Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **30** (2010) n. 1, 93–105. doi:10.1285/i15900932v30n1p93

Constructive definition of central collineations in projective Desargues planes

Artur Bergmann
Department of Mathematics, University of Düsseldorf, Germany
wisbauer@math.uni-duesseldorf.de

Received: 09.07.2009; accepted: 29.09.2009.

Abstract. In any projective incidence plane satisfying the Desargues Theorem, we introduce central collineations in a constructive way and show that these coincide with those defined in the usual way.

Keywords: projective Desargues plane, central collineation

MSC 2000 classification: 51A30, 51E15

1 (Z, a)-quadrangles

The construction of a bijective point mapping ((Z,a)-mapping) inducing a collineation employs suitable quadrangles ((Z,a)-quadrangles). The properties of these quadrangles imply the properties of the (Z,a)-collineations. The disadvantage of this access being more involved is compensated by the fact that choosing this way directly provides the possible central collineations.

In the projective minimal plane the results can be proved directly without any problem. For this some hints are given in the corresponding chapters.

Let us agree about the following notation and terminology.

Definition 1. (1) *Points* are denoted by capital letters (P, Q, ...) while small letters (g, h, ...) are used for *lines*.

(2) For a point P and a line g we write $P \mid g$ if P lies on g, and $P \nmid g$ if P does not lie on g.

(3) The connecting line of two points $P \neq Q$ is denoted by $g(P, Q)$.

(4) Lines are called *Z-perspective* if they go through the point Z.

(5) Two lines $g \neq h$ are said to be a-perspective if their intersection point lies on the line a.

Definition 2 (Central collineation)**.** In any projective incidence plane, a collineation κ (i.e. bijections of the set of points and the set of lines, respectively, preserving incidences) is called *central* with center Z provided Z is linewise fixed (i.e. all lines through Z are fixlines).

In this case there exists a line a, called *axis*, staying pointwise fixed ([1], [2]). For collineations different from the identity the center and the axis are uniquely determined.

Definition 3 ($D(Z, a)$-configuration (see figure 1 for the case $Z \nmid a$))**.** Given a point Z and a line g, the data

$$(Z, a, g_1, g_2, g_3, (P_1, P_2, P_3), (Q_1, Q_2, Q_3))$$

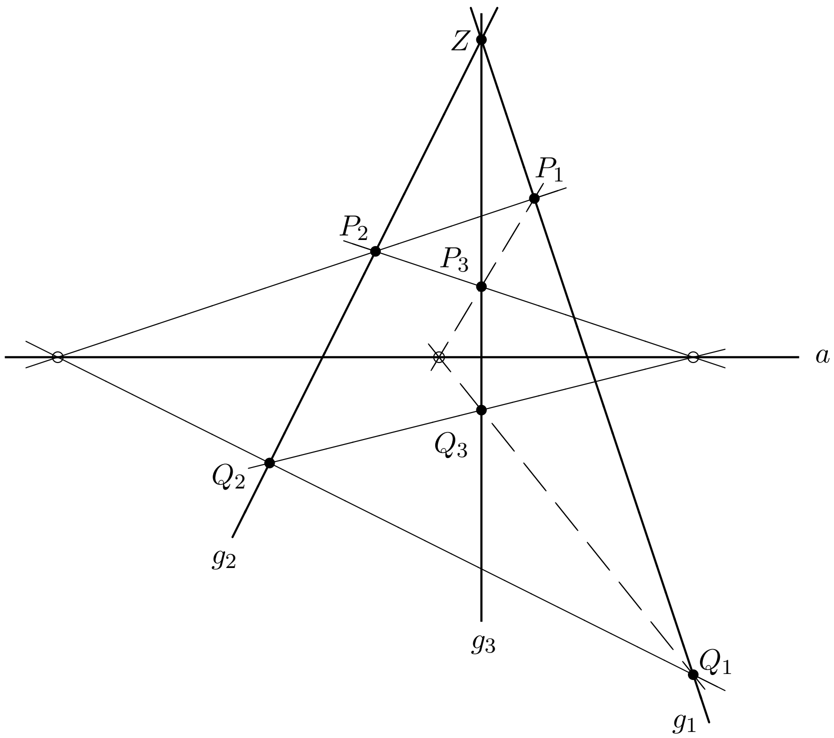

Figure 1

are called a $D(Z,a)$-*configuration* if

(i) g_1, g_2, g_3 are pairwise distinct lines going through Z;

(ii) for the points P_i, Q_i, $i \leq 3$,

   (a) $P_i, Q_i \neq Z, \nmid a$, for $i \leq 3$,

   (b) $P_i, Q_i \mid g_i$, for $i \leq 3$,

   (c) $P_i \neq Q_i$, for $i \leq 3$,

   (d) neither the points P_i nor the Q_i are collinear;

(iii) $g(P_1, P_2)$ and $g(Q_1, Q_2)$ as well as $g(P_2, P_3)$ and $g(Q_2, Q_3)$ are a-perspective.

Z is called the *center* and a the *axis* of the $D(Z,a)$-configuration.

Definition 4. Special Desargues' Postulate $D(Z,a)$**.** Given a point Z and a line a, the *special Desargues' Postulate* says:

In any $D(Z,a)$*-configuration,* $g(P_1, P_3)$ *and* $g(Q_1, Q_3)$ *are a-perspective.*

Definition 5. Desargues' Postulate (D)**.** *Desargues' Postulate* in an incidence plane, also called *Desargues' Theorem*, requires that the postulate $D(Z,a)$ holds for any center Z and any axis a in the plane.

Definition 6. A projective incidence plane in which the Special Desargues' Postulate $D(Z,a)$ holds is said to be a $D(Z,a)$-*plane* and if Desargues' Postulate holds it is said to be a (D)-*plane*.

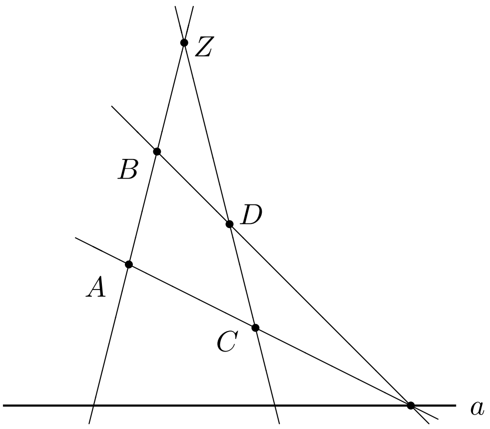

Figure 1 a $(Z \nmid a)$

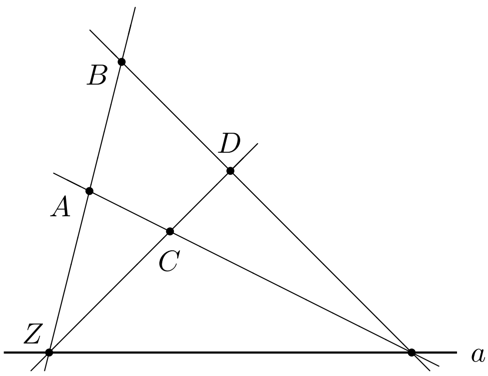

Figure 1 b $(Z \mid a)$

Unless specified otherwise, in what follows we will only consider $D(Z,a)$-planes and (D)-planes.

2 (Z,a)-quadrangles

Definition 7 ((Z,a)-quadrangles)**.** Let Z be a point and a a line in a projective incidence plane. A quadruple (A,B,C,D) of points with $A,B,C,D \neq Z, \nmid a$ is called

(a) a *proper* (Z,a)*-quadrangle* provided (see Figure 1a, 1b)

  (i) $A \neq B$,

  (ii) A,B,C,D are not collinear,

  (iii) A,B,Z and C,D,Z are collinear (that is, $g(A,B)$ and the line on which C and D are lying are Z-perspective),

  (iv) $g(A,C)$ and $g(B,D)$ are a-perspective (intersect on a);

(b) an *improper* (Z,a)*-quadrangle* provided (see Figure 2a, 2b)

  (i) $A \neq B$,

  (ii) A,B,C,D are collinear,

  (iii) there is a pair of points (U,V) such that (U,V,A,B) as well as (U,V,C,D) are proper (Z,a)-quadrangles;

(c) a *degenerated* (Z,a)*-quadrangle* provided if $A = B$ and $C = D$;

(d) a (Z,a)*-quadrangle* if (A,B,C,D) is either a proper, an improper or a degenerated (Z,a)-quadrangle.

Remark 1. (1) The definition of (Z,a)-quadrangles is given for arbitrary projective incidence planes. In what follows we will assume $D(Z,a)$ since important properties only hold in $D(Z,a)$-planes (e.g., the independence from the pair (U,V) of auxiliary points in the definition of improper (Z,a)-quadrangles).

(2) Z may lie on a.

(3) It follows from the definition of proper (Z,a)-quadrangles that $C \neq D$. Hence condition (a.iii) can be written as

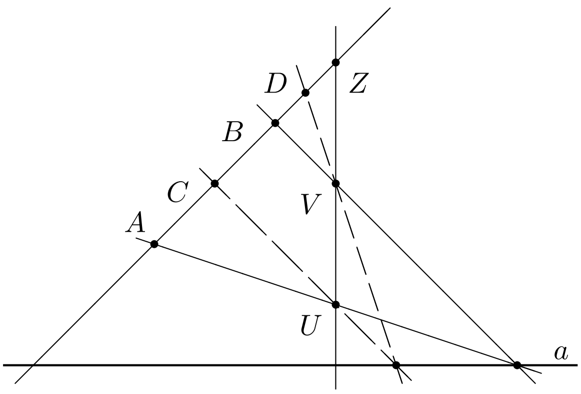

Figure 2 a $(Z \nmid a)$

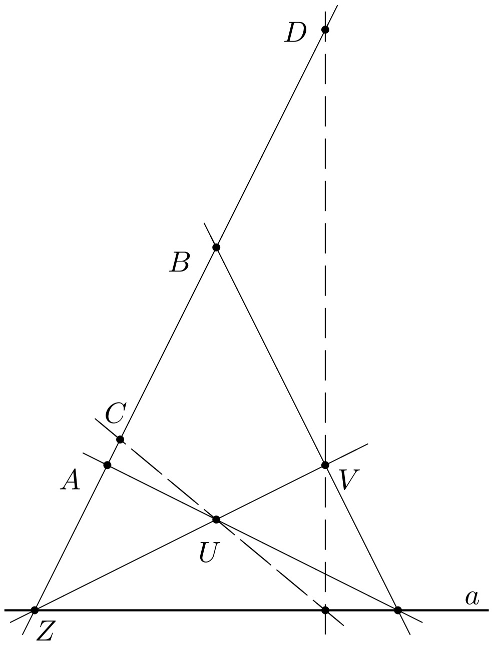

Figure 2 b $(Z \mid a)$

$g(A,B)$ and $g(C,D)$ are Z-perspective.

(4) The definition of improper (Z,a)-quadrangles (A,B,C,D) refers to proper (Z,a)-quadrangles (U,V,C,D). Hence $C \neq D$ by Remark (3).

(5) In the minimal plane all (Z,a)-quadrangles can be listed explicitly and hence the results can be checked for those. Note that for $Z \nmid a$ there are neither proper nor improper but only degenerated (Z,a)-quadrangles, since for any line through Z there is only one point not lying on the axis. If, in what follows, hints will be given to proofs for (Z,a)-quadrangles in the minimal plane, we will always assume $Z|a$.

Theorem 1 (Symmetry)**.** *If (A,B,C,D) is a (Z,a)-quadrangle then so are (C,D,A,B), (B,A,D,C) and (D,C,B,A).*

This follows - for any projective incidence plane - immediately from the definition.

Theorem 2 (Transitivity/Composition)**.** *If (A,B,C,D) and (C,D,E,F) are (Z,a)-quadrangles in a $D(Z,a)$-plane, then so is (A,B,E,F).*

Reformulation by Theorem 1:

If (A,B,C,D) and (A,B,E,F) are (Z,a)-quadrangles in a $D(Z,a)$-plane, then so is (C,D,E,F).

The proof is based on the following three lemmas.

Lemma 1. *If (A,B,C,D) and (C,D,E,F) are proper (Z,a)-quadrangles, then (A,B,E,F) is a (Z,a)-quadrangle. It is proper if A,B,E,F are not collinear and it is improper if A,B,E,F are collinear.*

Proof. 1. Assume A,B,E,F to be collinear.

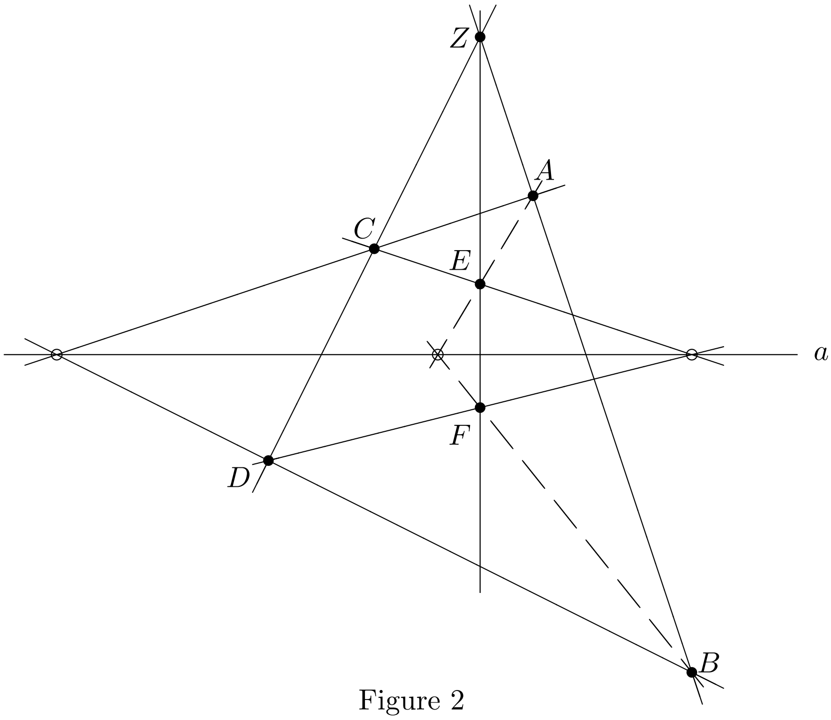

Figure 2

By assumption (C, D, E, F) and (A, B, C, D), and by Theorem 1 (C, D, A, B), are proper (Z, a)-quadrangle. Then, by definition, (A, B, E, F) is an improper (Z, a)-quadrangle.

2. Now let A, B, E, F be not collinear. (In the minimal plane there are only two lines through $Z|a$ distinct from a, hence this cannot happen.)

Then $A \neq E$ and $B \neq F$ and by assumption, the lines $g(A, B)$, $g(C, D)$ and $g(E, F)$ through Z are pairwise distinct and the points A, B, C, D, E, F are distinct from Z and are not lying on a.

Firstly assume that (A, C, E) and (B, D, F) are not collinear (see Figure 2).

We claim that the data

$$(Z,\ a,\ g(A, B),\ g(C, D),\ g(E, F),\ (A, C, E),\ (B, D, F))$$

form a $D(Z, a)$-configuration.

Condition (i) is satisfied for the Z-perspective lines. For the points (A, C, E), respectively (B, D, F), the conditions (ii.a) - (ii.c) are satisfied by the properties of (Z, a)-quadrangles.

Since, by assumption, we have proper (Z, a)-quadrangles, $g(A, C)$ and $g(B, D)$ as well as $g(C, E)$ and $g(D, F)$ are a-perspective showing that condition (iii) for $D(Z, a)$-configurations is satisfied. Postulate $D(Z, a)$ implies that $g(A, E)$ and $g(B, F)$ are a-perspective. Thus (A, B, E, F) is a proper (Z, a)-quadrangle.

Now consider the case when (A, C, E) or (B, D, F) are collinear. Since the conditions (i), (ii)(a)-(c), and (iii) for $D(Z, a)$-configurations are satisfied we conclude that $g(A, E) = g(A, C)$ and $g(B, F) = g(B, D)$ are a-perspective. QED

Lemma 2. *Let (A, B, C, D) and (C, D, E, F) be two (Z, a)-quadrangles with one of them proper and the other improper. Then (A, B, E, F) is a proper (Z, a)-quadrangle.*

Proof. Without restriction, let (A,B,C,D) be an improper and (C,D,E,F) a proper (Z,a)-quadrangle. Under these assumptions there is a pair of points (U,V) such that (U,V,A,B) as well as (U,V,C,D) are proper (Z,a)-quadrangles.

Case 1. U,V,E,F are not collinear. (Not possible in the minmal plane, see proof of 1,(2)).

Then (U,V,E,F) is a proper (Z,a)-quadrangle by Lemma 1. Since A,B,C,D are collinear and C,D,E,F are not, it follows that A,B,E,F are not collinear. Since (U,V,A,B) is a proper (Z,a)-quadrangle, the same holds for (A,B,U,V) by Theorem 1. Hence Lemma 1 implies that (A,B,E,F) is a proper (Z,a)-quadrangle.

Case 2. U,V,E,F are collinear.

The lines $g(A,B)=g(C,D)$ and $g(U,V)=g(E,F)$ are Z-perspective since (A,B,U,V) is a proper (Z,a)-quadrangle. We choose a point $U' \neq Z$ which lies neither on a nor on $g(A,B)$ nor on $g(E,F)$ and construct a point V' such that (A,B,U',V') form a proper (Z,a)-quadrangle. For this let S be the intersection of $g(A,U')$ and a. Then the intersection of $g(B,S)$ with $g(Z,U')$ yields such a point V'. Since (U,V,A,B) and (A,B,U',V') are proper (Z,a)-quadrangles, the same holds for (U,V,U',V') by Lemma 1. By construction, C,D,U',V' are not collinear and (U,V,C,D) - hence, by Theorem 1, also (C,D,U,V) - are proper (Z,a)-quadrangles. Now Lemma 1 implies that (C,D,U',V') is a proper (Z,a)-quadrangle and, by Theorem 1, the same holds for (U',V',C,D).

Replacing (U,V) by (U',V'), the conditions of Case 1 are satisfied showing that (A,B,E,F) is a proper (Z,a)-quadrangle.

In the minimal plane, the choice of such an auxiliary point U' is not possible; here we can refer to the complete list of (Z,a)-quadrangles for the proof. QED

Lemma 3. *Let (A,B,C,D) and (C,D,E,F) be improper (Z,a)-quadrangles. Then (A,B,E,F) is also an improper (Z,a)-quadrangle.*

Proof. By the definition of improper (Z,a)-quadrangles, there exists a pair of points (U,V) such that (U,V,A,B) and - by Theorem 1 - also (A,B,U,V) and (U,V,C,D) are proper (Z,a)-quadrangles. Then, by Lemma 2, (U,V,E,F) is also a proper (Z,a)-quadrangle. Now Lemma 1 implies that (A,B,E,F) is a (improper) (Z,a)-quadrangle. QED

For degenerated (Z,a)-quadrangles the composition/transitivity can only be formulated with degenerated (Z,a)-quadrangles and hence the assertion of Theorem 2 also holds in this case. This completes the proof of Theorem 2.

Lemma 2 now implies:

Theorem 3. *For an improper (Z,a)-quadrangle (A,B,C,D) in a $D(Z,a)$-plane, the following holds for any pair (U,V) of points: If (U,V,A,B) is a proper (Z,a)-quadrangle, then so is (U,V,C,D).*

This means that in the definition of improper (Z,a)-quadrangles, the assertion "such that (U,V,A,B) as well as (U,V,C,D)" in Definition 7(b.iii) is independent from the pair (U,V) of auxiliary points.

Proof. Since (A,B,C,D) is an improper and (U,V,A,B) a proper (Z,a)-quadrangle, it follows by Lemma 2 that (U,V,C,D) is a proper (Z,a)-quadrangle. QED

Remark 2. The independence from the choice of the auxiliary points shown in Theorem 3 is just a different formulation of the transitivity of improper and proper (Z,a)-quadrangles.

Theorem 4. *(A,B,A,X) is an improper (Z,a)-quadrangles if and only if $X=B$.*

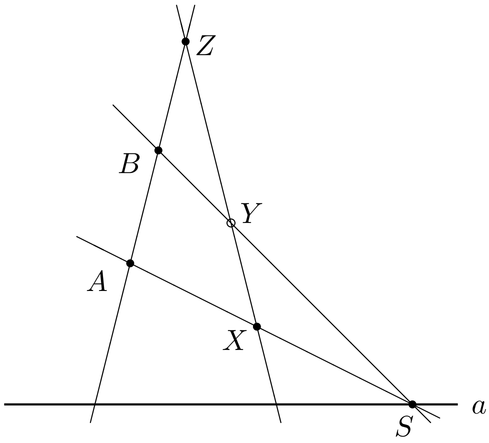

Figure 3

Proof. 1) The definition of improper (Z, a)-quadrangles requires $A \neq B$ and implies that (A, B, A, B) is an improper (Z, a)-quadrangle: Choose a point $U \neq Z, \nmid a, g(A, B)$ and construct a point V such that (A, B, U, V) - and hence (U, V, A, B) form a proper (Z, a)-quadrangle (compare the proof of Lemma 2, Case 2)

2) Now assume (A, B, A, X) to be an improper (Z, a)-quadrangle. Then, by definition, there is a pair of points (U, V) such that (U, V, A, B) as well as (U, V, A, X) are proper (Z, a)-quadrangles. Let S be the intersection of $g(U, A)$ with a. For the proper (Z, a)-quadrangle (U, V, A, B), B is the intersection of $g(S, V)$ with $g(Z, A)$. Similarly, for the proper (Z, a)-quadrangle (U, V, A, X), the point X is the intersection of $g(S, V)$ and $g(Z, A)$, that is $X = B$. QED

3 (Z, a)-mappings

The basis for the definition of (Z, a)-point mappings is

Theorem 5. *In a projective $D(Z, a)$-plane, let Z be a point, a a line, and (A, B) points distinct from Z and not on a, but $Z \mid g(A, B)$.*

For every point $X \neq Z$, $X \nmid a$, there is a unique point Y extending the triple (A, B, X) to a (Z, a)-quadrangle (A, B, X, Y) which is

- *proper if $A \neq B$ and A, B, X are not collinear,*
- *improper if $A \neq B$ and A, B, X are collinear, and*
- *degenerateed if $A = B$.*

Hereby, $Y \neq Z, \nmid a$, and $Y = X$ if and only if $A = B$.

Proof. If $A = B$, then $Y = X$ is the only point extending (A, B, X) to a (degenerated) (Z, a)-quadrangle.

Hence in what follows we assume $A \neq B$.

Case 1. Let A, B, X not be collinear (see Figure 3).

As in the proof of Lemma 2 (Case 2), let S be the intersection of $g(A, X)$ and a, and Y the intersection of $g(S, B)$ with $g(Z, X)$ $(\neq g(Z, A))$. Then (A, B, X, Y) is a proper (Z, a)-quadrangle. For a proper (Z, a)-quadrangle (A, B, X, Y'), Y' is by definition ($g(A, X)$ and $g(B, Y')$

are a-perspective) the intersection of $g(S,B)$ and $g(Z,X)$. In any projective incidence plane this intersection is unique. Hence for the present case (A,B,X not collinear), existence and uniqueness of the forth point leading to a (proper) (Z,a)-quadrangle (A,B,X,Y) is shown for arbitrary projective incidence planes.

Case 2. A,B,X are collinear.

Existence: Choose a point $U \neq Z, \nmid a, g(A,B)$, i.e. not collinear with A,B. By Case 1, there exists a unique point V such that (A,B,U,V) and hence also (U,V,A,B) are proper (Z,a)-quadrangles. Again by Case 1, for (U,V,X) there exists a unique point Y yielding a proper (Z,a)-quadrangle (U,V,X,Y). Hence, by definition, (A,B,X,Y) is an improper (Z,a)-quadrangle.

Uniqueness. Besides (A,B,X,Y) let (A,B,X,Y') be an improper (Z,a)-quadrangle, where Y' is constructed similarly to Y but using a distinct pair (U',V'). Then, by the transitivity property (Theorem 2), (X,Y,X,Y') is an improper (Z,a)-quadrangle. Now Theorem 4 implies $Y'=Y$.

Remarks. This proof of uniqueness holds for the first as well as for the second case. However, while in Case 1, in any projective incidence plane, uniqueness follows directly by the construction, in Case 2 we have to exhibit a relation between the improper (Z,a)-quadrangles (A,B,X,Y) and (A,B,X,Y') constructed independently employing two pairs of points (U,V) and (U',V'). This was performed by applying the transitivity property (Theorem 2) which only could be shown by using the Postulate $D(Z,A)$. Seen differently, the coincidence of Y and Y' expresses the independence from the choice of auxiliary points in the definition of improper (Z,a)-quadrangles. This, in turn, means the transitivity of proper and improper (Z,a)-quadrangles (Lemma 2), whose proof in non-minimal projective incidence planes needs the Postulate $D(Z,a)$. Compare also the reformulation of the $D(Z,a)$-configurations and of the Postulate $D(Z,a)$ by referring to (Z,a)-quadrangles before Theorem 14.

The assertion $Y \neq Z, \nmid a$ follows from the definion of the (Z,a)-quadrangles, and so does the assertion that $Y = X$ if and only if $A = B$. In this case the (Z,a)-quadrangles are degenerated. QED

Definition 8. Let Z be a point and a a line in a projective $D(Z,a)$-plane. Let A and B be points satisfying

$$A,B \neq Z; \quad A,B \nmid a; \quad A,B,Z \text{ are collinear.}$$

(a) Let $\kappa_{AB}^{Z,a}$ denote the point mapping
   (i) for $X \neq Z, \nmid a$, let $\kappa_{AB}^{Z,a}(X)$ be the unique point (by Theorem 5) making $(A,B,X,\kappa_{AB}^{Z,a}(X))$ a (Z,a)-quadrangle;
   (ii) $\kappa_{AB}^{Z,a}(Z) := Z$;
   (iii) for any point $X \mid a$ put $\kappa_{AB}^{Z,a}(X) := X$.

   a is called the *axis* and Z the *centre* of $\kappa_{AB}^{Z,a}$.

(b) For any point Z and line a, a point mapping κ is said to be a (Z,a)-*mapping* provided there exists a pair of points (A,B) such that $\kappa = \kappa_{AB}^{Z,a}$, called a *presentation of* κ.

(c) The set of all (Z,a)-mappings with fixed axis a and centre Z will be denoted by $\mathcal{Z}(Z,a)$.

Remark 3. Let the conditions on a,Z,A,B be as in the definition above.

(1) $\kappa_{AB}^{Z,a}$ maps the points on a line g through the center Z to points on the same line g (Z stays linewise fixed). The points on the axis a are fixpoints (a stays pointwise fixed).

(2) $\kappa_{AB}^{Z,a}(A) = B$ (follows by Theorem 4).

(3) $\kappa_{AB}^{Z,a} = id$ if and only if $A = B$ (compare Theorem 5).

(4) For any point $X \neq Z, \nmid a$, $\kappa_{AB}^{Z,a}(X) \neq Z$, $\kappa_{AB}^{Z,a}(X) \nmid a$ (by Theorem 5).

(5) If $A \neq B$, then Z and the points on a are the only fixpoints (by Theorem 5).

(6) In the minimal plane, where all (Z,a)-quadrangles are known, Theorem 5 can be verified directly and one obtains the following (Z,a)-mappings:

(i) for $Z \nmid a$, only the identity map;

(ii) for $Z \mid a$ there are precisely two maps: the identity map and the map leaving Z linewise and a pointwise fixed and interchanging the two points on each of the two lines through Z different from the axis. These maps preserve collinearity.

Theorem 6. *Let Z be a point and a a line in a projective $D(Z,a)$-plane.*

(a) *For any points $A,B,C,D \neq Z, \nmid a$ with A,B,Z and C,D,Z collinear,*

$$\kappa_{AB}^{Z,a} = \kappa_{CD}^{Z,a} \quad \text{if and only if } (A,B,C,D) \text{ is a } (Z,a)\text{-quadrangle.}$$

(b) *If κ is a (Z,a)-mapping, then for any point $P \neq Z, \nmid a$,*

$$\kappa = \kappa_{P\,\kappa P}^{Z,a},$$

hence, for a presentation of a (Z,a)-mapping, any point $P \neq Z, \nmid a$ can be chosen as a first point.

(c) *If two (Z,a)-mappings produce the same image for some point $P \neq Z, \nmid a$, then they coincide.*

(d) *If κ is a (Z,a)-mapping, then for any points $U,V \neq Z, \nmid a$, $(U,\kappa U,V,\kappa V)$ is a (Z,a)-quadrangle.*
Hence the lines $g(U,\kappa U)$ and $g(V,\kappa V)$ are Z-perspective, and for $U \neq V$ and Z,U,V not collinear, the lines $g(U,V)$ and $g(\kappa U,\kappa V)$ are a-perspective.

Proof. (a) $\Rightarrow$ By Remark 3(2), $\kappa_{AB}^{Z,a}(C) = \kappa_{CD}^{Z,a}(C) = D$. Hence (A,B,C,D) is, by definition of $\kappa_{AB}^{Z,a}$, a (Z,a)-quadrangle.

$\Leftarrow$ For $X = Z$ or $X \mid a$, $\kappa_{AB}^{Z,a}(X) = X = \kappa_{CD}^{Z,a}(X)$.

For $X \neq Z, \nmid a$, $(A,B,X,\kappa_{AB}^{Z,a}(X))$ and $(C,D,X,\kappa_{CD}^{Z,a}(X))$ are (Z,a)-quadrangles. By the assumption that (A,B,C,D) is a (Z,a)-quadrangle, it follows by transitivity (Theorem 2) that $(A,B,X,\kappa_{CD}^{Z,a}(X))$ is also a (Z,a)-quadrangle. Now Theorem 5 implies $\kappa_{CD}^{Z,a}(X) = \kappa_{AB}^{Z,a}(X)$.

(b) If $\kappa = \kappa_{AB}^{Z,a}$ is a (Z,a)-mapping, then for $P \neq Z, \nmid a$, $(A,B,P,\kappa P)$ is a (Z,a)-quadrangle. Now by (a), $\kappa = \kappa_{AB}^{Z,a} = \kappa_{P\,\kappa P}^{Z,a}$.

(c) Let κ, κ' be (Z,a)-mappings and $S \neq Z, \nmid a$ a point with $\kappa(S) = \kappa'(S) =: T$. Then by (b), $\kappa = \kappa_{ST}^{Z,a} = \kappa'^{Z,a}_{ST} = \kappa'$.

(d) By (b), $\kappa = \kappa_{U\,\kappa U}^{Z,a} = \kappa_{V\,\kappa V}^{Z,a}$. Now it follows from (a) that $(U,\kappa U,V,\kappa V)$ is a (Z,a)-quadrangle. The remaining assertions follow from the definition of (Z,a)-quadrangles. QED

Remark. Assertion (d) is a direct consequence of the transitivity properties of (Z,a)-quadrangles. For its proof the parts (a), (b), (c) are not needed. They could be derived from (d) which also gives a good idea about the action of (Z,a)-mappings.

Theorem 7. *Every (Z,a)-mapping in a projective $D(Z,a)$-plane is bijective and its inverse is again a (Z,a)-mapping. More precisely, for all $A,B \neq Z, \nmid a$,*

$$(\kappa_{AB}^{Z,a})^{-1} = \kappa_{BA}^{Z,a}.$$

Proof. 1) Let $X \neq Z, \nmid a$. For $Y := \kappa_{AB}^{Z,a}(X)$, (A,B,X,Y) is a (Z,a)-quadrangle by definition. Hereby $Y \neq Z, \nmid a$ (Theorem 5). By Theorem 1, (B,A,Y,X) is a (Z,a)-quadrangle with $\kappa_{BA}^{Z,a}(Y) = X$.

2) For $X = Z$ or $X \mid a$ we have $\kappa_{AB}^{Z,a}(X) = X = \kappa_{BA}^{Z,a}(X)$. QED

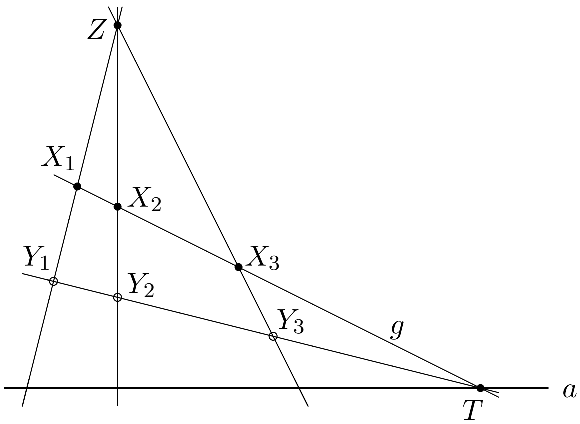

Figure 4

4 (Z,a)-collineations

In projective planes the point mapping ψ of a collineation is characterized by the fact that it is bijective and that ψ as well as ψ^{-1} preserve collinearity ([2]). The conditions can be weakened but this is not needed for what is following. By Theorem 7 every (Z,a)-mapping is bijective. Now we show:

Lemma 4. *Let κ be a (Z,a)-mapping in a projective $D(Z,a)$-plane. Then for any collinear points X_1, X_2, X_3, the images $\kappa X_1, \kappa X_2, \kappa X_3$ are also collinear and vice versa.*

Proof. $\Rightarrow$ Without restriction, assume that $\kappa \neq \mathrm{id}$ and the points X_1, X_2, X_3 are pairwise distinct.

Case 1. X_1, X_2, X_3 lie on a line g passing through Z. By Remark 3(1), the image points also lie on g. This includes the cases that one of the X_1, X_2, X_3 is equal to Z or is lying on a.

Case 2. X_1, X_2, X_3 lie on a. Then they are fixpoints and the images also lie on a.

Case 3. X_1, X_2, X_3 lie on a line $g \neq a$ and not going through Z. Then at least two of these points are not on a. Without restriction assume $X_1, X_2 \nmid a$. By Theorem 6(b), κ can be represented as $\kappa^{Z,a}_{X_1,Y_1}$ for $Y_1 := \kappa(X_1)$. Then $Y_2 := \kappa X_2 = \kappa^{Z,a}_{X_1,Y_1}(X_2)$ is the intersection of $g(Z, X_2)$ and $g(Y_1, T)$ where T denotes the intersection of $g(X_1, X_2)$ with a. For $X_3 \nmid a$, the point $Y_3 := \kappa Y_3 = \kappa^{Z,a}_{X_1,Y_1}(X_3)$ also lies on $g(Y_1, T)$. For $X_3 \mid a$, we have $X_3 = T$ and thus $Y_3 = \kappa(X_3) = T$.

The converse conclusion follows by the fact that, by Theorem 7, κ^{-1} is again a (Z,a)-mapping. QED

By Lemma 4, together with the hint to the literature before this proposition, we obtain:

Theorem 8. *In a projective $D(Z,a)$-plane, any (Z,a)-mapping κ induces a collineation. The corresponding line mapping is given by*

$$g(P,Q) \longmapsto g(\kappa(P), \kappa(Q)).$$

From now on we will always consider (Z,a)-mappings as collineation and hence we will call them (Z,a)-*collineations*.

Corollary 1. *In any projective $D(Z,a)$-plane the following hold:*

(1) *Every (Z,a)-collineation is a central collineation (Definition 2).*

(2) *For any (Z,a)-collineation $\kappa \neq \mathrm{id}$, the points on the axis a together with $\{Z\}$ form the set of all fixpoints of κ, and the lines through Z together with the axis a is the set of all fixlines.*

Proof. (1) follows from Remark 3(1).

(2) The assertion about the fixpoints follwos by Theorem 5 and the definition of (Z,a)-mappings.

For any point $X \neq Z, \nmid a$, the points $Z, X, \kappa_{AB}^{Z,a}(X)$ are collinear. Hence any line through Z is a fixline. On the other hand, if $g \neq a$ is a fixline of $\kappa_{AB}^{Z,a}$ and $X \neq Z, \nmid a$ a point on g, then $g(X, \kappa_{AB}^{Z,a}(X)) = g$, that is, g is a line passing through Z. QED

5 Central collineations and (Z,a)-collineations

Theorem 9. *Given a point Z and a line a in a $D(Z,a)$-plane, then for any $A, B \neq Z, \nmid a$ with A, B, Z collinear there exists exactly one (Z,a)-collineation κ with $\kappa A = B$.*

Proof. By definition, $\kappa = \kappa_{AB}^{Z,a}$ is such a (Z,a)-collineation and it is unique by Theorem 6(c). QED

Theorem 10. *In (D)-planes, any central collineation (Definition 2) is a (Z,a)-collineation and vice versa.*

Proof. It is known from the literatur (e.g. [2]) that for points $A, B \neq Z, \nmid a$, there is at most one central collineation κ with center Z and axis a transferring A in B. The (Z,a)-collineation $\kappa_{AB}^{Z,a}$ is of this type, hence $\kappa = \kappa_{AB}^{Z,a}$.

On the other hand, by Corollary 1, $\kappa_{AB}^{Z,a}$ is such a central collineation. QED

As a consequence of this theorem, in (D)-planes we need not distinguish between central collineations and (Z,a)-collineations.

Theorem 11. *In $D(Z,a)$-planes, the group $\mathcal{Z}(Z,a)$ of central collineations with center Z and axis a is linear transitive, that is, for any $A, B \neq Z, \nmid a$ with A, B, Z collinear, there is precisely one central collineation with center Z and axis a.*

Proof. This follows from the Theorems 9 and 10. QED

Theorem 12. *In a projective incidence plane, let κ be a central collineation with centre Z and axis a. Then for any points $U, V \neq Z, \nmid a$, the quadruple $(U, \kappa U, V, \kappa V)$ is a (Z,a)-quadrangle.*

Proof. For such a central collineation κ, Z is pointwise fixed by definition. Hence the image points κU of a point $U \neq Z, \nmid a$ lies on the line $g(Z,U)$, such that $U, \kappa U, Z$ are collinear. The same holds for $V \neq Z, \nmid a$.

For $\kappa = \mathrm{id}$ the assertion is true. So assume $\kappa \neq \mathrm{id}$.

Case 1. Let U, V be distinct points with $U, V \neq Z, \nmid a$ and U, V, Z not collinear. Then $\kappa U \neq U$ and $\kappa V \neq V$ (since κ is a central collineation $\neq \mathrm{id}$ [1]) and by bijectivity of κ also $\kappa U \neq \kappa V$. Let $S = S(g(U,V), a)$ be the intersection of $g(U,V)$ and a. The line $g(U,V)$ is mapped to the line $g(\kappa U, \kappa V)$. Lying on a, S is a fixpoint. Hence $\kappa\, g(U,V) = \kappa\, g(U,S) =$

[1] for such central collineations only Z and points on a are fixpoints

$g(\kappa U, \kappa S) = g(\kappa U, S)$. Thus the lines $g(U,V)$ and $\kappa\, g(U,V)$ are a-perspective. This implies that $(U, \kappa U, V, \kappa V)$ is a proper (Z,a)-quadrangle.

Case 2. Let $U, V \neq Z, \nmid a$ with U, V, Z collinear. Choose a point $X \neq Z, \nmid a$ such that U, X, Z and hence V, X, Z are not collinear. By Case 1, we know that $(X, \kappa X, U, \kappa U)$ as well as $(X, \kappa X, V, \kappa V)$ are proper (Z,a)-quadrangles. Hence by definition $(U, \kappa U, V, \kappa V)$ is an improper (Z,a)-quadrangle. QED

Remark. By this theorem, in any projective incidence plane, the image points of a central collineation can be constructed, starting with a pair $(U, \kappa U)$, in the same way as for (Z,a)-mappings.

6 $\mathcal{Z}(Z,a)$ and $D(Z,a)$

A projective incidence plane, in which the group $\mathcal{Z}(Z,a)$ of all central collineations with center Z and axis a is linear transitive (see Theorem 11), is a $D(Z,a)$-plane. Before proving this we give a formulation for $D(Z,a)$-configurations in terms of (Z,a)-quadrangles.

The definition of a $D(Z,a)$-configuration (Definition 3)

$$(Z, a, g_1, g_2, g_3, (P_1, P_2, P_3), (Q_1, Q_2, Q_3))$$

implies that (P_1, Q_1, P_2, Q_2) and (P_2, Q_2, P_3, Q_3) are proper (Z,a)-quadrangles. In view of this we show:

Theorem 13. *Given a point Z and a line g, the data (see Figure 1)*

$$(Z, a, g_1, g_2, g_3, (P_1, Q_1, P_2, Q_2), (P_2, Q_2, P_3, Q_3))$$

are a $D(Z,a)$-configuration *if and only if the following hold:*

(i) *g_1, g_2, g_3 are pairwise distinct lines going through Z;*

(ii') (a) *the points P_i, Q_i satisfy $P_i, Q_i \mid g_i$, for $i \leq 3$;*

(b) *neither P_1, P_2, P_3 nor Q_1, Q_2, Q_3 are collinear;*

(iii') *(P_1, Q_1, P_2, Q_2) and (P_2, Q_2, P_3, Q_3) are proper (Z,a)-quadrangles.*

Proof. (i) is the same as in Definition 3.

(ii), (a), (b) and (c) follow from the definition of proper (Z,a)-quadrangles.

(ii)(d) corresponds to (ii')(b).

(iii) follows from the definition of proper (Z,a)-quadrangles.

The converse implications are obtained in a similar way. QED

With the characterisation of $D(Z,a)$-configurations in Theorem 13, the Postulate $D(Z,a)$ now has the form:

In any $D(Z,a)$-configuration the quadrupel (P_1, Q_1, P_3, Q_3) is a proper (Z,a)-quadrangle.

That is, the Postulate $D(Z,a)$ just expresses transitivity/composition of proper (Z,a)-quadrangles and vice versa.

Theorem 14 (R. Baer [1])**.** *If in a projective incidence plane the group $\mathcal{Z}(Z,a)$ of central collineations with center Z and axis a is linear transitive, then the special Desargues Postulate $D(Z,a)$ holds.*

Proof. Let $(Z, a, g_1, g_2, g_3, (P_1, Q_1, P_2, Q_2), (P_2, Q_2, P_3, Q_3))$ be a $D(Z, a)$-configuration (as in Theorem 13). By assumption, there is a central collineation $\kappa \in \mathcal{Z}(Z, a)$ with $\kappa P_1 = Q_1$. By Theorem 12, $(P_1, \kappa P_1, P_2, \kappa P_2) = (P_1, Q_1, P_2, \kappa P_2)$ is a proper (Z, a)-quadrangle. By Theorem 5 (Case 1, i.e. P_1, Q_1, P_2 not collinear), there is a unique point extending the triple (P_1, Q_1, P_2) to a (Z, a)-quadrangle. κP_2 is such a point. By assumption (i.e. $D(Z, a)$-configuration), (P_1, Q_1, P_2, Q_2) is also a proper (Z, a)-quadrangle. Now Theorem 5 (Case 1) implies that $Q_2 = \kappa P_2$.

Furthermore, by Theorem 12, $(P_2, \kappa P_2 = Q_2, P_3, \kappa P_3)$ is a proper (Z, a)-quadrangle and by assumption ($D(Z, a)$-configuration) (P_2, Q_2, P_3, Q_3) is of the same type. Again Theorem 5 implies that $Q_3 = \kappa P_3$.

Once more referring to Theorem 12, we know that $(P_1, \kappa P_1, P_3, \kappa P_3)$ is a proper (Z, a)-quadrangle. Since $\kappa P_1 = Q_1$ and $\kappa P_3 = Q_3$, the quadruple (P_1, Q_1, P_3, Q_3) is a proper (Z, a)-quadrangle showing that Postulate $D(Z, a)$ holds. QED

The proof shows that the given $D(Z, a)$-configuration can be fitted into the system of (Z, a)-quadrangles $\{(X, \kappa X, Y, \kappa Y) \mid X, Y \neq Z, \nmid a\}$ attached to a central collineation $\kappa \in \mathcal{Z}(Z, a)$.

References

[1] R. Baer: *Homogeneity of projective planes*, Amer. J. Math. 64, No. 1 (1942), 137-152

[2] G. Pickert: Projektive Ebenen, Grundlehren Band LXXX, Springer, Berlin e.a., 2. edition (1975)