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Bol planes of orders 3^4 and 3^6

Norman L. Johnson
Mathematics Dept., University of Iowa,
Iowa City, Iowa 52242, USA
njohnson@math.uiowa.edu

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Abstract. Finite Bol planes of order not 3^4 or 3^6 are known to be nearfield planes. This article resolves the remaining cases; Finite Bol planes are nearfield.

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1 Introduction

A Bol translation plane is a translation plane with two distinguished components L and M such that for any other component N, there is an involutory collineation σ_N which fixes N pointwise and interchanges L and M. For example, every nearfield plane is a Bol translation plane, and it has been conjectured that these are the only possibilities. On the other hand, Burn [3] has shown that there are infinite Bol planes that are not nearfield planes.

Bol planes are connected to flocks of hyperbolic quadrics in the following way: Given a flock of a hyperbolic quadric in $PG(3,q)$, Thas [13] shows that there is a set of involutions that act on the flock. Using the Thas-Walker construction, Bader and Lunardon [1] realized that there are associated Bol translation planes. Of course, these Bol spreads are in $PG(3,q)$, so one does not require complete knowledge of all Bol spreads to classify spreads arising from flocks of hyperbolic quadrics. From the translation plane point of view, a flock of a hyperbolic quadric is equivalent to a translation plane of order q^2 with spread in $PG(3,q)$, which is a union of $q+1$ reguli sharing exactly two lines (components). The associated translation plane becomes a Bol translation plane.

The famous theorem of Thas [13]–Baer–Lunardon [1] shows that the associated translation planes are all nearfield planes. This result would immediately follow from a complete classification of Bol translation planes as nearfield planes (given that one knows that the set of involutions exists by Thas [13]).

The group G generated by the involutory collineations can be either solvable or non-solvable, of course, and Kallaher and Ostrom [11], [12] have shown the solvable case can almost be resolved. The group G_L stabilizing L and M is transitive on the non-zero vectors and if the group is solvable either G_L is a subgroup of $\Gamma L(1, q^n)$ if the plane has order q^n, or the order is in $\{ 3^2, 5^2, 7^2, 11^2, 23^2, 3^4 \}$. The structure of the Bol group shows that except for these sporadic orders the Bol plane must be a generalized André plane.

1 Theorem (Kallaher and Ostrom [11], [12]). *A finite generalized André plane which is a Bol plane is a nearfield plane; a Bol plane with associated group in $A\Gamma L(1, q^n)$ is a nearfield plane.*

2 Theorem (Kallaher and Ostrom [11], [12]). *A finite solvable Bol plane is a nearfield plane or the order is in* $\{ 3^2, 5^2, 7^2, 11^2, 23^2, 3^4 \}$.

In general, Kallaher was able to resolve most of the non-solvable cases and proved

3 Theorem (Kallaher [10]). *A finite Bol translation plane is either a nearfield plane or the order is in* $\{ 5^2, 7^2, 11^2, 19^2, 23^2, 59^2, 3^4, 3^6 \}$.

More recently, Kallaher and Hanson have resolved most of these remaining cases.

4 Theorem (Kallaher and Hanson [5]). *A finite Bol plane of order in* $\{ 5^2, 7^2, 11^2, 19^2, 23^2, 59^2 \}$ *is a nearfield plane.*

Hence, the remaining two orders 3^4 and 3^6 remain in question, and the order-3^4 case is quite problematic.

However, recently, the work of Johnson and Prince [9] and of Jha and Johnson [6] proves the following:

5 Theorem (Johnson–Prince [9] and Jha–Johnson [6]). *There are exactly* 14 *translation planes of order* 81 *admitting* $SL(2, 5)$ *as a collineation group in the translation complement as listed in Johnson and Prince [9].*

It turns out that this result may be utilized to study Bol planes of order 3^4. Recently, the non-solvable triangle transitive translation planes have been classified (a triangle transitive translation plane is a translation plane admitting an autotopism collineation group which acts transitively on the non-vertex points of each leg of the triangle).

6 Theorem (Johnson [8]). *Let π be a finite non-solvable triangle transitive plane. Then π is an irregular nearfield plane of order* $11^2, 29^2$ *or* 59^2.

Indeed, the cases of order 3^4 and 3^6 must be analyzed in the triangle transitive planes in a manner similar to that of Bol planes, as Bol planes admit a group which acts transitively on two legs of an autotopism triangle. These analysis of planes of order 3^4 and 3^6 of Jha, the author and Prince may be applied for the study of Bol planes of these orders. Using these ideas, the two remaining orders are resolved to complete the study of Bol translation planes.

Our main theorem is

7 Theorem. *If π is a Bol translation plane of order 3^4 or 3^6 then π is a nearfield plane.*

8 Corollary. *Finite Bol translation planes are nearfield planes.*

2 Background

It is useful to state the classification theorem of all finite doubly transitive groups.

Let v denote the degree of the permutation group.

The possibilities are as follows:

(A) G has a simple normal subgroup N, and $N \leq G \leq \operatorname{Aut} N$ where N and v are as follows:

(1) A_v, $v \geq 5$,

(2) $PSL(d,z)$, $d \geq 2$, $v = (z^d - 1)/(z-1)$ and $(d,z) \neq (2,2),(2,3)$,

(3) $PSU(3,z)$, $v = z^3 + 1$, $z > 2$,

(4) $Sz(w)$, $v = w^2 + 1$, $w = 2^{2e+1} > 2$,

(5) ${}^2G_2(z)'$, $v = z^3 + 1$, $z = 3^{2e+1}$,

(6) $Sp(2n,2)$, $n \geq 3$, $v = 2^{2n-1} \pm 2^{n-1}$,

(7) $PSL(2,11)$, $v = 11$,

(8) Mathieu groups M_v, $v = 11, 12, 22, 23, 24$,

(9) M_{11}, $v = 12$,

(10) A_7, $v = 15$,

(11) HS (Higman-Sims group), $v = 176$,

(12) .3 (Conway's smallest group), $v = 276$.

(B) G has a regular normal subgroup N which is elementary Abelian of order $v = h^a$, where h is a prime. Identify G with a group of affine transformations $x \longmapsto x^g + c$ of $GF(h^a)$, where $g \in G_0$. Then one of the following occurs:

(1) $G \leq A\Gamma L(1,v)$,

(2) $G_0 \trianglerighteq SL(n,z)$, $z^n = h^a$,

(3) $G_0 \trianglerighteq Sp(n,z)$, $z^n = h^a$,

(4) $G_0 \trianglerighteq G_2(z)'$, $z^6 = h^a$, z even,

(5) $G_0 \trianglerighteq A_6$ or A_7, $v = 2^4$,

(6) $G_0 \trianglerighteq SL(2,3)$ or $SL(2,5)$, $v = h^2$, $h = 5, 11, 19, 23, 29$, or 59 or $v = 3^4$,

(7) G_0 has a normal extraspecial subgroup E of order 2^5 and G_0/E is isomorphic to a subgroup of S_5, where $v = 3^4$,

(8) $G_0 = SL(2,13)$, $v = 3^6$.

Given a finite Bol plane, the stabilizer of two designated components within the group generated by the set of involutions is in the translation complement and acts transitively on the non-zero vectors on each of the components. The translation groups with corresponding fixed centers relative to the components produce doubly transitive groups on the associated vector spaces. Hence, in our situation, we will only be using part (B).

We shall also require the theorem of Foulser–Johnson in the odd order case.

9 Theorem (Foulser–Johnson [4])**.** *Let π denote a translation plane of odd order q^2 that admits a group G isomorphic to $SL(2,q)$ that induces a non-trivial collineation group acting in the translation complement (G need not act faithfully, but does act non-trivially).*

Then π is one of the following planes:

(1) Desarguesian,

(2) Hall,

(3) Hering, or

(4) one of three planes of Walker of order 25.

Some of our arguments require the 'p-planar bound of Jha'.

10 Theorem (see 21.2.8 of [2]) (The p-planar bound)**.** *A quasifield of order p^n cannot admit an automorphism p-group S unless $|S| \le p^{n-1}$, and where equality holds, $n = 2$ or $p^n = 16$.*

At one point, we require knowledge of the nature of Baer groups.

11 Theorem (see, e.g., Jha and Johnson [7])**.** *Let π be a finite translation plane of order q^2 and let B be a Baer group of order prime to q. Let $\pi_0 = \mathrm{Fix}\, B$. Then in the net N_{π_0} of degree $q+1$ and order q^2 defined by the components of π_0, there is a unique Baer subplane π_0^2 of N_{π_0}, which is fixed by B, and $\pi_0^2 \neq \pi_0$.*

3 The main theorem

12 Theorem. *If π is a Bol translation plane of order 3^4 or 3^6 then π is a nearfield plane.*

The proof of the theorem shall be given as a series of lemmas. We assume the hypothesis of the statement throughout. The first lemma is well known in the quasifield formulation. Here we wish a spread argument.

13 Lemma. *The spread may be represented in the form*

$$x = 0, y = 0, y = xA; A \in S,$$

where A is a 4×4 matrix over $GF(3)$ and $x = 0, y = 0$ are the special components inverted by the involutions. We may assume that $I \in S$. The involutions may be represented in the following form:

$$(x, y) \to (yA^{-1}, xA),$$

where this element fixes $y = xA$ pointwise. The group

$$G_0 : \big\langle (x, y) \to (xA, yA^{-1}); A \in S \big\rangle$$

is transitive on each of the non-zero vectors of $x = 0$ and $y = 0$.

Proof. An involution which inverts $x = 0$ and $y = 0$ has the form $(x, y) \to (yB, xB^{-1})$. In order that $y = xA$ is fixed pointwise, it then follows that $AB = I$ and $B^{-1} = A$. Since $y = x$ is a component, we compose $(x, y) \to (y, x)$ and $(x, y) \to (yA^{-1}, xA)$ to obtain a group element $(x, y) \to (xA, yA^{-1})$ which fixes both $x = 0$ and $y = 0$. Since the group $\langle x \to xA; A \in S\rangle$ is transitive on non-zero vectors of the underlying vector space, we see that the lemma follows. $\boxed{QED}$

We also denote $\{ x = 0, y = 0 \}$ by $\{L, M\}$.

3.1 Order 3^6

Assume that the order is 3^6. Then, since we have a doubly transitive group (combine the above group with the associated translation group), it follows from the classification theorem of finite doubly transitive groups that either the group is in $A\Gamma L(1, 3^6)$ or the group is non-solvable.

14 Lemma. *If the group is solvable then the plane is a nearfield plane.*

Proof. Apply Theorem 2. $\boxed{QED}$

Hence, assume that the group is non-solvable.

15 Lemma.
(1) The stabilizer G_0 of L and M is $SL(2, 13)$.
(2) The 3-elements in G_0 are planar.
(3) The stabilizer of a non-zero vector on L has order 3.

Proof. The stabilizer G_0 is non-solvable and we must have one of the following three cases:

(2) $G_0 \trianglerighteq SL(n, z)$, $z^n = h^a$, (3) $G_0 \trianglerighteq Sp(n, z)$, $z^n = h^a$, (8) $G_0 = SL(2, 13)$, $v = 3^6$. Assume situation (2). Then $z^n = 3^6$, so $n = 2, 3, 6$ and G_0 is $SL(2, 3^6)$, $SL(3, 3^2)$ or $SL(6, 3)$. In the first of these cases, the planes are determined by Theorem 9 and in fact the plane is Desarguesian. In the second case, when G_0 is $SL(3, 9)$, we have a 3 -group of order 3^6, which is planar. This is a contradiction to the p -planar bound Theorem 10. Similarly, in the case $SL(6, 3)$ there is a planar 3-group of order 3^{15}, again a contradiction to the p -planar bound. In case (3), we have $S_p(2, 3^6)$, $S_p(3, 3^2)$ or $S_p(6, 3)$. We have dealt the the first two of these groups; in the case of $S_p(6, 3)$, the Sylow 3-group has order 3^{3^2}, violating the planar p-bound. Hence, we can only have situation (8), which is our statement (1). Since $SL(2, 13)$ has order $13 \cdot 7 \cdot 3 \cdot 8$, it follows that $SL(2, 13)$ is generated by planar 3-elements. Furthermore, the stabilizer of a non-zero point has order exactly 3, since $3^6 - 1 = 13 \cdot 7 \cdot 8$. This proves all parts of the lemma. $\boxed{QED}$

16 Lemma. *G_0 cannot be $SL(2, 13)$.*

Proof. Under our representation, consider a collineation, necessarily of the form

$$(x, y) \to (xA_1A_2 \cdots A_t, yA_1^{-1}A_2^{-1} \cdots A_t^{-1}), \text{ where } A_i \in S,\ i = 1, 2, \ldots, t.$$

Let $A_1 \cdots A_t = C$. Suppose that this collineation fixes $y = x$. Then, $(x, x) \to (xC, xC^{-1})$, which implies that $C = C^{-1}$ or rather that $C^2 = I$. This means the collineation has order 2. There is a unique involution in $SL(2, 13)$. Hence, the stabilizer of $y = x$ has order 2. Therefore, the orbit length of $y = x$ is $|SL(2, 13)|/2 = 13 \cdot 7 \cdot 3 \cdot 8/2 > 3^6 - 1 = 13 \cdot 7 \cdot 8$, a contradiction. $\boxed{QED}$

3.2 Triangle transitive planes of order 3^6

17 Remark. There is a corresponding situation in triangle transitive planes where the question is asked if there could be a non-solvable triangle transitive plane of order 3^6. The following analogous argument gives an simple proof that there are no such translation planes other than the Desarguesian plane.

Proof. Again, we may reduce to situation in lemma 15. In this setting, we have two fixed components L and M and a group G_0 which acts transitively on the non-vertex points of each side of the autotopism triangle. By similar arguments as above we reduce to the case where $G_0 \mid L$ is isomorphic to $SL(2,13)$. But $SL(2,13)$ must induce faithfully on M as well, as there can be no affine homology subgroup of $SL(2,13)$ (that is, the involution in $SL(2,13)$ also acts faithfully on L and/or M). So, G_0 must be isomorphic to $SL(3,13)$. Furthermore, the 3-elements are planar. Since the group is transitive on the non-vertex points of the infinite side, it follows that the orbit length of a component $y = x$ is exactly $13 \cdot 7 \cdot 8 = 3^6 - 1$. However, there is a unique involution in $SL(2,13)$, which must either be Baer or the kernel involution and hence must fix parallel classes other than on L or M. But then, the group could not be triangle transitive. $\boxed{QED}$

3.3 Order 3^4

Thus, we consider the order 3^4. First assume that the group is non-solvable. By the classification theorem, we have the following possibilities: (2) $G_0 \trianglerighteq SL(n,z)$, $z^n = h^a$, (3) $G_0 \trianglerighteq Sp(n,z)$, $z^n = h^a$, (6) $G_0 \trianglerighteq SL(2,3)$ or $SL(2,5)$, $v = h^2$, $h = 5, 11, 19, 23, 29$, or 59 or $v = 3^4$, (7) G_0 has a normal extraspecial subgroup E of order 2^5 and G_0/E is isomorphic to a subgroup of S_5, where $v = 3^4$.

18 Lemma. *Cases* (2) *and* (3) *cannot occur.*

Proof. In case (2), we have $z^n = 3^4$, so we have $n = 2$ or 4. If we have the group $SL(2,9)$ acting on a component L, Theorem 9 applies to classify the plane and since M is also invariant, the plane must be Hall, which, of course, is not Bol. If we have $SL(4,3)$, we have a 3-group of order 3^6, which must be planar due to the nature of the group action in contradiction to the p-planar bound of Theorem 10. Part (3) similarly does not occur since we would have $S_p(4,3)$ or $S_p(2,9)$ acting, where the 3-groups are planar. $\boxed{QED}$

19 Lemma. *Non-solvable case* (6) *cannot occur.*

Proof. If we have case (6), we have $SL(2,5)$ as a normal subgroup in the translation complement acting on L and/or M. Note that the stabilizer of the two special components of the group generated by the central involutions must act faithfully on each of the two components. However, by Johnson–Prince [9] and Jha–Johnson [6], we know all possible translation planes of order 81 admitting $SL(2,5)$ as a collineation group, and it turns out that none of these are Bol. $\boxed{QED}$

So consider case (7).

20 Lemma. *In case* (7)*, solvable or non-solvable,* G_0 *has a normal extraspecial subgroup* E *of order* 2^5 *containing at least* 10 *Baer involutions. Furthermore,* *the center $Z(E) = E'$ is the kernel involution. These* 10 *Baer collineations generate E.*

Proof. In this case, we note that E/E' is an elementary Abelian 2 -group of order 16. Since E has order 32 and acts on 80 points, which is $16 \cdot 5$, E must have an orbit of length dividing 16. That is, not all orbits can have length 32. Therefore, there must be a collineation of E which fixes at least two points on a component L. But then there is an involution σ in E which is not the kernel involution of the translation plane. However, the nature of the group shows that this involution must be planar and hence E contains a Baer involution.

Now consider the center involution τ. Since τ is either Baer or the kernel homology, as it acts isomorphically on both L and M, assume that τ is Baer. Since E' is characteristic in E, which is normal in G, then, on L the fixed points of τ are permuted by G. However, this means that there is an invariant set of 9 points on L, a contradiction to transitivity. Hence, τ is the kernel involution.

Let π_1 be a Baer subplane fixed pointwise by an involution σ_1 of E. Whether G_0 is solvable or non-solvable, G_0 contains a collineation of order 5. Let θ_5 denote an element of order 5 in G_0, which then normalizes E. If θ_5 leaves π_1 invariant then since 5 is a 3-primitive divisor of $3^4 - 1$, it follows that θ_5 must leave L pointwise fixed. But this implies that θ_5 also fixes M pointwise, a contradiction. Hence, there are at least 5 Baer involutions in E. Each Baer involution fixes pointwise a Baer line on L and leaves invariant a second Baer line. That is, it follows from Theorem 11 that corresponding to π_1 is a second Baer subplane π_1^2, which shares its parallel classes with π_1 and which is fixed by σ_1. Since σ_1 fixes all parallel classes of π_1^2 and fixes an affine point, it follows that σ_1 induces the kernel involution on π_1^2. Hence, $\sigma_1\tau$ is also a Baer involution in E. But θ_5 cannot map π_1 onto π_1^2, since if it did, it would be forced to map π_1^2 onto π_1, implying that θ_5^2 fixes both subplanes, a contradiction. Hence, there are at least 10 subplanes in two orbits of length 5 under $\langle\theta_5\rangle$. It remains to show that these generate E. Let B denote the subgroup generated by the Baer involutions. The above argument shows that τ is in B. Since E/E' is elementary Abelian of order 16, then B/E' has at least 10 involutions in E/E', implying that $B = E$. $\boxed{QED}$

21 Lemma. *Neither solvable nor non-solvable case* (7) *can occur.*

Proof. Any element of G_0 is generated by a finite number of elements of the form $(x, y) \to (xA, yA^{-1})$, where A is in the corresponding spread. If a finite number of such elements generate a Baer involution, then we have a collineation of the form $(x, y) \to (xA_1A_2 \cdots A_t, yA_1^{-1}A_2^{-1} \cdots A_t^{-1})$, where A_i is in the spread for $i = 1, 2, \ldots, t$. If we let $A_1 \cdots A_t = C$, then such a collineation $(x, y) \to (xC, yC^{-1})$ is an involution if and only if $C^2 = I$. But then $C = C^{-1}$, implying that this collineation fixes $y = x$. Therefore, we have shown that any Baer involution in E fixes $y = x$, and since E is generated by Baer involutions we see that E fixes $y = x$. However, E is not elementary Abelian, so there are elements g of E of order 4, and of the general form $(x, y) \to (xT, yT^{-1})$, which fix $y = x$, implying again that $T = T^{-1}$ but $T^4 = I$ and $T^2 = -I$, a contradiction. $\boxed{QED}$

Thus, we must have that the group G_0 is solvable. We therefore have one of the following: (1) $G \leq A\Gamma L(1, v)$, (6) $G_0 \trianglerighteq SL(2,3)$ or $SL(2,5)$, $v = h^2$, $h = 5, 11, 19, 23, 29$, or 59 or $v = 3^4$, as the solvable (and non-solvable) case (7) has been resolved. If we have case (1), Theorem 2 applies.

22 Lemma. *Solvable case* (6) *case cannot apply.*

Proof. Hence, consider case (6), where we have a normal subgroup isomorphic to $SL(2,3)$. The 3-elements must be planar and hence fix a subplane of order 3 or 3^2. There are four planar 3-groups and the group is transitive and hence permutes these subplanes restricted to the axes. Hence, there are at most $4(3^2 - 1) + 1$ permuted points on each axis—not enough to be transitive (that is, in general, case (6) and $v = 3^4$ forces $SL(2,5)$ in the transitive group). $\boxed{QED}$

This completes the proof that Bol planes of orders 3^4 and 3^6 are nearfield planes.

References

[1] L. Bader, G. Lunardon: *On the flocks of* $Q^+(3, q)$, Geom. Dedicata, **29** (1989), n. 2, 177–183.

[2] M. Biliotti, V. Jha, N. L. Johnson: Foundations of Translation Planes, Monographs and Textbooks in Pure and Applied Mathematics, vol. 243, Marcel Dekker, New York, Basel 2001.

[3] R. P. Burn: *Finite Bol loops*, Math. Proc. Cambridge Philos. Soc., **84** (1978), n. 3, 377–385.

[4] D. A. Foulser, N. L. Johnson: *The translation planes of order* q^2 *that admit* SL(2, q) *as a collineation group, II: Odd order*, J. Geom., **18** (1982), n. 2, 122–139.

[5] J. Hanson, M. J. Kallaher: *Finite Bol quasifields are nearfields*, Utilitas Math., **37** (1990), 45–64.

[6] V. Jha, N. L. Johnson: *The translation planes of order* 81 *admitting* SL(2, 5), Note Mat., **24** (2005), 59–73.

[7] V. Jha, N. L. Johnson: *Structure theory for point-Baer and line-Baer collineations in affine planes*, Mostly Finite Geometries (Iowa City, IA, 1996), Lecture Notes in Pure and Appl. Math., vol. 190, Dekker, New York-Basel-Hong Kong 1997, pp. 235–273.

[8] N. L. Johnson: *The non-solvable triangle transitive planes*, Note Mat., **23** (2004), 39–46.

[9] N. L. Johnson, A. Prince: *The translation planes of order* 81 *that admit* SL(2, 5)*, generated by elations*, J. Geom., **84** (2005), 73–82.

[10] M. J. Kallaher: *On finite Bol quasifields*, Proceedings of the conference on groups and geometry, Part A (Madison, Wis., 1985), Algebras Groups Geom., **2** (1985), n. 3, 300–312.

[11] M. J. Kallaher, T. G. Ostrom: *Fixed point free linear groups, rank three planes, and Bol quasifields.*, J. Algebra, **18** (1971), 159–178.

[12] M. J. Kallaher, T. G. Ostrom: *Bol quasifields and generalized André systems*, J. Algebra, **58** (1979), n. 1, 100–116.

[13] J. A. Thas: *Maximal exterior sets of hyperbolic quadrics: the complete classification*, J. Combin. Theory Ser. A, **56** (1991), 303–308.