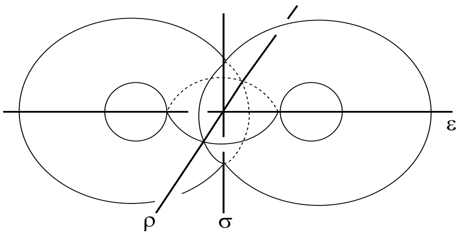

Figure 2. Genus two handlebody with involutions $I = \{\epsilon, \sigma, \rho\}$

In particular, the quotient space of D_2 by σ, call a $(1, 1)$-decomposition induced by the 6-tuple (d, a, b, c, r, s) is defined as follows. Let D_1 be the quotient space of D_2 by σ and K_1 a trivial arc with branching index 2 in D_1. Then there is a pair (V_1, K_1) of a solid torus and a trivial arc such that $\partial V_1 = D_1$ and $K_1 \cap V_1 = \partial K_1 = \{P, Q\}$. See Figure 3.

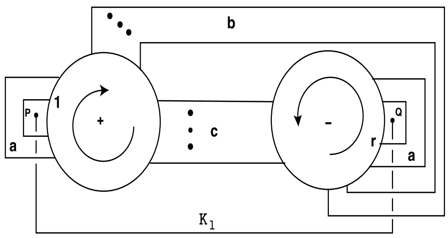

Figure 3. $(1, 1)$-decomposition consisting of D_1 and a trivial arc K_1

Note that $D_1(d, a, b, c, r, s)$ is independent of a parameter s, and let $d = 2a + b + c$.

As results, we have following theorems:

2 Theorem. *Let $d = 2a + b + c$ be odd, and D_2 a Heegaard splitting determined by the 6-tuple (d, a, b, c, r, s). Then the quotient spaces of D_2 by σ and ρ admit $(1, 1)$-decomposition of lens spaces, and the quotient space of D_2 by ϵ admits a $(0, 3)$-decomposition of the 3-sphere.*

3 Lemma. *The quotient spaces of $D_2(d, a, b, c, r, s)$ by σ admit $(1, 1)$-decompositions of the 3-sphere if and only if it satisfies condition $p = \pm 1$, where p is the defined number in Theorem 1.*

Proof. Sufficient condition is a clear from the standard Heegaard splitting of the 3-sphere.

To prove necessary condition, suppose that D_2 is a genus two Heegaard splitting determined by the 6-tuple (d, a, b, c, r, s). Then we have an orientation preserving homeomorphism on D_2 induced by σ. The rotation by an angle $(\frac{2\pi}{2})$ of σ defines an action of the cyclic group $Z_2 =< \sigma | \sigma^2 = 1 >$ on D_2. The quotient space D_2/Z_2 admit a $(1, 1)$-decomposition consisting of D_1 and a trivial arc with branching index 2 as in Figure 3. The axis of the rotation is drawn as a trivial arc that lies below the diagram as in Figure 3. Let l_1 be the number of arrows pointing down the page in D_1, l_2 the number of arrows pointing up, l_3 the number of arrows pointing from left to right, and l_4 the number of arrows pointing from right to left. We then obtain that $l_1 - l_2 = p$ and $l_3 - l_4 = q$. Furthermore it be seen that $l_1 + l_2 = b + c$ and $l_3 + l_4 = 2a + b$. We can reduce D_1 in order to remove a edges completely by means of Whitehead algorithm(also called the band move). As result we have the fact that the number of vertices in a disc pair $\{+, -\}$ is equal to $d - 2a = b + c$, and that $p \leq b + c$. Thus $\pi(D_1) =< x | x^{\pm p} >= Z_{|p|}$ for some $p \leq b + c$, and it is a group presentation of lens spaces. Therefore if $p = \pm 1$, then D_1 is the Heegaard splitting of the 3-sphere. QED

After this, we now use 4-tuple (a, b, c, r) briefly instead of 6-tuple (d, a, b, c, r, s) since $d = 2a + b + c$ and $s = (-q)/p$. We denote briefly such a $(1, 1)$-decomposition by $D_1(a, b, c, r)$.

4 Theorem. *A closed orientable 3-manifold M admitting a genus two Heegaard splitting $D_2(a, b, c, r)$ is the $(Z_2 \oplus Z_2)$-fold covering of the 3-sphere branched over a spatial theta-curve.*

Proof. Let h be a gluing homeomorphism of the two handlebodies W_1 and W_2 such that $W_1 \cap W_2 = D_2$, $W_1 \cup_{D_2} W_2 = M$, and $\sigma_{|\partial W_1} \circ h = h \circ \sigma_{|\partial W_2}$. Then M is a genus two closed orientable 3-manifold with $G = Z_2 \oplus Z_2$-symmetry

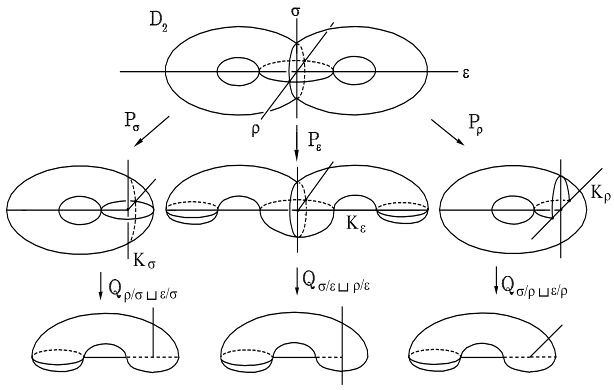

Figure 4. Genus two Handlebody and theta-curve

generated by ϵ and σ. In fact, it is a clear that h is compatible with the involution ϵ by [2], and that $\rho = \epsilon \circ \sigma = \sigma \circ \epsilon$. Let $I = \{\epsilon, \sigma, \rho\}$ be the set of involutions satisfying the above facts as shown in Figure 3. For each involution $i \in I$ on D_2, we has a double covering projection $P : D_2 \to D_2/i$ branched over $P(A_i) = K_i$, where A_i is the axis of the involution i. On D_2/I, there are three involutions $\rho/\sigma \cup \epsilon/\sigma$, $\sigma/\epsilon \cup \rho/\epsilon$ and $\sigma/\rho \cup \epsilon/\rho$ for A_σ, A_ϵ and A_ρ, respectively. We denote the set of three involutions by J. For each involution $i \in I$ and $j \in J$, we has a double covering projection $Q : D_2/i \to (D_2/i)/j = S^3$. The action of G on D_2 yields a covering projection $\Pi_G : M \to S^3$ branched over $\Pi_G(A_\epsilon \cup A_\sigma \cup A_\rho)$ defined by

$$\Pi_G = Q_{\rho/\sigma\cup\epsilon/\sigma} \circ P_\sigma = Q_{\sigma/\rho\cup\epsilon/\rho} \circ P_\rho = Q_{\sigma/\epsilon\cup\rho/\epsilon} \circ P_\epsilon.$$

Then $\Pi_G(A_\epsilon \cup A_\sigma \cup A_\rho)$ is a spatial theta-curve associate with (D_2, G) having $(\rho/\sigma \cup \epsilon/\sigma)$, $(\sigma/\epsilon \cup \rho/\epsilon)$, and $(\sigma/\rho \cup \epsilon/\rho)$ as constituent knots. We denote it by $\theta(I)$. *QED*

For example, in [7] the class of the theta-curves was used to compare two non-isotopic Heegaard decompositions of a closed 3-manifold.

5 Remark. We denote the set of constituent knots of $\theta(I)$ by $\{K_{12}, K_{23}, K_{31}\}$. Then we can construct the 2-fold branched covering of S^3 branched along K_{12}, and denote it by $M_2(K_{12})$. Let $\bar{3}$ be the lift of the edge 3 in $M_2(K_{12})$. Hence we can construct the 2-fold branched covering of $M_2(K_{12})$ branched along $\bar{3}$. Then, this covering coincides with the formerly defined $(Z_2 \oplus Z_2)$-fold branched covering M of $\theta(I)$. By the same way, we can also construct M by choosing $M_2(K_{23})$ or $M_2(K_{31})$ instead of $M_2(K_{12})$. That is, M is independent of the choice of the constituent knots.

6 Lemma. *([10]) Let M be the $(Z_2 \oplus Z_2)$-fold covering of S^3 branched over $\theta(I)$. Let K_{12}, K_{23}, and K_{31} be the three constituent knots of $\theta(I)$, and $M_2(K_{12})$, $M_2(K_{23})$, and $M_2(K_{31})$ the 2-fold branched coverings of K_{12}, K_{23}, and K_{31}, respectively. Then we have*

$$H_1(M, Z) \cong H_1(M_2(K_{12}), Z) \oplus H_1(M_2(K_{23}), Z) \oplus H_1(M_2(K_{31}), Z).$$

7 Theorem. *Let M be the $(Z_2 \oplus Z_2)$-fold covering of S^3 branched over $\theta(I)$. Then at least one of the following propositions holds:*
(1) *M is the 2-fold covering of 3-sphere branched over at most three different knots;*
(2) *M is the 2-fold branched covering of 3-sphere over at most two different knot and is the 2-fold branched covering of a lens space;*
(3) *$H_1(M, Z)$ is isomorphic to the trivial group, the homology group of a lens space, or the direct sum of the homology groups of two lens spaces.*

Proof. By the construction of standard involution ϵ on D_2, we can see that the quotient D_2/ϵ represents the 3-sphere. By Lemma 3, we can take 4-tuples (a, b, c, r) such that D_2/σ is homeomorphic to $D_1(a, b, c, r)$ representing the 3-sphere. In this case, at least two constituent knots of $\theta(I)$ are trivial and the remaining one is a 2-bridge knot. Therefore two 3-spheres and a lens space have M as the same 2-fold branched covering, hence (2) holds. If D_2/ρ represents the 3-sphere, then $\theta(I)$ is locally unknotted(equivalently it has all trivial constituent knots). Thus, three 3-spheres have M as the same 2-fold branched covering whose branching sets have at most three different knots. Then (1) is verified. Applying Lemma 6 gives (3). This completes the proof. QED

8 Remark. 1. It was proved in [16] that there exist infinitely many different triples of different knots coming from locally unknotted theta-curves such that the three knots of any triple have the same 2-fold branched covering M which is a homology 3-sphere. But we don't know whether the result of [16] occurs or not in our case.
2. Examples in [7] satisfy (1) and (3) of Theorem 7, i.e., $\theta(I)$ has all trivial constituent knots, but two different knots. For (2) and (3) we refer $D_2(7, 1, 2, 3, 3, 4)$ with two trivial knots and a 2-bridge knot as elements of $\theta(I)$.

In general, if D_2/σ is homeomorphic to D_1 representing Lens space, then we obtain the generalized results compared with preceding theorem:

9 Theorem. *Suppose that M is the closed orientable 3-manifold having a genus two Heegaard splitting with the set I of involutions. Then M is the $(Z_2 \oplus Z_2)$-fold covering of the 3-sphere branched over a spatial theta-curve, whose constituent knots are a trivial knots and two 2-bridge knots.*

3 Spatial theta-curves arising from $(1, 1)$-decompositions and unknotting tunnels

In this section, we introduce another theta-curve induced by $D_1(a, b, c, r)$ and an unknotting tunnel as follows. Let σ be the minor involution on the genus two Heegaard splitting $D_2(a, b, c, r)$. Then $D_2/\sigma = D_1(a, b, c, r)$ admits a $(1, 1)$-decomposition consisting of a genus one handlebody V and a 1-bridge(or trivial arc) K_1 such that $K_1 \cap \partial V = \partial K_1$. See Figure 3. In addition, we consider a trivial arc τ in $int(V)$ such that $\tau \cap int(K_1) = \partial\tau$ and $\tau \cap D = 1pt$, where D is a meridian disk of V. Then $(D_1, K_1 \cup \tau)$ admits a theta-curve in lens space, and denote it by $\theta(a, b, c, r)$.

We are interested in spatial theta-curves, and thus we assume that $\theta(a, b, c, r)$ is a theta-curve in the 3-sphere. For example, Figure 5 illustrates the $(1, 1)$-decomposition induced by 4-tuple $(2, 2, 3, 5)$, which is equivalent to $D_1(2, 2, 3, 5)$ and its spatial theta curve $\theta(2, 2, 3, 5)$.

Let $E(K)$ be the exterior of a knot K in S^3. A tunnel τ (which is a properly embedded arc in $E(K)$) is an unknotting tunnel if $Cl(E(K) - N(\tau))$ is a genus two handlebody. We denote a regular neighborhood of X in Y by $N(X; Y)$.

10 Theorem. *A trivial arc τ in spatial theta-curve $\theta(a, b, c, r)$ is an unknotting tunnel of the $(1, 1)$-knot K induced by $D_1(a, b, c, r)$.*

Proof. Let $(V_1, K_1) \cup_{D_1} (V_2, K_2)$ be a $(1, 1)$-decomposition of K such that $V_1 \cup_{D_1} V_2 = S^3$, $\partial V_1 = \partial V_2 = D_1$ and $K_1 \cup_{D_1} K_2 = K$. Then $W_1 = N(K_2; V_2) \cup V_1$ and $W_2 = cl(V_2 - N(K_2; V_2))$ are genus two handlbodies such that $W_1 \cup W_2 = S^3$. Since $W_1 \cong N(K; S^3) \cup N(C_1; V_1)$, where C_1 is a core of V_1, the trivial arc $\bar{\tau}$ obtained by joining K_1 and C_1 with another trivial arc is an unknotting tunnel which is equivalent to τ. QED

11 Theorem. *A spatial theta-curve $\theta(a, b, c, r)$ consists of two $(1, 1)$-knots and a trivial knot as constituent knots.*

Proof. Let $\theta(a, b, c, r)$ be a theta-curve in S^3 consisting of two vertices and three edges and $\{1, 2, 3\}$ the set of numbers which label the edges. Suppose that the edges on τ with two vertices of ∂K_1, and remaining edge of K_1 separated by $\partial\tau$ are labelled by 1, 2, and 3, respectively. Then K_{13}, K_{23}, and K_{12} are constituent knots of the theta-curve $\theta(a, b, c, r)$. As seen in Figure 5, K_{13} is trivial because it is independent of 6-tuple (d, a, b, c, r, s), and K_{23} is the $(1, 1)$-knot induced by $D_1(a, b, c, r)$. Finally, K_{12} is a trivial arc in a genus one handlebody V_1 such that $V_1 \cap K_{12} = \partial K_{12}$. Moreover, it meets a meridian disk of V_1 in exactly one point. To keep form of Figure 3, we perform isotopy moves on the (V_1, K_{12}) (like the Singer moves on Heegaard diagram). This produces a $(1, 1)$-decomposition, denoted by $D_1(\bar{a}, \bar{b}, \bar{c}, \bar{r})$ or $\bar{D_1}$ briefly, such that $\bar{d} = 2\bar{a} + \bar{b} + \bar{c} <$

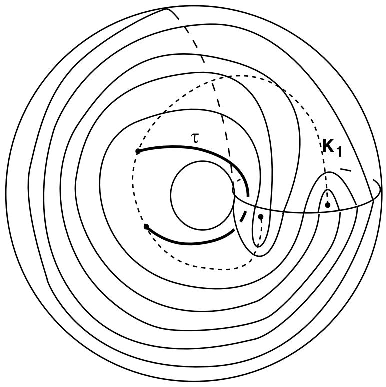

Figure 5. $(1,1)$-decomposition $D_1(2,2,3,5)$ and theta curve $\theta(2,2,3,5)$

$d = 2a + b + c$. QED

Let $(S^3, \bar{2})$ be the 2-fold covering of S^3 branched along K_{13}, where $\bar{2}$ is the lift of the edge 2 of the spatial theta-curve $\theta(a,b,c,r)$. Then we can also construct the 2-fold covering of S^3 branched over $\bar{2}$, and denote it by $M(\theta)$. This covering coincides with the $(Z_2 \oplus Z_2)$-fold branched covering of the theta-curve $\theta(a,b,c,r)$. Since $M(\theta)$ is independent of the choice of the constituent knots, we can also construct $M(\theta)$ by choosing $(M_2(K_{23}), \bar{1})$ or $(M_2(K_{12}), \bar{3})$ instead of $(S^3, \bar{2})$. In fact, $M_2(K_{23})$ and $M_2(K_{12})$ are equivalent to $D_2(a,b,c,r)$ and $D_2(\bar{a},\bar{b},\bar{c},\bar{r})$, respectively. Thus we have the followings:

12 Theorem. *Let $\theta(a,b,c,r)$ be the spatial theta-curve defined in previous section. Then the closed orientable 3-manifold $M(\theta)$, constructed above, is the $(Z_2 \oplus Z_2)$-fold covering of the 3-sphere branched over $\theta(a,b,c,r)$. Furthermore, the spatial theta-curve $\theta(a,b,c,r)$ has three 2-fold coverings of S^3 branched over itself.*

The following result is obtained from Theorem 4 and Theorem 12:

13 Corollary. *The closed orientable 3-manifold M admitting a genus two Heegaard splitting $D_2(a,b,c,r)$ is the $(Z_2 \oplus Z_2)$-fold (resp. 2-fold) covering of the 3-sphere branched over $\theta(I)$ (resp. $\theta(a,b,c,r)$). Moreover, the closed orientable 3-manifold $\bar{M}$ admitting a genus two Heegaard splitting $D_2(\bar{a},\bar{b},\bar{c},\bar{r})$ is the $(Z_2 \oplus Z_2)$-fold (resp. 2-fold) covering of the 3-sphere branched over $\bar{\theta}(I)$ (resp. $\theta(\bar{a},\bar{b},\bar{c},\bar{r})$).*

As similar results of Lemma 6 and Theorem 7, we have the following corollary:

14 Corollary. *Let $M(\theta)$ be the $(Z_2 \oplus Z_2)$-fold covering of S^3 branched over*

$\theta(a,b,c,r)$. Then at least one of the following propositions holds:
(1) $M(\theta)$ is the 2-fold covering of the 3-sphere branched over at most three different knots;
(2) $M(\theta)$ is a 2-fold covering of the 3-sphere branched over at most two different knots, and it is a 2-fold branched covering of $D_2(a,b,c,r)$;
(3) $M(\theta)$ is the 2-fold branched covering of 3-sphere, $D_2(a,b,c,r)$, and $D_2(\bar{a},\bar{b},\bar{c},\bar{r})$;
(4) $H_1(M(\theta),Z)$ is isomorphic to $H_1(D_2(a,b,c,r),Z)$ or $H_1(D_2(a,b,c,r),Z) \oplus H_1(D_2(\bar{a},\bar{b},\bar{c},\bar{r}),Z)$.

For two theta-curves θ_1 and θ_2 in S^3, they are equivalent if and only if there exists an orientation-preserving homeomorphism of S^3 that maps θ_1 to θ_2.

15 Theorem. *If two $(1,1)$-knots of the spatial theta-curve $\theta(a,b,c,r)$ as its constituent knots are non-isotopic by Reidemeister moves, then the spatial theta-curves $\bar{\theta}(I)$ and $\theta(I)$ in Corollary 13 are non-equivalent.*

Proof. As stated in Theorem 12, we assume that the two $(1,1)$-knots of $\theta(a,b,c,r)$ are K_{23} and K_{12}. Suppose that $\bar{\theta}(I)$ and $\theta(I)$ are equivalent. Then standard arguments of graph theory imply that sets of constituent knots of $\bar{\theta}(I)$ and $\theta(I)$ are equivalent. Let $M_2(K_{23})$ and $M_2(K_{12})$ be the 2-fold coverings of S^3 branched along K_{23} and K_{12}, respectively. Since they are the $(Z_2 \oplus Z_2)$-fold coverings of S^3 branched over the two theta-curves, they have to be equivalent. This contradicts the hypothesis on K_{23} and K_{12}. QED

For each of the three trivial simple closed curves J_i in a locally unknotted spatial theta-curve θ, the arc $\overline{\theta - J_i}$ lifts to a knot K_i in the two-fold cyclic cover of S^3 branches over J_i. From facts about branched covers([4]), it is clear that the unordered triple (K_1,K_2,K_3) of knots in S^3 is an isotopy invariant of θ.

16 Lemma. *([15]) Let θ be a locally unknotted spatial theta-curve with K_1, K_2 and K_3 obtained as above. Then the following are equivalent:*

(i) θ is planar;

(ii) All the knots K_1, K_2 and K_3 are trivial;

(iii) At least one of K_1, K_2 and K_3 is trivial.

The following theorems are further results obtained from Lemma 16.

17 Theorem. *If the theta-curves $\theta(I)$ and $\bar{\theta}(I)$ are planar graphs, then the spatial theta-curve $\theta(a,b,c,r)$ is locally unknotted.*

Proof. Assume that $\theta(I)$ and $\bar{\theta}(I)$ are two theta-curves induced by $D_2(a,b,c,r)$ and $D_2(\bar{a},\bar{b},\bar{c},\bar{r})$, respectively. By the hypothesis of theorem, Lemma 16 implies that $D_2(a,b,c,r)$ and $D_2(\bar{a},\bar{b},\bar{c},\bar{r})$ are homeomorphic to the 3-sphere. Therefore, $\theta(a,b,c,r)$ is locally unknotted. QED

18 Theorem. *Let the theta-curve $\theta(a,b,c,r)$ be a planar graph. Then two theta-curves $\theta(I)$ and $\bar{\theta}(I)$ are planar graphs.*

Proof. Suppose that (K_1, K_2, K_3) (resp. $(\bar{K}_1, \bar{K}_2, \bar{K}_3)$) is obtained by lifting the trivial simple closed curves J_i (resp. $\bar{J}_i$), $i = 1, 2, 3$, of $\theta(I)$ (resp. $\bar{\theta}(I)$). Since $\theta(a,b,c,r)$ is a planar graph, (K_1, K_2, K_3) and $(\bar{K}_1, \bar{K}_2, \bar{K}_3)$ are formed by trivial knots. Thus the result follow from Lemma 16. QED

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