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André flat flocks

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Abstract. It is shown that the every plane of the class of André planes of order q^n corresponds to a flat flock of a Segre variety $S_{n-1,n-1}$ over $GF(q)$ if and only if n is divisible by $q-1$. In addition, there are large numbers of generalized André planes constructed that produce flat flocks.

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1 Introduction

The connections of the theories of finite geometries is widely expanding and many diverse geometries are intrinsically related to 'translation planes' or rather to 'spreads' in projective spaces. It is now well known that there are connections of the theories of flocks of hyperbolic quadrics in $PG(3, q)$ and translation planes with spreads that are unions of reguli mutually sharing two common lines. A wonderful illustration of a symbiosis between these two theories is illustrated by the beautiful theorem of Thas [12] and Bader-Lunardon [2], which completely characterizes 'hyperbolic flocks'. The hyperbolic flocks corresponding to the regular nearfield planes were constructed by Thas [12] by geometric methods. Also, it was shown independently that there are flocks corresponding to certain irregular nearfields by Bader [1], Baker and Ebert [5] for $p = 11$ and 23, and Johnson [10].

1 Theorem (Thas, Bader-Lunardon). *A flock of a hyperbolic quadric in $PG(3, q)$ is one of the following types:*

(i) *linear,*

(ii) *a Thas flock, or*

(iii) *a Bader/Baker-Ebert/Johnson flock of order p^2 for $p = 11, 23, 59$.*

Further contributing to the interconnections between various geometries, Bader, Cossidente, and Lunardon ([3], [4]) recently generalized the notions of hyperbolic flocks that connects with a theory of spreads or translation planes depending on the notion of an (A, B)-regular spread.

2 Definition. An (A, B)-regular spread in $PG(2n+1, q)$ is a spread S such that for C in $S - \{A, B\}$, the q-regulus generated by $\{A, B, C\}$ is in S.

The authors have recently studied translation planes of order q^n corresponding to certain flocks of Segre varieties $S_{n-1,n-1}$ over $GF(q)$. Each associated spread admits a 'regulus 'hyperbolic cover' which is a union of $(q^n - 1)/(q - 1)$ reguli that share two components.

In Jha and Johnson [8], the emphasis was on semifield planes that can produce flocks of certain Segre varieties. In this case, there is a cover of the Segre variety by caps and there is a cyclic group that acts transitively on the caps of the covers. The flocks obtained are called 'flat flocks'. In this article, the connections between the Segre varieties and the translation planes were based on algebraic techniques involving the so-called 'right, middle, and left nuclei' of an associated semifield.

However, if a translation plane exists of order q^n and kernel containing $GF(q)$ that admits two affine homology groups of order $q - 1$ such that the kernel homology group of order $q - 1$ is contained in the product of these two groups then there is a corresponding flat flock.

In Bader, Cossidente and Lunardon [4], it is shown that the class of Dickson nearfield planes of orders q^n and kernel containing $GF(q)$ always produce such flat flocks. In this situation, all Dickson nearfield planes are generalized André planes but not conversely.

In the present article, we exploit this idea and consider the following question: **Is there a class of translation planes of order q^n and kernel containing $GF(q)$ such that each plane of this class corresponds to and produces a flat flock of a Segre variety $S_{n-1,n-1}$?**

Since we will be guided by such considerations in André planes, we ask if there is a fixed order q^n such that every André plane of this order produces a flat flock of a Segre variety?

In this note, we show that flat flocks are ubiquitous in André planes of order $q^{t(q-1)}$. That is, every such André plane in this class produces a flat flock.

2 Background

Part of this background section is taken from Jha and Johnson [8] and the reader is referred to this article for additional information.

As mentioned in the introduction, Bader, Cossidente, and Lunardon ([3] and [4]), showed that there are connections between what are called '(A, B)-regular' spreads and flocks of Segre varieties.

We have defined 'regulus hyperbolic covers' in the introduction. We again merely point out that the definition of a spread admitting a regulus hyperbolic cover is equivalent to an (A, B)-regular spread. Since every three mutually skew lines in $PG(2n-1, 2)$ determine a regulus, we shall usually assume that $q > 2$ to avoid this trivial case.

3 Definition. Let $N = (k+1)(m+1)+1$. The 'Segre variety' of $PG(m, q)$ and $PG(k, q)$ is the variety $S_{m,k}$ of $PG(N, q)$ consisting of the points given by the vectors $v \otimes u$ as v and u vary over $PG(m, q)$ and $PG(k, q)$ respectively (the decomposable tensors in the associated tensor product of the vector spaces).

When $m = k = n$, a 'flock' of $S_{n,n}$ is a partition of the point-set into caps (a set of points no three of which are collinear) of size $(q^{n+1}-1)/(q-1)$. Note that there would also be $(q^{n+1}-1)/(q-1)$ such caps.

A flock is 'flat' if its caps are Veronese varieties $\mathcal{V}_n$ obtained as sections of $S_{n,n}$ by dimension $n(n+3)/2$ projective subspaces of $PG(N, q)$.

The reader is referred to Hirschfeld and Thas [7], chapter 25 for background on Veronese and Segre varieties. In particular, the variety considered in the previous definition is the Veronesian of quadrics of $PG(n, q)$, and sits naturally in $PG(n(n+3)/2, q)$. There is a bijection from the Veronesian of quadrics $\mathcal{V}_n$ onto $PG(n, q)$ so that $|\mathcal{V}_n| = (q^{n+1}-1)/(q-1)$. Using [7], Theorems (25.1.8), (25.5.4) and (25.5.8), we see that (i) $\mathcal{V}_n$ is a cap of $PG(n(n+3)/2, q)$,(ii) $|S_{n,n}| = ((q^{n+1}-1)/(q-1))^2$, and (iii) the intersection of appropriate projective subspaces of dimension $n(n+3)/2$ of $PG((n+1)^2-1, q)$ with $S_{n,n}$ are Veronesian's of quadrics of type $\mathcal{V}_n$.

When $n = 1$, $S_{1,1}$ is a hyperbolic quadric in $PG(3, q)$ and the Veronesian is a conic of $q+1$ points in $PG(2, q)$ considered as a section of $S_{1,1}$ by a $PG(2, q)$. Hence, the concept of a flock of Segre varieties $S_{n,n}$ by caps that are Veronese varieties of type $\mathcal{V}_n$ is a direct generalization of flocks of hyperbolic quadrics in $PG(3, q)$.

Bader, Cossidente, and Lunardon show that the Veronese varieties correspond to $GF(q)$-reguli and furthermore, show that there is a corresponding spread.

4 Theorem (Bader, Cossidente, Lunardon [4], Theorem 3.4). *Flat flocks of $S_{n,n}$ and (A, B)-regular spreads in $PG(2n+1, q)$ are equivalent.*

Hence, we note from Jha and Johnson [8]:

5 Corollary. *Flat flocks of $S_{n,n}$ are equivalent to regulus hyperbolic q-covers in translation planes of order q^{n+1}.*

3 André planes

Let π be an André plane of order q^n and kernel $GF(q)$ that corresponds to a flat flock of a Segre variety $S_{n-1,n-1}$ over $GF(q)$. Let Σ denote a Desarguesian affine plane of order q^n coordinatized by a field F isomorphic to $GF(q^n)$ and let K denote the subfield of F isomorphic to $GF(q)$. We consider the André nets defined by the following partial spreads:

$$A_\delta = \left\{ y = xm \mid m \in F;\ m^{(q^n-1)/(q-1)} = \delta \right\}.$$

This partial spread has a set of replacement partial spreads A_δ^i, defined as follows:

$$A_\delta^i = \left\{ y = x^{q^i} m \mid m \in F;\ m^{(q^n-1)/(q-1)} = \delta \right\},$$

for $i = 0, 1, 2, \ldots, n-1$.

For each δ in $K - \{0\}$, choose a element integer $i(\delta)$ between 0 and $n-1$.

6 Definition. The 'multiply-replaced André plane' with components

$$x = 0,\ y = 0,\ y = x^{q^{i(\delta)}} m;\ m^{(q^n-1)/(q-1)} = \delta$$

shall be denoted by $\cup_{i=0}^{n-1} A_\delta^{i(\delta)}$.

We note that the André planes of order q^n, for $n > 2$ have been characterized by Johnson and Pomareda [11] as exactly those translation planes of order q^n that admit symmetric affine homology groups of orders $(q^n-1)/(q-1)$ (symmetric simply means that the axis and coaxis of one homology group is the coaxis and axis of a second homology group).

First assume that $n = t(q-1)$. Then

$$\begin{aligned}(q^{t(q-1)} - 1)/(q-1) &= 1 + q + q^2 + \cdots + q^{t(q-1)-1} \\ &= \sum_{j=1}^{t(q-1)-1} (q^j - 1) + t(q-1) - 1 + 1.\end{aligned}$$

Hence, $q-1$ divides $(q^{t(q-1)}-1)/(q-1)$. Consider the Desarguesian group $\langle (x,y) \longmapsto (x, y\alpha);\ \alpha \in K - \{0\} \rangle$. This group is 'regulus-inducing' in the sense that each orbit of a n-dimensional K-subspace disjoint from $x = 0$, $y = 0$ union $x = 0$, $y = 0$ is a K-regulus in the associated projective space.

Now we claim that we also have $\langle (x,y) \longmapsto (x, y\alpha);\ \alpha \in K - \{0\} \rangle$ as a collineation group of any André spread. To see this, we note that

$$y = x^q m;\ m^{(q^{t(q-1)}-1)/(q-1)} = \delta$$

is mapped under $(x, y) \longmapsto (x, y\alpha)$ to

$$y = x^q m\alpha;\ m^{(q^{t(q-1)}-1)/(q-1)} = \delta = (m\alpha)^{(q^{t(q-1)}-1)/(q-1)},$$

since $q-1$ divides $\left(q^{t(q-1)}-1\right)/(q-1)$. Hence, it follows that we have a 'regulus hyperolic cover'. Thus, we obtain the following theorem and its corollary.

7 Theorem. *Any André spread of order $q^{t(q-1)}$ and kernel containing $GF(q)$ admits a regulus hyperbolic cover.*

8 Corollary. *Any André spread of order $q^{t(q-1)}$ and kernel containing $GF(q)$ produces a flat flock of a Segre variety $S_{t(q-1)-1,t(q-1)-1}$ over $GF(q)$.*

We now consider the converse. We first establish that the 'carriers' of a regulus hyperbolic cover are normally uniquely defined in generalized André planes.

9 Theorem. *Let π be a non-Desarguesian and non-Hall generalized André plane of finite order q^n. If $q > 2$ and π admits a regulus hyperbolic cover then the set of two common components is uniquely defined.*

PROOF. By Foulser [6], if π is not either Desarguesian or Hall then the full collineation group leaves invariant a unique set of two components. Hence, if there is a regulus hyperbolic cover, there is a corresponding regulus-inducing homology group of order $q-1$. If $q > 2$, then $q-1 > 1$, implying that the two fixed components under the regulus-inducing homology group are uniquely defined. This then implies that the regulus hyperbolic cover is also uniquely determined. $\boxed{QED}$

Now assume that an André spread of order q^n and kernel containing $GF(q)$ corresponds to a flat flock of a Segre variety. Assume that the plane is not Desarguesian or Hall. Then, there is a homology group of order $q-1$ as above, and $q-1 > 2$ by assumption. Furthermore, there is a homology group of order $(q^n-1)/(q-1)$. By the previous theorem, we may assume that the axes and centers of both homology groups are the same (note there are 'symmetric' homology groups of both orders).

Assume that every André plane π of this order has a regulus-inducing homology group. Then, there exists an André spread where exactly one André partial spread is replaced. Since the plane is not Hall, then $n > 2$. There are $q^n+1-(q^n-1)/(q-1)$ remaining components of π. Suppose that this replaced partial spread is moved by an element σ of the regulus-inducing homology group of order $q-1$. Since the components of π are either of the form $y = x^{q^i}m$ or $y = xn$, it follows that the regulus-inducing group must leave invariant the set of all components of the general form $y = x^{q^i}m$; that is this group must leave the replaced net invariant. Since this affine homology group of order $q-1$ must leave this net invariant then $q-1$ divides $(q^n-1)/(q-1) = 1+n-1+\sum_{j=1}^{n-1}(q^j-1)$, implying that $q-1$ divides n. Hence, we obtain the following result:

10 Theorem. *Every André plane of the class of André planes of order q^n and kernel containing $GF(q)$ corresponds to a flat flock of a Segre variety in $S_{n-1,n-1}$ over $GF(q)$ if and only if $q-1$ divides n.*

11 Remark. It is possible to determine exactly the number of isomorphic planes within the class of André planes of order q^n and kernel containing $GF(q)$. Roughly speaking, we have $(q-1)$ choices for δ and n choices for the nets A^i_δ. Hence, potentially, there are $(q-1)^n$ possible planes. Thus, as a rough count, there are approximately $(q-1)^{t(q-1)}$ corresponding flat flocks of Segre varieties $S_{t(q-1)-1,t(q-1)-1}$.

4 Generalized André planes

We now vary the subfield required in the construction of André nets and construct a tremendous number of generalized André planes of order q^{q-1} and kernel containing $GF(q)$ that produce flocks of Segre varieties.

The idea is to construct André nets of degree $\left(q^{t(q-1)}-1\right)/(q^z-1)$ with kernel containing $GF(q)$ for a set of subfields $GF(q^z)$ of $GF(q^{q-1})$ that admit the homology group of order $q-1$. We require that $(q-1)$ divides $(q^{t(q-1)}-1)/(q^z-1)$. Since $(q-1)$ divides $\left(q^{t(q-1)}-1\right)/(q-1)$. But, this requirement is satisfied provided z divides t.

Hence, take a subset of fields $\{GF(q^{t_i});\ t_i \mid t \text{ for } i=1,2,\dots,k\}$. For each such field, consider an André net A_{δ_j} where $\delta_j \in GF(q^{t_j})$ and choose a replacement net $A^{i(\delta_j)}_{\delta_i}$. Assume that the André nets chosen are mutually disjoint. Now if we replace by $\cup_{j=1}^k A^{i(\delta_j)}_{\delta_j}$, we obtain a generalized André plane that admits the homology group of order $q-1$ mentioned previously; the regulus-inducing homology group. Hence, we obtain generalized André planes that are not necessarily André planes whose spreads are unions of $GF(q)$-reguli sharing two components. We formalize this as follows.

12 Theorem. *Let π be an André plane of order $q^{t(q-1)}$ and kernel containing $GF(q)$. Using a set $\{GF(q^{t_i});\ t_i \mid t$ for $i=1,2,\dots,k\}$, choose k mutually disjoint André nets A_{δ_j} corresponding to the subfields $GF(q^{t_i})$ where $\delta_i \in GF(q^{t_i})$. Then, choose any replacement net $A^{i(\delta_j)}_{\delta_j}$ where $i(\delta_j)$ is an integer between 0 and t_i-1. Let $\mathcal{N}$ denote the net defined by the components of π not in one of the chosen André nets.*

Then $\cup_{j=1}^k A^{i(\delta_j)}_{\delta_j} \cup \mathcal{N}$ is a generalized André plane and corresponds to a flat flock of a Segre variety in $S_{q-2,q-2}$ over $GF(q)$.

13 Remark. In Johnson [9], there are some constructions of non-André replacements of André nets. Any of these with the restrictions of the previous theorem will produce another class of generalized André planes that correspond to flat flocks.

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