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A Darboux theorem for generalized contact manifolds [i]

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Abstract. We consider a manifold M equipped with 1-forms $\eta_1, \ldots, \eta_s$ which satisfy certain contact like properties. We prove a generalization of the classical Darboux theorem for such manifolds.

Keywords: contact manifold, Darboux theorem, parallelized null bundle

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1 Introduction

In the present paper we consider a smooth manifold M of dimension $(2n+s)$ admitting a set of 1-forms $\eta_1, \ldots, \eta_s$ such that: $\eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_1)^n$ is nowhere vanishing, the form $d\eta_1$ is of constant rank equal to $2n$ and $d\eta_1 = \cdots = d\eta_s$. We call such manifolds *generalized contact manifolds with a parallelizable null bundle*. We observe that such manifolds carry a transverse symplectic foliation $\mathcal{F}$ determined by the null distribution of $d\eta_1$. The idea of our proof is to apply the Darboux theorem for the transverse part of the foliation and then extend it along the leaves. Our version of the Darboux theorem is clearly local and may be expressed as a property of certain forms on an open subset of $\mathbb{R}^{2n+s}$. However, we present here this theorem in the framework of the generalized

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contact manifolds with a parallelized null bundle and almost $\mathcal{S}$-manifolds to underline its geometric applications.

The notion of the generalized contact manifold with a parallelized null bundle arises in the context of the so called (almost) $\mathcal{S}$-manifolds. These are manifolds which, besides the 1-forms $\eta_1, \ldots, \eta_s$, are equipped with an f-structure, a compatible Riemannian metric and appropriate integrability conditions on the f-structure, cf. [2]. The (almost) $\mathcal{S}$-manifolds were studied by various authors and are currently under intensive research, cf. [4] and the references there.

The classical theorems of Darboux for symplectic and contact manifolds are the starting point for study of these manifolds. Hence there are many possible references. The standard proofs and some generalizations may be found for example in [5, 6, 8, 11]; the modern approach including the so called Moser's trick may be found in [7]. Further generalizations are given in [1, 10, 12].

All manifolds, maps, distributions considered here are smooth i.e. of the class C^∞; we denote by $\Gamma(-)$ the set of all sections of a corresponding bundle. We use the convention that $2u \wedge v = u \otimes v - v \otimes u$.

2 Main Theorem

Throughout all of this paper we assume that there is given a $(2n+s)$-dimensional manifold M with 1-forms $\eta_1, \ldots, \eta_s$ such that $d\eta_1 = \cdots = d\eta_s =: F$ and

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge F^n \neq 0 \tag{1}$$

at each point of M. We suppose also that $\operatorname{rank}(F) = \text{const.} = 2n$. Then we call (M, η_i), $(i = 1, \ldots, s)$ a *generalized contact manifold with a parallelizable null bundle*. We denote by $\mathcal{D} := \bigcap_{i=1}^s \ker \eta_i$. Then from (1) it follows that $\mathcal{D}$ is a vector subbundle of TM of rank $2n$ and F restricted to $\mathcal{D}$ is non-degenerate, i.e. $(\mathcal{D}, F|_{\mathcal{D}\times\mathcal{D}})$ is a symplectic vector bundle over M. By $\mathrm{Null}(F_p)$ we denote the set of all $X \in T_pM$, $(p \in M)$, such that $X \lrcorner F = 0$; it is called the *null subspace associated with* F_p. Then the *null subbundle associated with* F is $\mathrm{Null}(F) := \bigcup \mathrm{Null}(F_p)$ where the union is taken over all $p \in M$. $\mathrm{Null}(F)$ is a vector bundle over M of rank s. Moreover, we have that $\mathrm{Null}(F) \oplus \mathcal{D} = TM$.

In the following lemma we generalize the existence theorem of the Reeb vector field on contact manifolds.

1 Lemma. *There exist unique* $\xi_1, \ldots, \xi_s \in \Gamma(\mathrm{Null}(F))$ *such that* $\eta_i(\xi_j) = \delta_{ij}$ *for all* $i, j = 1, \ldots, s$. *Moreover,* $[\xi_i, \xi_j] = 0$, $L_{\xi_i} X \in \Gamma(\mathcal{D})$ *and* $L_{\xi_i}\eta_j = 0$ *for all* $i, j = 1, \ldots, s$, *and* $X \in \Gamma(\mathcal{D})$.

Proof. We consider a section Ψ of the bundle $(\mathrm{Null}(F))^* \otimes \mathbb{R}^s$ given by $\Psi_p(X) := (\eta_1(X), \ldots, \eta_s(X))$ for each $p \in M$ and each $X \in \mathrm{Null}(F_p)$. Since (1) holds and $\dim(\mathrm{Null}(F_p)) = s$, it follows that Ψ_p is an isomorphism between $\mathrm{Null}(F_p)$ and $\mathbb{R}^s$. Then we put

$$(\xi_i)_p := \Psi_p^{-1}(e_i) \tag{2}$$

where $e_1, \ldots, e_s$ is the canonical basis of $\mathbb{R}^s$ and p varies over all of M. Such fields ξ_i are smooth and satisfy $\eta_i(\xi_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, s\}$. Then the uniqueness of the existence of $\xi_1, \ldots, \xi_s$ follows from the fact they are uniquely defined by condition (2).

We have that for each $k \in \{1, \ldots, s\}$

$$\begin{gathered}\eta_k([\xi_i, \xi_j]) = -2F(\xi_i, \xi_j) + \xi_i \eta_k(\xi_j) - \xi_j \eta_k(\xi_i) = 0\\ [\xi_i, \xi_j] \lrcorner F = L_{\xi_i}(\xi_j \lrcorner F) - \xi_j \lrcorner (L_{\xi_i} F) = -\xi_j \lrcorner (\xi_i \lrcorner dF + d(\xi_i \lrcorner F)) = 0.\end{gathered}$$

This implies that $[\xi_i, \xi_j]$ annihilates η_k and F for all k. Therefore $[\xi_i, \xi_j]$ vanishes. If $X \in \Gamma(\mathcal{D})$ and $i, j = 1, \ldots, s$, then $[\xi_i, X] \lrcorner \eta_j = -2F(\xi_i, X) + \xi_i \eta_j(X) - X\eta_j(\xi_i) = 0$ hence it follows that $[\xi_i, X] \in \Gamma(\mathcal{D})$. Since

$$\begin{gathered}L_{\xi_i} \eta_j(\xi_k) = \xi_i(\eta_j(\xi_k)) - \eta_j([\xi_i, \xi_k]) = 0\\ (L_{\xi_i} \eta_j)(X) = \xi_i(\eta_j(X)) - \eta_j([\xi_i, X]) = 0\end{gathered}$$

for all $i, j, k \in \{1, \ldots, s\}$ and $X \in \Gamma(\mathcal{D})$, it follows that $L_{\xi_i} \eta_j = 0$. $\boxed{QED}$

We shall call $\xi_1, \ldots, \xi_s$ the *Reeb vector fields* associated with the structure (M, η_i), $(i = 1, \ldots, s)$.

The generalized contact structure with a parallelized null bundle (M, η_i), $(i = 1, \ldots, s)$ may be characterized in another way. If $C_s(M) := M \times \mathbb{R}^s_+$ is the *s-cone over M* where $\mathbb{R}_+$ denotes the positive real numbers and $r = (r_1, \ldots, r_s)$ then we put

$$\omega_F := d\left(\sum_{i=1}^{s} r_i^2 \eta_i\right) = \|r\|^2 F + \sum_{i=1}^{s} 2r_i dr_i \wedge \eta_i.$$

It is straightforward to observe that $(C_s(M), \omega_F)$ is a symplectic manifold, cf. [3].

2 Remark. There exists on $\mathcal{D}$ a Riemannian metric g_0 and a compatible complex structure J_0 such that $d\eta_1$ is the associated Kähler form; actually J_0 may be obtained via the polar decomposition of $d\eta_1$ with respect to any Riemannian metric. Then we extend g_0 to a Riemannian metric g on M by assuming that $g|_{\mathcal{D}} = g_0$, $\xi_1, \ldots, \xi_s$ are orthonormal and $\mathcal{D} \perp \mathrm{Null}(F)$. We extend also J_0 to an endomorphism ϕ of TM by assuming that $\phi|_{\mathcal{D}} = J_0$ and $\phi|_{\mathrm{Null}(F)} = 0$. As a result we obtain the set $(M, g, \phi, \eta_i, \xi_j)$, $(i, j = 1, \ldots, s)$ which is an almost $\mathcal{S}$-manifold. If a certain integrability condition, the so called *normality condition*, for ϕ is satisfied, then the manifold is called an $\mathcal{S}$-manifold, cf. [2, 4].

3 Theorem (Darboux Theorem). *Let (M^{2n+s}, η_i), $(i \dots, s)$, be a generalized contact manifold with a parallelized null bundle, then locally on M there exist coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s)$ such that for each $i = 1, \dots, s$*

$$\eta_i = dz_i + \sum_{\alpha=1}^{n} x^\alpha dy^\alpha.$$

PROOF. From the Lemma we get the Reeb vector fields $\xi_1, \dots, \xi_s$. We denote by $\mathcal{F}$ the foliation on M determined by the distribution $\operatorname{span}\{\xi_1, \dots, \xi_s\} = \operatorname{Null}(F)$. Since $\xi_i \lrcorner F = 0$ and $dF = 0$ then it follows that F is a basic 2-form on M, cf. [9]. Since $\mathcal{D}$ is complementary to $T\mathcal{F}$ and $(\mathcal{D}, F|_{\mathcal{D}\times\mathcal{D}})$ is a symplectic vector bundle then the foliated manifold $(M, \mathcal{F}, F)$ is transverse symplectic, cf. [9, p.53]. Let $p_0 \in M$ then there exists an open connected neighbourhood U of p_0 and a trivial fibration $\pi_U : U \to V$ on a $2n$-dimensional symplectic manifold (V, ω) with $\pi_U^*(\omega) = F$ and such that the fibres of π_U coincide with the intersection of $\mathcal{F}$ with U, cf. [9]. From the classical Darboux theorem, after possible compression of V, we may assume that V is an open neighbourhood of $0 \in \mathbb{R}^{2n}$ with $\pi_U(p_0) = 0$ and

$$\omega = \sum_{\alpha=1}^{n} d\tilde{x}_\alpha \wedge d\tilde{y}_\alpha = d\left(\sum_{\alpha=1}^{n} \tilde{x}_\alpha d\tilde{y}_\alpha\right)$$

where $(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{y}_1, \dots, \tilde{y}_n)$ are the canonical coordinates on $V \subset \mathbb{R}^{2n}$. We denote by $(x_1, \dots, x_n, y_1, \dots, y_n) := \pi_U^*(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{y}_1, \dots, \tilde{y}_n)$ which are coordinates on the transverse part to $\mathcal{F}$. Let $\sigma : V \to U$ be a smooth section of π_U such that $\sigma(0) = p_0$.

Let ψ_t^i be a 1-parameter subgroup of local transformations associated with ξ_i for each $i \in \{1, \dots, s\}$. Since $[\xi_i, \xi_j] = 0$ then $\psi_{t_1}^i$ and $\psi_{t_2}^j$ commute with each other within their domains. Then we define a map Φ from a neighbourhood of $0 \in \mathbb{R}^{2n+s}$ into M by setting $\Phi(x, y, \tilde{z}) := \psi_{\tilde{z}_1}^1 \circ \dots \circ \psi_{\tilde{z}_s}^s(\sigma(x, y))$ where x stands for $(x_1, \dots, x_n)$, y stands for $(y_1, \dots, y_n)$ and $\tilde{z}$ stands for $(\tilde{z}_1, \dots, \tilde{z}_s)$. Since Φ is smooth and $\Phi(0) = p_0$, there exists an open subset W of $0 \in \mathbb{R}^{2n+s}$ such that $\Phi(W) \subset U$. It follows from the construction of Φ that

$$d_0\Phi\left(\tfrac{\partial}{\partial \tilde{z}_i}|_0\right) = (\xi_i)_{p_0}, \quad d_0\Phi\left(\tfrac{\partial}{\partial x_\alpha}|_0\right) = d_0\sigma\left(\tfrac{\partial}{\partial x_\alpha}|_0\right), \quad d_0\Phi\left(\tfrac{\partial}{\partial y_\alpha}|_0\right) = d_0\sigma\left(\tfrac{\partial}{\partial y_\alpha}|_0\right)$$

for each $i \in \{1, \dots, s\}$ and $\alpha \in \{1, \dots, n\}$; here $d_0\Phi$ and $d_0\sigma$ denote the differential of the corresponding maps taken at 0. Since $(\xi_i)_{p_0}$, $(i = 1, \dots, s)$ give a basis of the vertical space of π_U and $d_0\sigma(\frac{\partial}{\partial x_\alpha}|_0)$, $d_0\sigma(\frac{\partial}{\partial y_\beta}|_0)$, $(\alpha, \beta = 1, \dots, n)$ give a basis of the horizontal space, it follows that $d_0\Phi$ is an isomorphism and Φ is a diffeomorphism in a neighbourhood of $0 \in \mathbb{R}^{2n+s}$. Hence we may assume, after possible compression of W, that $\Phi : W \to \Phi(W)$ is a diffeomorphism of W onto $\Phi(W) \subset U$ an open neighbourhood of p_0. From the construction of Φ it follows that for each $i = 1, \dots, s$ we have that $\Phi_*(\frac{\partial}{\partial \tilde{z}_i}) = \xi_i$. We denote by φ the inverse of Φ i.e. $\varphi = \Phi^{-1} : \Phi(W) \to W$. The map $\varphi = (x_1, \dots, x_n, y_1, \dots, y_n, \tilde{z}_1, \dots, \tilde{z}_s)$ is a chart on M with respect to which $\xi_i = \frac{\partial}{\partial \tilde{z}_i}$ and $d\tilde{z}_i(\xi_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, s\}$. The 1-form

$$\eta_i - d\tilde{z}_i - \sum_{\alpha=1}^{n} x_\alpha dy_\alpha \tag{3}$$

is defined on $\Phi(W)$ and is closed for each $i = 1, \dots, s$ since $d\eta_i = F$. From the Poincaré lemma, after possible compression on W, we may assume that there exist smooth functions $f_1, \dots, f_s \in C^\infty(\Phi(W))$ such that

$$df_i = \eta_i - d\tilde{z}_i - \sum_{\alpha=1}^{n} x_\alpha dy_\alpha \tag{4}$$

for all $i \in \{1, \dots, s\}$. Since the 1-forms in (3) are basic with respect to the foliation $\mathcal{F}$ so are df_i for all i. This means that $\frac{\partial f_i}{\partial \tilde{z}_j} = 0$ for all $i, j = 1, \dots, s$. Then we consider the new coordinates

$$x_\alpha := x_\alpha, \quad y_\beta := y_\beta, \quad z_i := \tilde{z}_i + f_i \tag{5}$$

where $\alpha, \beta \in \{1, \dots, n\}$ and $i \in \{1, \dots, s\}$. We need to comment that equations (5) actually define new coordinates; in fact the Jacobian matrix between φ and the new coordinates is the following

$$\begin{pmatrix} (\delta_{\alpha\beta}) & & \\ & (\delta_{\alpha\beta}) & \\ -(\frac{\partial f_i}{\partial x_\alpha}) & -(\frac{\partial f_i}{\partial y_\alpha}) & (\delta_{ij}) \end{pmatrix}. \tag{6}$$

Therefore our assertion follows from the definition of the coordinates (x, y, z) in (5) and equation (4). $\boxed{QED}$

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