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Symmetries of linear programs

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Abstract. Every finite (permutation) group is the full symmetry group of a suitable linear program.

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1 Introduction

The symmetric groups S_n and the cyclic groups C_n essentially are the only examples for symmetry groups of linear or integer programs that have been discussed in the literature, see e.g. [5] and [6]. In [4], Bödi, Herr, and Joswig developed some ideas to tackle linear and integer programs with arbitrary groups of symmetries. However, the question remained whether or not there are linear (integer) programs with groups of symmetries other than S_n and C_n. Indeed, we show in this short note that every finite permutation group is the full symmetry group of a suitable linear or integer program. Some of our constructions are based on graph theory.

2 Symmetries of linear programs

A linear (or integer) program Λ of dimension n is usually described in the form

$$\Lambda : \qquad \max c^t x \quad \text{such that } Ax \leq b \text{ and } x \in \mathbb{R}^n_{\geq 0} \qquad (\text{or } x \in \mathbb{Z}^n_{\geq 0}),$$

where $A \in \mathbb{R}^{m \times n}$ is a matrix and $b \in \mathbb{R}^m$. The vector $c \in \mathbb{R}^n \setminus \{0\}$ is called the utility vector of Λ. The linear inequality system $Ax \leq b$ comprises the m linear constraints $\sum_j a_{ij} x_j \leq b_i$ with $1 \leq i \leq m$; each of these m linear constraints defines an affine half-space in $\mathbb{R}^n$ (except if all entries in the corresponding row of A are 0).

The symmetric group S_n acts naturally on $\mathbb{R}^n$ by permuting the n coordinates. This action is described by permutation matrices and preserves the cone $\mathbb{R}^n_{\geq 0}$ as well as $\mathbb{Z}^n_{\geq 0}$. In fact, every orthogonal matrix preserving $\mathbb{Z}^n_{\geq 0}$ is a permutation matrix. An affine half-space $\{ x \in \mathbb{R}^n \mid \sum_j a_j x_j \leq \beta \}$ is mapped by $g \in S_n$ onto

$$\begin{aligned} \{ x^g \in \mathbb{R}^n \mid \sum_j a_j x_j \leq \beta \} &= \{ x^g \in \mathbb{R}^n \mid \sum_j a_{j^g} x_{j^g} \leq \beta \} \\ &= \{ x \in \mathbb{R}^n \mid \sum_j a_{j^g} x_j \leq \beta \}. \end{aligned}$$

This describes the action of S_n on the linear constraints.

We say that $g \in S_n$ is a *symmetry of* Λ if $c^g = c$ and the m linear constraints (or affine half-spaces) described by $Ax \leq b$ are permuted by g. The latter condition means that there exists a permutation $h \in S_m$ such that $b^h = b$ and $P_h A P_g = A$, where P_h and P_g are the permutation matrices corresponding to h and g. The *symmetry group of* Λ is the subgroup of S_n consisting of all symmetries of Λ. This definition of symmetry is most commonly used in the literature, see e.g. [7].

The symmetry group of Λ acts on the feasible region $\{ x \in \mathbb{R}^n \mid Ax \leq b \}$ and on the set of all solutions x of Λ. One can show that a solvable linear program Λ has always a solution which is fixed by the symmetry group of Λ; see [4], compare also [8].

In all our constructions and examples we always use the utility vector $c = (1, 1, \ldots, 1)^t$; then the condition $c^g = c$ is automatically satisfied, and Λ is described by specifying only the linear constraints $Ax \leq b$.

3 All finite groups occur

The following results extend the list of 'group-universal' structures in the sense of Funk, Kegel and Strambach [3] (see also [1]).

Theorem 1. *Every abstract finite group is the full symmetry group of a suitable linear program.*

Proof. Choose $\beta \in \mathbb{R}$. According to a result of Frucht [2], every finite group G is isomorphic to the full automorphism group of some finite graph Γ (which is undirected and free of loops); in fact, Frucht shows that there are infinitely many possibilities for Γ. We use the vertices of Γ as indices of our variables and consider the linear program Λ with the constraints

$$x_v + x_w \leq \beta, \quad \text{where } \{v, w\} \text{ is an edge of } \Gamma.$$

The symmetry group of Λ coincides with the automorphism group of Γ and is therefore isomorphic to G. $\boxed{QED}$

Theorem 2. *Let Γ be a directed or undirected finite graph. We colour the edges of Γ in an arbitrary fashion. Then the permutation group of all colour-preserving automorphisms of Γ coincides with the full symmetry group of some linear program Λ (of dimension n if Γ has n vertices).*

Proof. Identify the distinct colours of edges with distinct real numbers $\beta_1, \beta_2, \ldots$. If Γ is directed, then we take for Λ all constraints

$$x_v + 2x_w \leq \beta_k,$$

where (v, w) is an edge of Γ with colour β_k. If Γ is undirected, then we take the constraints $x_v + x_w \leq \beta_k$, where $\{v, w\}$ is an edge with colour β_k. $\boxed{QED}$

Chamber systems, and buildings in particular, can be defined as undirected graphs with coloured edges, where the colours correspond to the types. Hence Theorem 2 implies that the group of all type-preserving automorphisms of every finite building (or chamber system) is the full symmetry group of some linear program.

Corollary 1. *Every finite sharply transitive permutation group G is the full symmetry group of a suitable linear program (of dimension $n = |G|$).*

Proof. Choose any generating set S of G with $1 \notin S$ (for example, $S = G \setminus \{1\}$). The corresponding directed Cayley graph Γ has G as its set of vertices, and the edges (g, sg) with $g \in G$, $s \in S$; we say that such an edge has colour s.

Then the colour-preserving automorphisms of Γ are the maps $x \mapsto xg$ with $g \in G$, which afford the right regular representation of G (the left regular and the right regular representations are equivalent via inversion $x \mapsto x^{-1}$). Thus Theorem 2 yields the assertion. $\boxed{QED}$

The above graph-theoretic constructions lead to linear constraints each involving only two variables. However, not every permutation group is the full automorphism group of a graph (for example, a doubly transitive group distinct from the full symmetric group is not). In order to realize all finite permutation groups, we use the following construction.

Theorem 3. *Let G be a subgroup of the symmetric group S_n. Then G is the full symmetry group of a suitable linear program Λ of dimension n with $m = |G|$ linear constraints.*

Proof. Choose $\beta \in \mathbb{R}$ and let Λ be described by the linear constraints

$$\sum_{j=1}^{n} j x_{j^g} \leq \beta \quad \text{with } g \in G.$$

(Instead of the coefficients $1, 2, 3, \dots, n$ we can use any sequence of n distinct real numbers.) By construction, every element of G permutes these constraints, hence G is contained in the symmetry group of Λ.

Conversely, every symmetry h of Λ maps the constraint (or half-space) $\sum_j j x_j \leq \beta$ to $\sum_j j x_{j^g} \leq \beta$ for some $g \in G$. Since we have chosen distinct coefficients, this implies that $h = g^{-1} \in G$. $\boxed{QED}$

4 Example

The following example illustrates the differences between the sparse inequality systems that originate from graphs and the dense systems constructed in the proof of Theorem 3. In the example, β is an arbitrary real number.

The *dihedral group D_{2k} of order* $2k$ is the automorphism group of the undirected cycle (in the sense of graph theory) of length k, hence the proof of Theorem 1 produces the following system of inequalities for D_{2k}:

$$x_i + x_{i+1} \leq \beta \quad \text{for } 1 \leq i < 2k, \quad \text{and } x_{2k} + x_1 \leq \beta.$$

D_{2k} has the presentation $\langle a, b \mid a^k = b^2 = 1, bab = a^{-1} \rangle$ in terms of generators and relations. The corresponding Cayley graph has $2k$ vertices and $4k$ directed edges, with two colours corresponding to a and b. Since b is an involution, we let the two edges (g, bg) and (bg, g) produce only one inequality, namely $x_g + x_{bg} \leq \beta_2$; with this modification, Theorem 2 or its Corollary lead to a sparse system of $3k$ inequalities in $2k$ variables for D_{2k}.

The two permutations $i \mapsto i + 1 \mod k$ and $i \mapsto -i \mod k$ generate a subgroup of S_k isomorphic to D_{2k}. The proof of Theorem 3, applied to this permutation representation of

D_{2k}, leads to a dense system of $2k$ inequalities in k variables for D_{2k}. For $k = 4$ we obtain the following system for D_8:

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + 4x_4 &\leq \beta & \qquad x_1 + 4x_2 + 3x_3 + 2x_4 &\leq \beta \\
2x_1 + 3x_2 + 4x_3 + x_4 &\leq \beta & \qquad 2x_1 + x_2 + 4x_3 + 3x_4 &\leq \beta \\
3x_1 + 4x_2 + x_3 + 2x_4 &\leq \beta & \qquad 3x_1 + 2x_2 + x_3 + 4x_4 &\leq \beta \\
4x_1 + x_2 + 2x_3 + 3x_4 &\leq \beta & \qquad 4x_1 + 3x_2 + 2x_3 + x_4 &\leq \beta
\end{aligned}
$$

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