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Rigidity of Iwasawa nilpotent Lie groups via Tanaka's theory

Alessandro Ottazzi
Università di Milano–Bicocca,
Milano 20126, Italy alessandro.ottazzi@unimib.it

Ben Warhurst
School of Mathematics, University of New South Wales,
Sydney NSW 2052, Australia
Institute of Mathematics of the Polish Academy of Sciences
Warszawa, Poland
benwarhurst68@gmail.com

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Abstract. We provide a new proof to the known result on rigidity of Iwasawa nilpotent Lie groups [5, 12]. More precisely, we use Tanaka's prolongation theory for establishing the rigidity type of those nilpotent groups. This note aims to complement [8], where we use the point of view of Tanaka prolongations for studying rigidity in the general setting of nilpotent stratified Lie groups. When the group is of Iwasawa type, a special formalism occurs, which is related to the theory of semisimple Lie groups, namely the formalism of root systems. We use this language in order to classify the rigidity types.

Keywords: Simple Lie groups and algebras, contact map, prolongation, H-type algebras, differential system

MSC 2000 classification: 22E25, 22E60, 53C17, 58A17, 58D05

1 Introduction

In [8], we consider the question of rigidity of stratified nilpotent Lie groups. This is the study of those diffeomorphisms on such a group whose differential preserves the horizontal bundle, that is the left invariant subbundle corresponding with the generating layer in the algebra. A classical problem is to investigate the family of contact mappings. Rigidity is the property that the family of local contact mappings form a finite dimensional space. In more precise terms this means that the dimension of the space of vector fields which generate local contact maps is finite. In [8] we apply an algebraic method due to Tanaka [10] as a unified technique to determine rigidity or nonrigidity of various classes of Carnot groups, such as H-type groups, Iwasawa groups, Métivier groups [6], groups satisfying the rank one condition [7]. Due to the special properties of Iwasawa nilpotent Lie groups coming from the formalism of the root systems, we decided to dedicate this separated note to these cases.

2 Notation and Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra with Killing form B and Cartan involution θ. Let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\mathfrak{a}'$ its dual. For $\alpha \in \mathfrak{a}'$, set

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X\},$$

where $H \in \mathfrak{a}$. When $\alpha \neq 0$ and $\mathfrak{g}_\alpha$ is not trivial, α is said to be a restricted root of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and $\mathfrak{g}_\alpha$ is the root space of α. We denote by Σ the set of the restricted roots and call it the root system of $\mathfrak{g}$. Choose an ordering $\succ$ on $\mathfrak{a}'$, thus defining the subsets Σ_+ and $\Delta = \{\delta_1, \ldots, \delta_r\}$ of positive and simple positive restricted roots. It is well-known that there is exactly one root ω, called the highest root, that satisfies $\omega \succ \alpha$ (strictly) for every other root α. Since we shall always work with the restricted root spaces, we forget the adjective "restricted" when refering to roots. Every root $\alpha \in \Sigma_+$ can be written as $\alpha = \sum_{i=1}^r n_i\delta_i$ with uniquely defined non-negative integers $n_1, \ldots, n_r$, and the positive integer $\mathrm{ht}(\alpha) = \sum_{i=1}^r n_i$ is called the height of α. The root space decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$, where $\mathfrak{m} = \{X \in \mathfrak{k} : [X, H] = 0,\ H \in \mathfrak{a}\}$. Writing $\Sigma_- = -\Sigma_+$, one has that $\Sigma = \Sigma_+ \cup \Sigma_-$ and the Iwasawa nilpotent Lie algebra

$$\mathfrak{n} = \bigoplus_{\gamma\in\Sigma_-} \mathfrak{g}_\gamma,$$

is stratified in the usual sense, that is $[\mathfrak{n}_i, \mathfrak{n}_j] = \mathfrak{n}_{i+j}$, where $\mathfrak{n}_i = \bigoplus_{\mathrm{ht}(\gamma)=i} \mathfrak{g}_{-\gamma}$, $i = -\mathrm{ht}(\omega)$, $\ldots, -1$. To any Iwasawa nilpotent Lie algebra $\mathfrak{n}$ there is a root system Σ associated.

The subspace $\mathfrak{n}_{-1}$ generates the whole algebra $\mathfrak{n}$ via Lie brackets, and it identifies with a subspace of the tangent space at the identity to N. By left translation, $\mathfrak{n}_{-1}$ defines a subbundle of the tangent bundle called the horizontal bundle which we denote by $\mathcal{H}$. A contact map is a diffeomorphism from an open subset of N into N whose differential preserves $\mathcal{H}$. Contact vector fields are vector fields on N that generate one parameter families of contact mappings. A contact vector field $V \in \mathfrak{X}(N)$ satisfies $[V, \mathcal{H}] \subset \mathcal{H}$. The group N is rigid if the Lie algebra of contact vector fields has finite dimension. We call N nonrigid otherwise. The condition $[V, \mathcal{H}] \subset \mathcal{H}$ yields a system of partial differential equations for the coefficients of V. The space of solutions of this system has been studied in [5] and [12]. In this article, unless otherwise stated, $\mathfrak{n}$ will always be an Iwasawa nilpotent Lie algebra.

3 Tanaka prolongation

Let $\mathfrak{n} = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_{-1}$ be a stratified nilpotent (not necessarily Iwasawa) Lie algebra. The Tanaka prolongation of $\mathfrak{n}$ is the graded Lie algebra $\mathrm{Prol}(\mathfrak{n})$ given by the direct sum $\mathrm{Prol}(\mathfrak{n}) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k(\mathfrak{n})$, where $\mathfrak{g}_k(\mathfrak{n}) = \{0\}$ for $k < -s$, $\mathfrak{g}_k(\mathfrak{n}) = \mathfrak{n}_k$ for $-s \le k \le -1$, and for each $k \ge 0$, $\mathfrak{g}_k(\mathfrak{n})$ is inductively defined by

$$\mathfrak{g}_k(\mathfrak{n}) = \Big\{u \in \bigoplus_{p<0} \mathfrak{g}_{p+k}(\mathfrak{n}) \otimes \mathfrak{g}_p(\mathfrak{n})^* \mid u([X, Y]) = [u(X), Y] + [X, u(Y)]\Big\},$$

with $\mathfrak{g}_0(\mathfrak{n})$ consisting of the strata preserving derivations of $\mathfrak{n}$. If $u \in \mathfrak{g}_k(\mathfrak{n})$, where $k \ge 0$, then the condition in the definition becomes the Jacobi identity upon setting $[u, X] = u(X)$ when $X \in \mathfrak{n}$. Furthermore, if $u \in \mathfrak{g}_k(\mathfrak{n})$ and $v \in \mathfrak{g}_\ell(\mathfrak{n})$, where $k, \ell \ge 0$, then $[u, v] \in \mathfrak{g}_{k+\ell}(\mathfrak{n})$ is defined inductively according to the Jacobi identity, that is

$$[u, v](X) = [u, [v, X]] - [v, [u, X]].$$

Define the subalgebra

$$\mathfrak{h} = \bigoplus_{k \geq -1} \mathfrak{h}_k \subset \mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0)$$

where the subspaces $\mathfrak{h}_k \subset \mathfrak{g}_k$ are defined as follows: set

$$\hat{\mathfrak{n}} = \mathfrak{n}_{-s} \oplus \cdots \oplus \mathfrak{n}_{-2}$$

and for $k \geq -1$ define

$$\mathfrak{h}_k = \{u \in \mathfrak{g}_k \mid [u, \hat{\mathfrak{n}}] = \{0\}\}\,. \tag{1}$$

It follows that $[\mathfrak{h}_k, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{k-1}$ for $k \geq 0$.

In [10], Tanaka shows that the rigidity of a stratified nilpotent Lie group can be determined by studying the Tanaka prolongation of the Lie algebra. More precisely, the algebra of contact vector fields on N is finite dimensional if and only if $\mathrm{Prol}(\mathfrak{n})$ is finite dimensional. In fact more is true [10, Corollary 2, page 76]: the group N is rigid if and only if $\mathfrak{h}_k = 0$ for some integer $k \geq -1$. Tanaka's theory provides some relatively simple tests for rigidity, at least at the levels $\mathfrak{h}_{-1}$, $\mathfrak{h}_0$, and $\mathfrak{h}_1$, moreover it also provides a definition of rigidity type as the smallest integer k greater or equal to -1 such that $\mathfrak{h}_k = \{0\}$. If $\mathfrak{n}$ is nonrigid, we shall say that it is of infinite type.

4 Rigidity

Root systems are classified by means of their Dynkin diagrams (see [1] for an insight). The nonisomorphic standard root systems are A_n, $n \geq 1$, B_n, $n \geq 2$, C_n, $n \geq 3$, D_n, $n \geq 4$. Beside these, there are the exceptional systems E_6, E_7, E_8, F_4, G_2 and the reducible system BC_r, $r \geq 1$. The subindex counts the number of simple roots generating the system. Whenever α is a root, 2α is never a root, unless $\alpha \in BC_r$. To every root system there is associated one or more real simple Lie algebras, each coming with the relative Iwasawa nilpotent component. We classify the rigidity types of all Iwasawa nilpotent Lie algebras using this correspondence. Let $\mathfrak{n}$ be such a nilpotent Lie algebra with root system Σ.

Proposition 1. *If Σ is one of the following: A_n, $n \geq 4$, B_n, C_n, $n \geq 3$, D_n, $n \geq 4$, E_6, E_7, E_8, F_4, G_2, BC_r, $r \geq 2$, then $\mathfrak{n}$ is rigid with rigidity type -1. If $\Sigma = A_3$, then it is of type 0.*

Proof. Let $\Delta = \{\delta_1, \ldots, \delta_r\}$ be a system of simple roots of Σ and write $\Delta_- = -\Delta$. We fix a basis of $\mathfrak{g}_{-\delta_j}$ for every $j = 1, \ldots, r$ and thus a basis of $\mathfrak{n}_{-1}$. Then we choose a basis for every root space relative to a negative root. This yields a stratified basis of $\mathfrak{n}$. It is well known that if the sum of two roots α and β is still a root, then for every vector X in $\mathfrak{g}_\alpha$ there exists a vector Y in $\mathfrak{g}_\beta$ such that $[X, Y] \neq 0$, and viceversa. In order to show that $\mathfrak{h}_{-1} = \{0\}$, it is then sufficient to prove that every negative simple root can be summed to at least another root of height less or equal than -2.

In the cases A_n, $n \geq 4$, B_n, $n \geq 3$, C_n, $n \geq 3$, D_n, $n \geq 5$, E_6, E_7, E_8, F_4, investigation of the Dynkin diagrams shows directly that every (negative) simple root can be summed to a root of height minus two. In particular, this implies that every vector in the chosen basis for $\mathfrak{n}_{-1}$ does not commute with at least one vector in $\mathfrak{n}_{-2}$.

If $\Sigma = D_4$, then the same remark above holds for all vectors in the root spaces relative to δ_1, δ_3 and δ_4. We conclude that $\mathfrak{h}_{-1} = \{0\}$ by observing that δ_2 can be summed to $\delta_1 + \delta_2 + \delta_3 + \delta_4$.

If $\Sigma = G_2$, then δ_1 can be summed to $\delta_1 + \delta_2$ and δ_2 can be summed to $3\delta_1 + \delta_2$.

If $\Sigma = BC_r$ with $r \geq 2$, then it contains the root system B_r. So if $r \geq 3$, then the considerations made above for the case when the root system is of type B_r show that $\mathfrak{n}$ is of type -1. If $r = 2$ then $\Sigma_+ = \{\delta_1, \delta_2, \delta_1 + \delta_2, 2\delta_1, 2\delta_1 + \delta_2, 2(\delta_1 + \delta_2)\}$. So δ_1 can be summed to $\delta_1 + \delta_2$ and δ_2 can be summed to $2\delta_1$, thus showing that the corresponding $\mathfrak{n}$ is of type -1.

Finally if $\Sigma = A_3$ then $\Delta = \{\delta_1, \delta_2, \delta_3\}$ and $\mathfrak{h}_{-1} = \mathfrak{g}_{-\delta_2}$, because δ_2 can be summed to δ_1 and δ_3 only. We show that $\mathfrak{h}_0 = 0$. Pick $D \in \mathfrak{h}_0$. From (1) and the remark thereafter, $D\mathfrak{n}_{-1} \subset \mathfrak{g}_{-\delta_2}$ and $D = 0$ on $\mathfrak{n}_j$ with $j \leq -2$. Suppose that $DX \neq 0$ for some $X \in \mathfrak{n}_{-1}$. Assume first that $X \in \mathfrak{g}_{-\delta_2}$. Then there exists $Y \in \mathfrak{g}_{-\delta_1}$ such that $[DX, Y] \neq 0$. This implies that

$$0 = D[X, Y] = [DX, Y] + [X, DY] = [DX, Y],$$

because $2\delta_2$ is not a root, and so we get a contradiction. Therefore $D\mathfrak{g}_{-\delta_2} = 0$. Choose now $X \in \mathfrak{g}_{-\delta_3}$ with $DX \neq 0$. If $Y \in \mathfrak{g}_{-\delta_1}$ is such that $[DX, Y] \neq 0$, then $[DX, Y]$ is in $\mathfrak{g}_{-\delta_1-\delta_2}$. On the other hand, $[X, DY]$ is in $\mathfrak{g}_{-\delta_2-\delta_3}$ and so the contradiction $D[X, Y] \neq 0$ arises, and we conclude that $D\mathfrak{g}_{-\delta_3} = 0$. Finally, take $X \in \mathfrak{g}_{-\delta_1}$ and pick $Y \in \mathfrak{g}_{-\delta_3}$ such that $[DX, Y] \neq 0$. Then $D[X, Y] = [DX, Y] + [X, DY]$, where the two summands belong to disjoint root spaces, thus giving a contradiction and proving $\mathfrak{h}_0 = \{0\}$. $\boxed{QED}$

The remaining cases need to be studied explicitly, since their behavior with respect to the rigidity question may change, even for algebras corresponding to the same root system.

If $\Sigma = A_2$, then $\mathfrak{n}$ is an H-type Lie algebra by [4]. More precisely there are exactly four algebras with root system A_2. We know by [9] that H-type algebras are rigid when the dimension of the centre is greater than two, hence they are of infinite type if and only if the centre is of dimension two. Looking at [4, Proposition 4.1] we then conclude that there are two nonrigid nilpotent algebras relative to A_2, namely the three dimensional Heisenberg algebra and its complexification. The remaining two algebras are the nilpotent components in the Iwasawa decomposition of $\mathfrak{sl}(3, \mathbb{H})$, where $\mathbb{H}$ denotes the quaternions, and the Iwasawa decomposition of $\mathfrak{e}_{(6,-26)}$. By [8, Theorem 4], these nilpotent algebras are of type 1.

If $\Sigma = A_1$, then $\mathfrak{n}$ is the abelian Lie algebra $\mathbb{R}^n$, which is trivially of infinite type.

If $\Sigma = B_2$, then $\Delta = \{\delta_1, \delta_2\}$ and $\Sigma_+ = \{\delta_1, \delta_2, \delta_1 + \delta_2, \delta_1 + 2\delta_2\}$. So $\mathfrak{n}$ has step three and the following three cases occur [4]:

δ_1	δ_2	$\delta_1 + \delta_2$	$\delta_1 + 2\delta_2$
1	n	n	1
2	2	2	2
3	4	4	3

where each number indicates the dimension of the root space relative to the root in the same column. The nilpotent Lie algebras described in the first row correspond to the simple Lie algebras $\mathfrak{so}(2, 2+n)$. If $n = 1$, then $\mathfrak{n}$ is the Engel Lie algebra, which is well known to be of infinite type [11, 7, 8]. If $n > 1$, then using [2, Proposition 4.3] we can set bases $\mathfrak{g}_{-\delta_1} = \mathbb{R}X$, $\mathfrak{g}_{-\delta_2} = \operatorname{span}\{Y_1, \ldots, Y_n\}$ so that $[X, Y_i] \neq [X, Y_j]$ for every $i \neq j$ and $[X, Y_1], \ldots, [X, Y_n]$ are a basis of $\mathfrak{g}_{-\delta_1-\delta_2}$. Moreover $\mathfrak{h}_{-1} = \mathfrak{g}_{-\delta_1}$. We show that $\mathfrak{h}_0 = \{0\}$. If $D \in \mathfrak{h}_0$ then $D\mathfrak{n}_{-1} \subset \mathfrak{g}_{-\delta_1}$ and for every $i = 1, \ldots, n$ we have

$$0 = D[X, Y_i] = [DX, Y_i] + [X, DY_i] = [DX, Y_i],$$

since $2\delta_1$ is not a root. It follows that $DX = 0$. If $DY_i = \lambda_i X$ then for every $i \neq j$ we have

$$0 = D[Y_i, Y_j] = \lambda_i [X, Y_i] - \lambda_j [X, Y_j],$$

which yields $\lambda_i = 0$ for every i. It follows that $\mathfrak{h}_0 = \{0\}$ and $\mathfrak{n}$ is of type 0.

The second row of (4) corresponds with the complexified Engel Lie algebra, which is nonrigid (see e.g. [8, Theorem 3]).

The third row of (4) corresponds with the nilpotent component of the simple Lie algebra $\mathfrak{sp}(2,2)$. In this case we compute explicitly a basis of $\mathfrak{n}$ and the bracket relations. We do this relying on the fact that $\mathfrak{g}_{-\delta_2} + \mathfrak{g}_{-\delta_1-\delta_2} + \mathfrak{g}_{-\delta_1-2\delta_2}$ is a H-type algebra [3, Theorem 3.8]. This and the proof of Proposition 4.3 in [4] give $\mathfrak{n}_{-1} = \operatorname{span}\{X_1, \dots, X_7\}$, $\mathfrak{n}_{-2} = \operatorname{span}\{Y_1, \dots, Y_4\}$, $\mathfrak{n}_{-3} = \operatorname{span}\{Z_1, Z_2, Z_3\}$ and the bracket table is

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Y_1	Y_2	Y_3	Y_4
X_1	0	0	0	0	$-Y_1$	$-Y_2$	$-Y_3$	Z_1	Z_2	Z_3	0
X_2	0	0	0	0	$-Y_2$	Y_1	$-Y_4$	Z_2	$-Z_1$	0	Z_3
X_3	0	0	0	0	$-Y_4$	$-Y_3$	Y_2	0	Z_3	$-Z_2$	Z_1
X_4	0	0	0	0	$-Y_3$	Y_4	Y_1	Z_3	0	$-Z_1$	$-Z_2$
X_5	Y_1	Y_2	Y_4	Y_3	0	0	0	0	0	0	0
X_6	Y_2	$-Y_1$	Y_3	$-Y_4$	0	0	0	0	0	0	0
X_7	Y_3	Y_4	$-Y_2$	$-Y_1$	0	0	0	0	0	0	0
Y_1	$-Z_1$	$-Z_2$	0	$-Z_3$	0	0	0	0	0		0
Y_2	$-Z_2$	Z_1	$-Z_3$	0	0	0	0	0	0	0	0
Y_3	$-Z_3$	0	Z_2	Z_1	0	0	0	0	0	0	0
Y_4	0	$-Z_3$	$-Z_1$	Z_2	0	0	0	0	0	0	0

Given these bracket relations, a direct but rather long calculation shows that $\mathfrak{h}_0(\mathfrak{n}) = \{0\}$.

If $\Sigma = BC_1$, then $\mathfrak{n}$ is H-type by [2]. Except for the $2n+1$-dimensional Heisenberg algebra which is nonrigid, the remaining cases are H-type algebras with center of dimension 3 and 7, and so they are rigid of type 1 by [8].

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