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Limiting behaviour of moving average processes under ρ-mixing assumption

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Abstract. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed ρ-mixing random variables, $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we prove the complete convergence and Marcinkiewicz-Zygmund strong law of large numbers for the partial sums of the moving average processes $\{\sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$.

Keywords: moving average, ρ-mixing, complete convergence, Marcinkiewicz-Zygmund strong laws of large numbers.

MSC 2000 classification: primary 60F15

Introduction and Main Results.

Let $\{Y_i, -\infty < i < +\infty\}$ be a doubly infinite sequence of identically distributed random variables and $\{a_i, -\infty < i < +\infty\}$ be an absolutely summable sequence of real numbers. Next,

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let

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1$$

be the *moving average process based on the sequence* $\{Y_i, -\infty < i < +\infty\}$. As usual, we denote $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$, the sequence of partial sums.

Under the assumption that $\{Y_i, -\infty < i < +\infty\}$ is a sequence of independent identically distributed random variables, many limiting results have been obtained for the moving average process $\{X_n, n \geq 1\}$. For example, Ibragimov [3] established the central limit theorem, Burton and Dehling [4] obtained a large deviation principle assuming $E\exp\{tY_1\} < \infty$ for all t, and Li *et al.* [5] obtained the complete convergence result for $\{X_n, n \geq 1\}$.

Certainly, even if $\{Y_i, -\infty < i < +\infty\}$ is the sequence of independent identically distributed random variables, the moving average random variables $\{X_n, n \geq 1\}$ are dependent. This kind of dependence is called *weak dependence*. The partial sums of weakly dependent random variables $\{X_n, n \geq 1\}$ have similar limiting behaviour properties in comparison with the limiting properties of independent identically distributed random variables.

For example, we could present some the previous results connected with complete convergence. The following was proved in Hsu and Robbins [1].

Theorem A. *Suppose* $\{X_n, n \geq 1\}$ *is a sequence of independent identically distributed random variables. If* $EX_1 = 0$, $E|X_1|^2 < \infty$, *then* $\sum\limits_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} < \infty$ *for all* $\varepsilon > 0$.

Hsu-Robbins result was extended by Li *et al.* [5] for moving average processes.

Theorem B. *Suppose* $\{X_n, n \geq 1\}$ *is the moving average processes based on a sequence* $\{Y_i, -\infty < i < \infty\}$ *of independent identically distributed random variables with* $EY_1 = 0$, $E|Y_1|^2 < \infty$. *Then* $\sum\limits_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} < \infty$ *for all* $\varepsilon > 0$.

Very few results for a moving average process based on a dependent sequence are known. In this paper, we provide a result on the limiting behavior of a moving average process based on a ρ-mixing sequence.

Let $\{Z_i, -\infty < i < \infty\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and denote σ-algebras

$$\mathcal{F}_n^m = \sigma(Z_i, n \leq i \leq m), \quad -\infty \leq n \leq m \leq +\infty.$$

As usual, for a σ-algebra $\mathcal{F}$ we denote by $\mathcal{L}^2(\mathcal{F})$ the class of all $\mathcal{F}$-measurable random variables with the finite second moment.

A sequence of random variables $\{Z_i, -\infty < i < \infty\}$ is called *ρ-mixing* if the maximal correlation coefficient

$$\rho(m) = \sup_{k \geq 1} \sup \left\{ \left| \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} \right|, X \in \mathcal{L}^2(\mathcal{F}_{-\infty}^k), Y \in \mathcal{L}^2(\mathcal{F}_{m+k}^{\infty}) \right\} \to 0$$

as $m \to \infty$.

The following maximal inequality can be found in Shao ([7], Theorem 1.1) and plays a crucial role in the proof of our main result. As usual, the notation $[\cdot]$ is used for the integer part function.

Maximal Inequality. *Assume that* $\{Z_n, n \geq 1\}$ *is a sequence of ρ-mixing random variables with* $EZ_n = 0$ *and* $E|Z_n|^q < \infty$ *for all* $n \geq 1$ *and some* $q \geq 2$. *Then there is a positive* *constant $K = K(q, \rho(\cdot))$ depending only on q and $\rho(\cdot)$ such that for any $n \geq 1$,*

$$E \max_{1\leq k\leq n} |\sum_{i=1}^{k} Z_i|^q \leq K \left(n^{q/2} \exp\left\{ K \sum_{i=0}^{[\log n]} \rho(2^i) \right\} \right) \max_{1\leq k\leq n} (E|Z_k|^2)^{q/2}$$
$$+ n \exp\left\{ K \sum_{i=0}^{[\log n]} \rho^{q/2}(2^i) \right\} \max_{1\leq k\leq n} E|Z_k|^q.$$

Recall that a measurable function h is said to be *slowly varying* if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1.$$

We refer to Seneta [6] for other equivalent definitions and for detailed and comprehensive study of properties of such functions.

Now we can present the main result of the paper.

Theorem 1. *Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of identically distributed ρ-mixing random variables, and $\{X_n, n \geq 1\}$ be the moving average process based on the sequence $\{Y_i, -\infty < i < \infty\}$.*

Let $h(x)$ be a positive slowly varying function and $1 \leq p < 2, r \geq 1$. If $rp = 1$, additionally assume that $\sum_{i=1}^{\infty} |a_i|^\theta < \infty$ for some $\theta \in (0,1)$. If $rp < 2$ take $q = 2$, and if $rp \geq 2$ take any $q > \frac{2p(r-1)}{2-p}$. Set

$$\phi(t) = \sum_{i=0}^{[\log t]} \rho^{2/q}(2^i).$$

Let $K = K(q, \rho(\cdot))$ be the constant defined in the Maximal Inequality presented above. If $EY_1 = 0$ and $E|Y_1|^{rp} h(|Y_1|^p) \exp\{K\phi(|Y_1|^p)\} < \infty$ then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{\max_{k\leq n} |S_k| \geq \varepsilon n^{1/p}\} < \infty.$$

In particular, if $EY_1 = 0$ and $E|Y_1|^p \exp\{K\phi(|Y_1|^p)\} < \infty$, then the following Marcinkiewicz-Zygmund strong law of large numbers

$$n^{-1/p} S_n \to 0, \quad a.s.$$

holds.

Remark 1. If $\sum_{n=0}^{\infty} \rho^{2/q}(2^n) < \infty$, then

$$E|Y_1|^{rp} h(|Y_1|^p) \exp\{K\phi(|Y_1|^p)\} < \infty \quad \text{and} \quad E|Y_1|^{rp} h(|Y_1|^p) < \infty$$

are equivalent. Hence the first statement of Theorem extends and generalizes Theorem 3.1 of Shao [7]. Marcinkiewicz-Zygmund strong law of large number presented in Theorem generalizes Theorem 5.1 of Fazekas and Klesov [2] and Corollary 3.1 of Shao [7].

By the following Proposition, the function $\exp\{K\phi(t)\}$ is a slowly varying function. Hence the assumption about $\exp\{K\phi(t)\}$ in Theorem 5.1 of Fazekas and Klesov [2] is excessive

Proposition 1. *Let $\{b_n, n \geq 0\}$ be a sequence of real number with $\lim_{n\to\infty} b_n = 0$ and set $\Phi(t) = \sum_{n=0}^{[\log t]} b_n$. Then for any constant $K > 0$ the function* $\exp\{K\Phi(t)\}$ *is a slowly varying function.*

Proofs

Proof of Theorem 1. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and since $\sum_{i=-\infty}^{\infty} |a_i| < \infty$,

$$\begin{aligned}
&n^{-1/p} |E \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I\{|Y_j| \le n^{1/p}\}| \\
&\le n^{-1/p} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E|Y_j| I\{|Y_j| \le n^{1/p}\} \\
&\le n^{1-1/p} \left(\sum_{i=-\infty}^{\infty} |a_i| \right) E|Y_1| I\{|Y_1| > n^{1/p}\} \\
&\le CE (n^{1/p})^{p-1} |Y_1| I\{|Y_1| > n^{1/p}\} \\
&\le CE|Y_1|^p I\{|Y_1| > n^{1/p}\} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Hence for n large enough we have

$$n^{1/p} | \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I\{|Y_j| \le n^{1/p}\}| < \varepsilon/4.$$

Let $Y_{nj} = Y_j I\{|Y_j| \le n^{1/p}\} - EY_j I\{|Y_j| \le n^{1/p}\}$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} h(n) P\{\max_{1 \le k \le n} |S_k| \ge \varepsilon n^{1/p}\} \\
&\le C \sum_{n=1}^{\infty} n^{r-2} h(n) P\{\max_{1 \le k \le n} | \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I\{|Y_j| > n^{1/p}\}| \ge \varepsilon n^{1/p}/2\} \\
&+ C \sum_{n=1}^{\infty} n^{r-2} h(n) P\{\max_{1 \le k \le n} | \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}| \ge \varepsilon n^{1/p}/4\} \\
&=: I + J, \text{ say.}
\end{aligned}$$

For I, if $rp > 1$, by Markov inequality we have

$$\begin{aligned}
I &\le C \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-1/p} E \max_{1 \le k \le n} | \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I\{|Y_j| > n^{1/p}\}| \\
&\le C \sum_{n=1}^{\infty} n^{r-1-1/p} h(n) E|Y_1| I\{|Y_1| > n^{1/p}\} \\
&= \sum_{n=1}^{\infty} n^{r-1-1/p} h(n) \sum_{m=n}^{\infty} E|Y_1| I\{m < |Y_1|^p \le m+1\} \\
&= \sum_{m=1}^{\infty} E|Y_1| I\{m < |Y_1|^p \le m+1\} \sum_{n=1}^{m} n^{r-1-1/p} h(n)
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{m=1}^{\infty} m^{r-1/p}h(m)E|Y_1|I\{m<|Y_1|^p\le m+1\}\\
&\le CE|Y_1|^{rp}h(|Y_1|^p)\le CE|Y_1|^{rp}h(|Y_1|^p)\exp\{K\phi(|Y_1|^p)\}<\infty.
\end{aligned}
$$

If $rp=1$, by Markov inequality and the same argument as the case $rp>1$ we have

$$
\begin{aligned}
I &\le C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-\theta/p}E\max_{1\le k\le n}|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_jI\{|Y_j|>n^{1/p}\}|^{\theta}\\
&\le C\sum_{n=1}^{\infty} n^{r-1-\theta/p}h(n)E|Y_1|^{\theta}I\{|Y_1|>n^{1/p}\}<\infty.
\end{aligned}
$$

For J, if $rp<2$, by Markov, Hölder, and Maximal inequalities we have

$$
\begin{aligned}
J &\le C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-2/p}E\max_{1\le k\le n}|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}|^2\\
&\le C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-2/p}E\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{1-\ 1/2}\right)\left(|a_i|^{1/2}\max_{1\le k\le n}|\sum_{j=i+1}^{i+k}Y_{nj}|\right)\right)^2\\
&\le C\sum_{n=1}^{\infty} n^{r-2-2/p}h(n)(\sum_{i=-\infty}^{\infty}|a_i|)\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\le k\le n}|\sum_{j=i+1}^{i+k}Y_{nj}|^2\\
&\le C\sum_{n=1}^{\infty} n^{r-2-2/p}h(n)\left(n\exp\left\{K\sum_{i=0}^{[\log n]}\rho(2^i)\right\}\right)E|Y_1|^2I\{|Y_1|\le n^{1/p}\}\\
&= C\sum_{n=1}^{\infty} n^{r-1-2/p}h(n)\exp\{K\phi(n)\}E|Y_1|^2I\{|Y_1|\le n^{1/p}\}\\
&\le C\sum_{n=1}^{\infty} n^{r-1-2/p}h(n)\exp\{K\phi(n)\}\sum_{k=1}^{n}E|Y_1|^2I\{(k-1)^{1/p}<|Y_1|\le k^{1/p}\}\\
&\le C\sum_{k=1}^{\infty}E|Y_1|^2I\{(k-1)^{1/p}<|Y_1|\le k^{1/p}\}\sum_{n=k}^{\infty}n^{r-1-2/p}h(n)\exp\{K\phi(n)\}\\
&\le C\sum_{k=1}^{\infty}k^{r-2/p}h(k)\exp\{K\phi(k)\}E|Y_1|^2I\{(k-1)^{1/p}<|Y_1|\le k^{1/p}\}\\
&\le CE|Y_1|^{rp}h(|Y_1|^p)\exp\{K\phi(|Y_1|^p)\}<\infty.
\end{aligned}
$$

If $rp\ge 2$, by Markov, Hölder, and Maximal inequalities we have

$$
\begin{aligned}
J &\le C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-q/p}E\max_{1\le k\le n}|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}|^q\\
&\le C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-q/p}E\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{1-\ 1/q}\right)\left(|a_i|^{1/q}\max_{1\le k\le n}|\sum_{j=i+1}^{i+k}Y_{nj}|\right)\right)^q\\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)(\sum_{i=-\infty}^{\infty}|a_i|)^{q-1}\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\le k\le n}|\sum_{j=i+1}^{i+k}Y_{nj}|^q\\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p+q/2}h(n)\exp\left\{K\sum_{i=0}^{[\log n]}\rho(2^i)\right\}(E|Y_1|^2I\{|Y_1|\le n^{1/p}\})^{q/2}
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty} n^{r-1-q/p} h(n) \exp\{K\phi(n)\} E|Y_1|^q I\{|Y_1| \le n^{1/p}\} \\
&=: J_1 + J_2, \text{ say.}
\end{aligned}
$$

For J_1, since $r-2-q/p+q/2<-1$, we can take $t>0$ small enough such that $r-2-q/p+q/2+tq/(2p)<-1$. Then

$$
J_1 = C\sum_{n=1}^{\infty} n^{r-2-q/p+q/2} h(n) \exp\left\{K\sum_{i=0}^{[\log n]} \rho(2^i)\right\} (E|Y_1|^{2-t+t} I\{|Y_1| \le n^{1/p}\})^{q/2}
$$

$$
\le C\sum_{n=1}^{\infty} n^{r-2-q/p+q/2+tq/(2p)} h(n) \exp\left\{K\sum_{i=0}^{[\log n]} \rho(2^i)\right\} (E|Y_1|^{2-t} I\{|Y_1| \le n^{1/p}\})^{q/2} < \infty.
$$

For J_2, we have

$$
\begin{aligned}
J_2 &= C\sum_{n=1}^{\infty} n^{r-1-q/p} h(n) \exp\{K\phi(n)\} E|Y_1|^q I\{|Y_1| \le n^{1/p}\} \\
&\le C\sum_{n=1}^{\infty} n^{r-1-q/p} h(n) \exp\{K\phi(n)\} \sum_{k=1}^{n} E|Y_1|^q I\{k-1 < |Y_1|^p \le k\} \\
&= C\sum_{k=1}^{\infty} E|Y_1|^q I\{k-1 < |Y_1|^p \le k\} \sum_{n=k}^{\infty} n^{r-1-q/p} h(n) \exp\{K\phi(n)\} \\
&\le C\sum_{k=1}^{\infty} k^{r-q/p} h(k) \exp\{K\phi(k)\} E|Y_1|^q I\{k-1 < |Y_1|^p \le k\} \\
&\le CE|Y_1|^{rp} h(|Y_1|^p) \exp\{K\phi(|Y_1|^p)\} < \infty.
\end{aligned}
$$

Now we show the almost sure convergence. By the first part of Theorem, $EY_1 = 0$ and $E|Y_1|^p \exp\{K\phi(|Y_1|^p)\} < \infty$ imply

$$
\sum_{n=1}^{\infty} n^{-1} P\left\{\max_{1\le k\le n} |S_n| \ge \varepsilon n^{1/p}\right\} < \infty, \text{ for all } \varepsilon > 0.
$$

Hence

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} |S_m| > \varepsilon n^{1/p}\} \\
&= \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k} n^{-1} P\{\max_{1\le m\le n} |S_m| > \varepsilon n^{1/p}\} \\
&\ge 1/2 \sum_{k=1}^{\infty} P\{\max_{1\le m\le 2^{k-1}} |S_m| > \varepsilon 2^{k/p}\}.
\end{aligned}
$$

By Borel-Cantelli lemma,

$$
2^{-k/p} \max_{1\le m\le 2^k} |S_m| \to 0 \text{ almost surely}
$$

which implies that $S_n/n^{1/p} \to 0$ almost surely.$\square$

Proof of Proposition 1. Since $\lim_{n\to\infty} b_n = 0$, for all $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that

$$-\varepsilon < b_n < \varepsilon \text{ for all } n > N.$$

For any $\lambda > 1$

$$\exp\{K\Phi(\lambda t)\}/\exp\{K\Phi(t)\} = \exp\{K \sum_{n=[\log t]+1}^{[\log(\lambda t)]} b_n\}.$$

Note that for $t > 1$

$$[\log(\lambda t)] - [\log t] \leq \log(\lambda t) - (\log t - 1) = \log\lambda + 1.$$

Hence for t large enough

$$\exp\{-K(\log\lambda + 1)\varepsilon\} \leq \exp\{K\Phi(\lambda t)\}/\exp\{k\Phi(t)\} \leq \exp\{K(\log\lambda + 1)\varepsilon\}$$

and by the arbitrariness of $\varepsilon > 0$

$$\lim_{t\to\infty} \exp\{K\Phi(\lambda t)\}/\exp\{K\Phi(t)\} = 1.\square$$

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