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Symplectic spreads and symplectically paired spreads

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Abstract. If π is a finite symplectic translation plane, it is shown that any affine homology group is cyclic and has order dividing the order of the kernel homology group. This criterion provides a means to ensure that a given spread is not symplectic. Furthermore, a variety of symplectically paired André spreads are constructed.

Keywords: translation planes, symplectic spreads, affine homologies, symplectically paired spreads

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1 Introduction

Let V be a $2d$-dimensional vector space over $GF(q)$, which admits a symplectic form. Then there is a basis for V so that the form is $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, since all forms are conjugate. If V admits a spread of totally isotropic subspaces, we call the spread a 'symplectic spread' and say that we also have a 'symplectic translation plane'. By noting that the symplectic group acts doubly transitively on totally isotropic subspaces, we see that we may choose

$$x = 0, y = 0,\ y = xM; M^t = M$$

as a matrix spread set (see also Biliotti, Jha, Johnson [3]). More generally, we may allow any two components to be called $x = 0, y = 0$, but we shall revisit this below. We call this a 'symmetric representation'. It is difficult to determine symplectic spreads at least from the point of view of the matrix spread set and there are very few such spreads in existence. However, Kantor [12] has shown that any commutative semifield plane transposes and dualizes to a symplectic semifield spread. Furthermore, it is noted in Biliotti, Jha and Johnson [3] an algebraic connection between semifield flock spreads and symplectic semifield spreads, the two spreads being connected by a five-step algebraic construction process. Such spreads are said to be 5th-cousins if they are related by the following chain of length 5.

1 Theorem (Biliotti, Jha, Johnson [3]). *Let S be a semifield flock spread in $PG(3,q)$.*

(1) Then applying the following sequence of construction operations produces a symplectic semifield spread S_{sym} in $PG(3,q)$:

$$\begin{aligned} S(flock) &\longmapsto dualize \longmapsto distort \longmapsto derive \\ &\longmapsto transpose \longmapsto dualize \longmapsto S_{sym}(symplectic) \end{aligned}$$

(2) Let S_{sym} be a symplectic semifield whose spread is in $PG(3,q)$. Then applying the following sequence of constructions produces a semifield flock spread in $PG(3,q)$:

$$\begin{aligned} S_{sym}(symplectic) &\longmapsto dualize \longmapsto transpose \longmapsto derive \\ &\longmapsto extend \longmapsto dualize \longmapsto S(flock). \end{aligned}$$

Furthermore, it is also noted for symplectic semifield spreads that we may assume that $GF(q)$ is the kernel of the semifield translation plane.

Indeed, Kantor [11] has recently shown that any symplectic spread remains symplectic when considered over the kernel of the translation plane. In other words, we may always assume that $K \simeq GF(q)$ is the kernel of the plane and that M is a set of $d \times d$ matrices over K such that $M^t = M$.

Hence, the known semifield flocks spreads provide us also with symplectic semifields. It is known also that the Lüneburg–Tits planes and the Hering planes of order 27 are symplectic. There is an infinite class of planes due to Suetake [15] that contain the Hering planes of order 27. In fact, the Suetake planes are known to be net replaceable from a certain class of generalized twisted field planes, which are, in fact, symplectic. That the replacement procedure preserves the symplectic form is a recent result of Ball, Bamberg, Lavrauw, and Penttila [1].

There are a variety of symplectic semifield planes and hence commutative semifield planes of even order in existence due to the 'slicing' and 'up and down' methods of Kantor [12]. However, the Suetake planes (and the Hering planes) are the only known planes of odd order and odd dimension over their kernel that are symplectic. For dimension 2 over the kernel, the known planes correspond to ovoids in $O(5,q)$-spaces. The corresponding translation planes are quite rare, being the Lüneburg–Tits planes, The Thas–Payne spread (the 5th cousin of the Cohen–Ganley semifield spreads), the Knuth spreads, and the Penttila–Williams spreads (5th cousin of the Bader–Lunardon–Pinneri spread of order 3^{10}), and a slice of the Ree–Tits ovoids in $O(7, 3^k)$ (see, e.g., Johnson [8] and Penttila, Williams [14]).

In general trying to determine if any given translation plane is also a symplectic plane is difficult without knowledge of a given symplectic form for then the question boils down to asking when there is a matrix spread set of symmetric matrices. For example, given an ovoid in $PG(3, q)$, for q even, the work of Thas [16] shows that there is a corresponding symplectic spread in $PG(3, q)$, hence providing a matrix spread set of symmetric 2×2 matrices. That is, there is a symmetric representation of the following form:

$$x = 0, y = x \begin{bmatrix} f(t,u) & t \\ t & u \end{bmatrix};\ t, u \in GF(q).$$

Brown [4] shows that if an ovoid in $PG(3, q)$, for q even, has at least one hyperplane intersection which is a conic then the ovoid is an elliptic quadric. Maschietti [13] points out that this is equivalent to having a regulus in the symmetric matrix representation. This means that coordinates may be chosen so that $f(0, u)$. So, it would potentially be possible to show that if $f(0, u) = u$ for all $u \in GF(q)$ then the spread is Desarguesian, which would be an algebraic proof of the theorem of Brown. This formulation of a deep result into a problem involving symmetric representations of symplectic spreads illustrates how potentially difficult it would be to use only the symmetric representations to attack problems on symplectic spreads.

Hence, it would be nice to have if not a criterion for the existence of a symplectic spread then perhaps a criterion for the non-existence of a symplectic spread. In other words, when can it be guaranteed that a spread is not symplectic? By again noting the connection with commutative semifield spreads and symplectic semifield spreads, this would speak to the problem of asking when a given semifield spread has an isotopic image which is commutative.

This note provides such a non-existence criterion.

Our result is:

2 Theorem. *Let π be a finite symplectic translation plane of order q^n and kernel $GF(q)$.*

(1) Then any affine homology must have order dividing $q - 1$.

(2) There is a coordinatization of π so that the middle and right associators (middle and right nuclei) are equal and contained in the kernel.

3 Corollary. *Let Σ be a finite commutative semifield plane. Then there is a coordinate semifield so that the left and right nuclei are equal and contained in the middle nucleus.*

4 Corollary. *Let Σ be a finite commutative and symplectic semifield plane. Then there is a coordinate semifield so that left, right and middle nuclei are equal.*

If a spread is left invariant by a polarity but not identically, the plane may not be symplectic but becomes of interest. One extreme example of this is when no component of the spread is fixed; that is, we must have $(q^n + 1)/2$ pairs of components each of which is left invariant by the polarity and the polarity has order 2 on each pair. We call such spreads 'symplectically paired'. Weintraub [17] originated this idea calling the pairs, 'symplectic pairs'. Weintraub [17] shows that these always occur and provides examples when q is $3, 5, 7$. We point out that the infinite class found by Weintraub [17] is the class of Hall planes. But, since symplectic spreads at least of dimension 2 are quite rare, the question arises if symplectically paired spreads are also rare. In a word, no! Specifically, we begin by showing that the Weintraub examples are Hall planes and then generalize this to show that essentially all André planes (see the restriction below) are symplectically paired.

2 Symplectic planes and affine homologies

If we have a symplectic plane, we may regard the kernel as the prime subfield and choose the symplectic form as $\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, which means if we consider the spread written over the prime field then we have a set of symmetric matrices forming a spread.

Let π be a symplectic plane. Then there is a matrix spread set so that the spread is $x = 0$, $y = 0$, $y = xM$, such that $M^t = M$. If we change bases by the mapping $(x, y) \longrightarrow (x, x(-Z) + y)$, where $Z^t = Z$ then the new spread has the form $x = 0, y = (N - Z)$, where N is either 0 or M such that $M^t = M$. Since $(N - Z)^t = N^t - Z^t = N - Z$. We still have a symplectic representation. We call this 'elation sliding'.

Similarly, if change bases by $(x, y) \longrightarrow (y, x)$ then the new representation is $x = 0, y = xM^{-1}$, such that $M^t = M$. But, $(M^{-1})^t = (M^t)^{-1} = (M)^{-1} = M^{-1}$, so the new representation is still symplectic. We call this 'inverting'. Hence, we have:

5 Theorem. *If we have a symmetric representation then any iteration of processes obtained by elation sliding and inverting preserves a symmetric representation.*

So, in particular, choose any two distinct components L and M of a symplectic spread in a symmetric representation. Then there is a basis change so that L may be represented by $x = 0$ and the matrix representation is still symmetric.

Proof. By elation sliding and inverting, we can assume that L is $x = 0$ and retain a symmetric representation. In this representation, we may apply elation sliding which fixes $x = 0$ and takes the new representation of M into $y = 0$. More precisely, if L is $y = xZ$ use the elation slide of $(x, y) \to (x, -xZ + y)$ to rewrite L as $y = 0$, while retaining the symmetric representation. Now use inverting to represent L as $x = 0$, while still maintaining a symmetric representation. Now M has the form $y = xT$ and use elation sliding to represent M as $y = 0$ while fixing $x = 0$ and maintaining a symmetric representation. $\boxed{QED}$

Now assume that π is a symplectic plane. Assume that we have an affine homology group. By the previous theorem, there is a symmetric representation so that $y = 0$ is the axis and $x = 0$ is the center. In such a plane, there is a representation so that components $x = 0$, $y = 0$ are chosen so that there is a homology group with axis $y = 0$ and co-axis $x = 0$. Considering that we are working over the kernel then the affine homology has the form $(x, y) \longrightarrow (x, yM)$, where we regard all of the matrices over the primitive field. That is, by the previous theorem, we may assume that we have a symplectic representation $x = 0, y = 0, y = xT$, such that $T^t = T$. Hence, we have the following conditions:

$$(TM)^t = TM = M^tT^t = M^tT,$$

for all T in S (of cardinality q^n) and for all M in a cyclic group G of order $(q^n - 1)/(q - 1)$. Thus, we have

$$T = M^tTM^{-1},$$

for all T in S. Now consider the mapping $(x, y) \to (xM^{-t}, yM^{-1})$, which maps $y = xT$ onto $y = M^tTM^{-1} = T$. This then is a kernel homology of the plane π of order the order of the matrix M, which in turn, is the order of the affine homology. That is, note that the square of the previous mapping is $(x, y) \longrightarrow (xM^{-2t}, yM^{-2})$. So we obtain a group of order the order of M, which acts as a kernel homology of the translation plane. Hence, we have proved the following theorem.

6 Theorem. *Let π be a symplectic translation plane of order q^n and kernel isomorphic to $GF(q)$. Then any affine homology of π must have order dividing $q - 1$.*

7 Theorem. *Let π be a non-Desarguesian symplectic plane of order $q^n - 1$. Then there cannot exist an affine homology of order a prime p-primitive divisor of $q^n - 1$.*

Proof. If there is such an affine homology then the kernel of π must be $GF(q^n)$, so that the planes is Desarguesian. $\boxed{QED}$

8 Corollary. *A non-Desarguesian symplectic plane of order q^n and $n > 1$ cannot be a $j \dots j$-plane.*

Proof. We now have an affine homology group of order $(q^n - 1)/(q - 1)$ which must divide $(q - 1)$, impossible unless $n = 1$. $\boxed{QED}$

Now assume that we have a symplectic plane of order q^n and kernel $GF(q)$. If $\tau_\alpha : (x, y) \to (x, y\alpha)$ is a collineation then so is $(x, y) \to (x\alpha^t, y\alpha)$, implying that $\rho_{\alpha^t} : (x, y) \to (x\alpha^t, y)$ is a collineation. Similarly if $(x, y) \to (x\beta, y)$ is a collineation so is $(x, y) \to (x\beta, y\beta^t)$, implying that $(x, y) \to (x, y\beta^t)$ is a collineation. Mapping τ_α to ρ_{α^t} is a monomorphism and mapping ρ_β to τ_{β^t} is a monomorphism. Hence, the two affine groups are isomorphic. Furthermore, since the groups define isomorphic subgroups of the kernel homology group, then are isomorphic. Hence, in symplectic planes the right and middle nuclei are isomorphic.

Now we may choose the kernel so that the kernel homology groups are represented as diagonal matrices. For example, we can certain do this if there is an element $y = x$ in the matrix spread set. In our situation, a basic change by $(x, y) \to (x, yT^{-1})$, produces such a matrix spread set (although we lose potentially the symmetric representation). By Kantor [11], we may assume that the matrix spread set are matrices over the kernel. When we have $y = x$ in the matrix spread set, then the corresponding affine collineations $(x, y) \to (x, yM)$ force M to be in the matrix spread set. In other words, in the symmetric representation $M = T_2^{-1}T_1$, where T_2 and T_1 are in the symmetric set S. In any case, M is a matrix over the kernel. Hence, this shows that as kernel homologies, the mappings are of the form $(x, y) \to (xG, yG)$, where G are diagonal matrices. So, there is a representation so that the right and middle associator groups (right and middle nuclei) are identical and contained in the kernel.

Hence, we have proved:

9 Theorem. *Let π be a symplectic translation plane. Then there is a coordinatization so that the right and middle associated groups (right and middle nuclei) are equal and contained in the left nucleus (coordinate kernel) of the translation plane.*

10 Corollary. *Let π be a symplectic semifield plane. Then there is a coordinatization so that the right and middle nuclei are equal fields and contained in the left nucleus (kernel) of the semifield.*

Now considered a symplectic semifield plane with semifield S. If we dualize S then the left nucleus and right nucleus are interchanged and the middle nucleus remains the same. Hence, we now have a semifield plane with left and middle nucleus equal and contained in the right nucleus. When we transpose, the middle and right nuclei are interchanged and the left nucleus remains the same. So, we have a semifield with left and right nuclei equal and contained in the middle nucleus. Since this set isotopically defines a commutative semifield plane, we have the following corollary.

11 Corollary. *Let Σ be a finite commutative semifield plane. Then there is a coordinate semifield so that the left and right nuclei are equal and contained in the middle nucleus.*

12 Corollary. *Let Σ be a finite commutative and symplectic semifield plane. Then there is a coordinate semifield so that left, right and middle nuclei are equal.*

The idea that one can 'fuse' (sub)nuclei of the same size is considered generally in Jha and Johnson [7] where is it shown that any semifield planes which has two or three subnuclei of the same order has a coordinate semifield when the two or three subnuclei are fused; that is, we may assume they are all equal in that coordinate semifield. For the example, the Hughes-Kleinfeld semifields of order q^n have right and middle nuclei equal and isomorphic to $GF(q)$. If it asked if these semifields might be symplectic, this would force the left nucleus to have order at least q and there is a fusion result that identifies the right and middle nuclei with a sub left nucleus isomorphic to $GF(q)$. For example, the above result shows that the Hughes-Kleinfeld semifields of order q^2 cannot be symplectic.

3 Self-transpose, set-transpose, symplectic pairs

A translation plane is said to be 'self-transpose' if it is isomorphic to its dual plane. Hence, a translation plane is self-transpose if and only if it is invariant under a correlation of the projective space within which lives the corresponding spread. But, a symplectic spread is invariant under a symplectic polarity of the space and there is a matrix spread set of symmetric matrices. It would be interesting to find translation planes which are invariant under a polarity and ask how far away from symplectic could be such associated spreads.

13 Definition. A finite translation plane is said to be 'set-transpose' if and only if the associated spread is invariant under a polarity of the associated projective space. Hence, there is an associated spread set $\mathcal{M}$, such that

$$M \in \mathcal{M} \implies M^t \in \mathcal{M},$$

where M^t denotes the transpose of M.

We note that our arguments for symplectic spreads and affine homologies will not quite work for set-transpose planes, but note the following connection with affine homology groups.

14 Theorem. *Let π set a set-transpose plane with spread*

$$x = 0, y = 0, y = xM,\ M \in \mathcal{M},\ \textit{such that}\ M^t \in \mathcal{M}.$$

(1) Then the homology group with axis $y = 0$ and coaxis $x = 0$ is isomorphic to the homology group with axis $x = 0$ and coaxis $y = 0$.

Proof. If $(x, y) \to (x, yB)$ is a collineation of π then $MB \in \mathcal{M}$, implies $B^tM^t \in \mathcal{M}$, for all $M^t \in \mathcal{M}$, which implies that $(x, y) \to (xB^{-t}, y)$ is a collineation of π. $\boxed{QED}$

Since symplectic spreads are difficult to construct, one wonders if this would be also true of set-transpose spreads. Actually, these are easy to construct from derivable symplectic spreads as follows.

15 Theorem. *Let π be a derivable symplectic translation plane with derivable net D.*

(1) Then derivation of π produces a set-transpose plane.

(2) More generally, any multiply derived plane from a set of mutually disjoint derivable nets produces a set-transpose plane.

(3) Any subregular translation plane is set-transpose.

Proof. Choose a matrix spread set with $x = 0$ and $y = 0$ not components of D and such that

$$x = 0, y = 0, y = xM,\ M \in \mathcal{M}, \text{ such that } M^t = M.$$

Now any Baer subplane of D incident with the zero vector will now have the general form $y = xN$, where N is a matrix over the prime field. If we choose the matrix spread set over the prime field, we may still have the properties mentioned. Consider $y = xN$. This subspace will now non-trivially intersect the components $y = xM$ of D non-trivially and hence $y = xN^t$ will non-trivially intersect the components $y = xM^t$ of D^t. Since this set is D again, we see that $y = xN^t$ becomes a Baer subplane of the net D. This means that upon derivation we have a set-transpose spread but not necessarily a symplectic spread. In the multiply derived case, we need only choose two components not in the set being replaced, which we may always do, to be called $x = 0, y = 0$.

This proves (1) and (2). Finally, any Desarguesian plane is symplectic so any subregular translation plane then is set-transpose. $\boxed{QED}$

The subregular spreads that we observe are set-transpose might better be called 'semi-symplectic of type i', where i is the number of components that are totally isotropic under the symplectic polarity. In the case of subregular planes, it is not at all clear when the type i might be. Still another variation of self-transposed spreads are what are called 'non-singular pairs' in Weintraub [17] (we shall call such pairs 'symplectic pairs'). These would be our type 0, or perhaps 'symplectically paired planes' and symplectic planes of order q^n would be of type $(q^n + 1)$. These are defined in our context as follows:

16 Definition. Let π be a finite translation plane whose spread is invariant under a symplectic polarity that permutes the components in orbits of length 2. Then the orbits are called 'symplectic pairs'. We shall call the plane a 'symplectically paired plane'.

Weintraub [17] shows that for any vector space V of dimension $2n$, n even over a field F isomorphic to $GF(q)$, q odd (or over a field F of characteristic not 2 which admits a cyclic Galois field extension of degree n), there is always a set of symplectic pairs that partition $V - \{0\}$. In the context of translation planes, this would mean that the translation plane is symplectically paired. We point out here that the infinite class that Weintraub found always determine the Hall planes of odd order q^2.

17 Theorem. *Let π_σ denote a Desarguesian affine translation plane of odd order q^2 with spread determined as follows:*

$$x = 0, y = x \begin{bmatrix} u & t \\ \gamma t & u \end{bmatrix};\ u, t \in GF(q),$$

where γ is a nonsquare. Define a symplectic form for the associated 4-dimensional $GF(q)$-vector space as follows:

$$\langle (x_1, x_2, y_1, y_2), (z_1, z_2, w_1, w_2) \rangle = (x_1, x_2, y_1, y_2) \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2\gamma \\ -2 & 0 & 0 & 0 \\ 0 & 2\gamma & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix},$$

$\forall x_i, y_i, z_i, w_i \in GF(q),$ *where* $i = 1, 2$.

Then

$$y = x \begin{bmatrix} u & t \\ \gamma t & u \end{bmatrix} \text{ and } y = \begin{bmatrix} u & -t \\ -\gamma t & u \end{bmatrix} \text{ for } t \neq 0$$

are orthogonal under the symplectic form and mutually disjoint, thus forming exactly $(q^2 - 1)/2$ symplectic pairs of mutually disjoint 2-dimensional $GF(q)$-subspaces.

Consider the regulus net

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix};\ u \in GF(q).$$

The opposite regulus has Baer subplanes of the following form:

$$\pi_\beta = \{ (x_1, \beta x_1, x_2, \beta x_2); x_i \in GF(q),\ i = 1, 2 \},$$
$$\pi_\infty = \{ (0, y_1, 0, y_2); y_i \in GF(q),\ i = 1, 2 \}\ \forall \beta \in GF(q).$$

Then

$$\{\pi_0, \pi_\infty\} \text{ and } \{\pi_\beta, \pi_{1/\gamma\beta}\} \text{ for } \beta \neq 0.$$

are symplectic $(q+1)/2$ pairs of mutually orthogonal 2-dimensional $GF(q)$-subspaces.

(1) Then the union of these pairs forms a symplectically paired spread which is clearly isomorphic to the spread obtained by the derivation of the Desarguesian spread π by the replacement of a single regulus. Hence, the resulting infinite class of symplectically paired spreads corresponds to the Hall spreads.

(2) In the infinite case, note that all of this may be accomplished for fields K of characteristic not 2 that admit a quadratic extension —obtain the infinite Hall spreads relative to K and the field extension.

Proof. Relative to the symplectic form given, it is a straightforward calculation to see that $y = x\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ will map to $y = x\left[\begin{smallmatrix} a & -c/\gamma \\ -b\gamma & d \end{smallmatrix}\right]$ under the symplectic form. If these two components belong to the same spread and if $-c \neq b\gamma$ we would have a symplectic pair. In our situation, consider

$$y = x\begin{bmatrix} u & t \\ \gamma t & u \end{bmatrix} \text{ and } y = \begin{bmatrix} u & -t \\ -\gamma t & u \end{bmatrix} \text{ for } t \neq 0.$$

Hence, we have symplectic pairs when t is non-zero. It is straightforward to see that

$$\{\pi_0, \pi_\infty\} \text{ and } \{\pi_\beta, \pi_{1/\gamma\beta}\} \text{ for } \beta \neq 0.$$

are symplectic $(q+1)/2$ pairs of mutually orthogonal 2-dimensional $GF(q)$-subspaces. Hence, the Hall plane is symplectically paired. $\boxed{QED}$

Since the Hall planes are André, a natural question would be to ask if there are other André planes which are symplectically paired. In fact, almost all André planes may be shown to be symplectically paired. Specially, we prove the following theorem.

18 Theorem. *Let π be a Desarguesian affine plane of odd order q^2. Choose a standard coordinatization and let R denote the regulus net coordinatized by $GF(q)$. Let K^* denote the kernel homology group of order q^2-1.*

(1) If $y = x^q m + xn$, for $m \neq 0$ is a Baer subplane of π then $K^*(y = x^q m + xn)$ is the opposite regulus of a regulus of π.

(2) Define a symplectic form as follows:

$$\langle (x,y), (x^*, y^*) \rangle = \mathrm{trace}_{GF(q)}\left(\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x^{*q} \\ y^{*q} \end{bmatrix} \right).$$

Then any subspace of the form $y = x^q m$ is totally isotropic and any subspace of the form $y = xn$ maps to $y = xn^q$ under the symplectic form. Hence

$$y = x^q m + xn \text{ and } y = x^q m + xn^q$$

are symplectic pairs if and only if $n \notin GF(q)$.

(3) $K^*(y = x^q m + xn) = \{\, y = x^q m d^{1-q} + xn;\ d \in GF(q^2)^* \,\}$.

(4) If $\mathcal{A}_R$ *is an André set of* $q-1$ *mutually disjoint reguli containing* R *then the Frobenius mapping* $(x,y) \longrightarrow (x^q, y^q)$ *leaves* A_R *invariant. Furthermore, this mapping maps* $K^*(y = x^q m + xn)$ *to* $K^*(y = x^q m + xn^q)$. *If* $n^q \neq n$, *we have a set of* $(q+1)/2$ *symplectic pairs.*

(5) If $\mathcal{A}_R$ *is an André set of* $q-1$ *mutually disjoint reguli containing* R *then there is a unique opposite regulus* $K^*(y = x^q m + xn)$ *such that* $n \in GF(q)$. *The set of remaining* $(q-3)$ *reguli corresponds to the paired set of opposite reguli of the following form:*

$$\mathcal{P}_R = \{\, (K^*(y = x^q m_i + xn_i), K^*(y = x^q m_i + xn_i^q)) \,\}, \qquad \text{where } n_i^q \neq n_i, \text{ for } 1, 2, \ldots, (q-3)/2,\ m_i \neq 0.$$

(6) Choose any subset λ *of* $\mathcal{P}_R$ *of symplectic pairs. Form the translation plane* $\pi_{R,\lambda}$ *by multiple derivation of* π *by replacement of the regulus nets corresponding to* λ *and also derive* R.

Then $\pi_{R,\lambda}$ *is a symplectically paired translation plane.*

Hence, there are at least $2^{(q-3)/2}$ *possible symplectically paired translation planes constructed (not all of these are necessarily non-isomorphic).*

(7) Specifically, assume that $K^*(y = xm_0; m_0^{q+1} = \alpha)$. *Choose a basis* $\{1, t\}$ *for* $GF(q^2)$ *over* $GF(q)$ *so that* $t^2 = \alpha$, α *a nonsquare. Then* $\mathcal{A}_R$ *may be chosen to have the following form:*

$$\mathcal{A}_R = \left\{ K^*\left(y = xm_0; m_0^{q+1} = \alpha\right), K^*\left(y = x^q m_\gamma \pm x\sqrt{\alpha\gamma^2}; m_\gamma^{q+1} = \alpha\gamma^2\right) \right\},$$

where $\gamma \neq 0$ *or* 1. *Note that*

$$(y = x^q m_\gamma d^{1-q} + x\sqrt{\alpha\gamma^2}; m_\gamma^{q+1} = \alpha\gamma^2) \leftrightarrows (y = x^q m_\gamma d^{1-q} - x\sqrt{\alpha\gamma^2}; m_\gamma^{q+1} = \alpha\gamma^2)$$

under the symplectic form, for all $d \in GF(q^2)^*$.

Proof. First note that every Baer subplane incident with the zero vector and disjoint from $x = 0$ has the form $y = x^q m + xn$, for $m \neq 0$. Suppose that A_R contains two reguli whose opposite reguli are of the form $K^*(y = x^q m + xn)$, where $n \in GF(q)$ in both opposite reguli. Note that the mapping $(x,y) \to (x, x\delta + y)$, for $\delta \in GF(q)$, leaves R invariant. Since there are $q-1-1$ remaining reguli distinct from R, there must be at least one such opposite reguli such that $n \in GF(q)$. This means we may take say $K^*(y = xm)$, where $m^{q+1} = \alpha$, is non-square (to ensure that this regulus is disjoint from R). We then need to show that all other reguli in the corresponding set have opposite reguli of the type indicated; that is $n's$ so that $n \neq n^q$. Note that n^{q+1} is square if $n = n^q$, so if we could show that n^{q+1} is non-square, we would have the proof. We claim that the mapping that fixes the André net

$$A_\alpha = \{\, y = xz; z^{q+1} = \alpha \,\}$$

componentwise is

$$\sigma_\alpha : (x, y) \to (y^q, x^q\alpha).$$

Now consider any other regulus in the same André net with opposite regulus of the form $K^*(y = x^q m_1 + xn)$. We note that this André net is

$$A_{\alpha+n} = \left\{ y = x(m_1 + n); m_1^{q+1} = \delta, \text{ for } \delta \text{ non-square} \right\},$$

and δ is square for the same reason as previous.

By conjugating σ_δ by the mapping $\tau_n : (x, y) \to (x, xn + y)$, we obtain

$$\rho = \tau_n^{-1}\sigma_\delta\tau_n : (x, y) \to ((-xn + y)^q, (-xn + y)^q n + x^q\delta).$$

Since these two nets are disjoint, it follows from Bruck [5], that there is a unique orbit of length 2 under σ and ρ. Note that $y = xt$ maps to $y = xt^{-q}\alpha$ under σ and if this is a common orbit under ρ, we must have the following condition:

$$(t - n)^q t^{-q}\alpha = (t - n)^q n + \delta.$$

The mapping $\omega : (x, y) \to (x^q, y^q)$ fixes R componentwise. Hence, it follows that there is a unique orbit of components under the three mappings σ_α, ρ and ω. Note that $y = xt$ maps to $y = xt^q$ under ω so we must have

$$t^q = t^{-q}\alpha,$$

from which it follows that

$$t^{2q} = \alpha \iff t^2 = \alpha.$$

Choose a basis $\{\,1, t\,\}$ for $GF(q^2)$ over $GF(q)$ noting that $y = xt$ cannot be a component of R, so that $t \notin GF(q)$. Hence, $t^q = -t$. Note that $y = xt$ maps to $y = x(t - n)^{-q}((t - n)^q n + \delta)$. Hence, we must have

$$n + (t - n)^{-q}\delta = t^q,$$

implying that we have have

$$n^q(t - n) + \delta = t(t - n) = t^2 - tn = \alpha - tn.$$

Therefore, we have the condition:

$$t(n^q + n) = \alpha - \delta + n^{q+1}.$$

However, all terms but t are in $GF(q)$, implying that

$$n^q = -n \text{ and } n^{q+1} = \delta - \alpha.$$

But, $n^q = -n$ implies that $n = t\gamma$ for $\gamma \in GF(q)$ and since n is not zero and $t - n$ is not zero then we must have $\gamma \neq 0$ or 1. Also then

$$n^{q+1} = -n^2, \text{ so } n^2 = \alpha - \delta = t^2\gamma^2 = \alpha\gamma^2.$$

So,

$$n = \pm\sqrt{\alpha\gamma^2}$$

noting that $n \notin GF(q)$. Hence, $n \neq n^q$.

However, now then all other André reguli $K^*(y = x^q m + xn)$ are paired with $K^*(y = x^q m + xn^q)$, for $n^q \neq n$, which gives distinct and hence mutually disjoint pairs. This means that there are at most $q - 2 - 1$ André regulus nets with corresponding n so that $n^q \neq n$ and since $q - 2$ is odd, it follows that there are exactly $q - 3$ André nets that are paired so we have exactly $(q - 3)/2$ pairs of Andr é nets such that each opposite regulus of one regulus maps under the symplectic form to the second regulus of the pair. Hence, by multiply deriving any subset of these pairs along with R, we obtain a symplectically paired André plane, providing each non-replaced set consists of pairs. But, two opposite reguli are paired if and only if there are images under the Frobenius homomorphism, noting that $K^*(y = x^q m + n) \to K^*(y = x^q m + xn^q)$, since $m^q = md^{1-q}$, for some $d \in GF(q^2)$. But, this means that the André reguli themselves are inverted by the Frobenius automorphism and since $y = xt$ of one André regulus will then map to $y = xt^q$ of the second André regulus, the set of all $q - 3$ André reguli are paired, as well as are all opposite reguli. Note that this says that the homology group of order $q + 1$ of any such André plane has axis and coaxis as pairs.

This completes all parts of the theorem. $\boxed{QED}$

There are many variations on this theme and many symplectic spreads of type i, and notice that the same plane can be symplectic of a variety of types. The symplectic group is doubly transitive on totally isotropic subspaces and transitive on pairs (Weintraub [17]). Hence, there is a symplectic form such that with a choice of basis, we may assume that $x = 0, y = 0$ are components that are either both totally isotropic or form a pair (if $x = 0, y = xT$ is a pair, a change of basis shows that $x = 0, y = 0$ is a pair for some symplectic form).

Note in the examples given in the previous result a basis change by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

will change π_0 and π_∞ to $x = (x_1, x_2) = 0$ and $y = (y_1, y_2) = 0$, respectively. Therefore, if L and M are components such that there is an affine homology group $H_{[L],M}$ with axis L and coaxis M as well as an affine homology group $H_{[M],L}$ with axis M and coaxis L then the only way that the spread could be symplectic of any type is if $H_{[L],M}$ is isomorphic to $H_{[M],L}$.

We note, for example, that all of the symplectically paired translation planes of order q^2 which we have found have isomorphic homology groups of order $(q+1)$.

4 Final remarks

Since there are a variety of translation planes that admit homology groups quite unrelated to the kernel of the translation plane, we have then a criterion to decide when a translation plane cannot be symplectic. For example, no André plane, generalized André plane, j-plane, $j \ldots j$-plane, nearfield planes, of order q^2 with affine homology groups of order not dividing $q-1$ such as those admitting cyclic homology groups of order $q+1$ obtained from a flock of a quadratic cone (see for example, Johnson [9] for this new connection with flocks of quadratic cones). Also, note that there are translation planes whose affine homology groups are not cyclic, such as the Heimbeck planes, and so forth, which therefore cannot be symplectic.

Furthermore, our results give criteria to decide if a given semifield has a commutative isotopic version. If the right and left nuclei are not the same size or one is larger than the middle nucleus, this cannot occur.

The only known commutative semifield planes which are also symplectic are generalized twisted field planes of order p^n with spreads given as follows:

$$x = 0, y = xm - cx^{p^a} m^{p^b},$$

where $c = -1$, $2a = b$, n and p of odd order and $z \to z^{p^a}$ of order 3 (see Kantor [10]). Biliotti, Jha and Johnson [2] have pointed out that the kernel (left nucleus) is isomorphic to $GF(p^{(n,a)})$, the middle nucleus is isomorphic to $GF(p^{(n,a-b)})$ and the right nucleus is isomorphic to $GF(p^{(n,b)})$. So, for such a generalized twisted field plane to be symplectic the necessary condition would be that $(n,b)=(n,a-b)\leq(n,a)$. And for, commutative and symplectic semifields, we would require $(n,b)=(n,a-b)=(n,a)$. In this context, we recall the following results:

19 Theorem (Jha and Johnson [6])**.**

(1) If a non-Desarguesian generalized twisted field plane of order q^n is one plane of a coupling then q is odd, n is odd and the second plane of the coupling is a Suetake plane.

(2) The generalized twisted field planes of order q^n corresponding to the Suetake planes are as follows:

$$x=0, y=0, y=xm-cx^{q^{2b^*}}m^{q^{b^*}},$$

where $(2b^*,n)=(b^*,n)=1$ and q is any odd prime power, n an odd integer.

(3) If a finite translation plane π is coupled with a non-Desarguesian semifield plane then π is a Suetake plane.

20 Theorem (Jha and Johnson [6])**.** *Let the following spread denote the generalized twisted field plane spread from which the Suetake planes may be constructed*

$$x=0, y=0, y=xm-cx^{q^{2b^*}}m^{q^{b^*}},\ \textit{for } m\in GF(q^n)-\{0\}. \tag{1}$$

where $(2b^*,n)=(b^*,n)=1$ and q is any odd prime power.

Let G denote the collineation group

$$\left\langle \begin{bmatrix} t^{-1} & 0 \\ 0 & t^{q^*} \end{bmatrix};\ t\in GF(q^n)^* \right\rangle.$$

Let $f(x)=x^{q^{-2b^*}}-c^{q^{-b^*}}x$ and note that f is injective on $GF(q^n)$. Let the two components orbits of length $(q^n-1)/2$ be denote by Γ_1 and Γ_2, where $y=x-cx^{q^{2b^*}}$ is in Γ_1 and $y=xm_0-cx^{q^{2b^*}}m_0^{q^{b^*}}$ is in Γ_2

Then the Suetake planes have the following spread:

$$x=0, y=0, (y=xm_0-cx^{q^{2b^*}}m_0^{q^{b^*}})G\cup(y=f^{-1}(x))G. \tag{2}$$

So, the generalized twisted field planes that may be net replaced to construct the Suetake planes are such that n and p are odd, $a=2b$. In this setting, $(n,b)=(n,a-b)=(n,a)=r$. As noted previously both the generalized twisted field planes and the Suetake planes are symplectic but not all of the semifield planes are commutative. In any case, in the generalized twisted field case, we have fusion among the nuclei.

We have also given various infinite classes of symplectically paired spreads from which it is very easy to construct various other classes of set-transpose spreads.

Again, perhaps it is of importance to ask when such constructions of set-transpose, symplectic, symplectically paired spreads are never possible. Therefore, if L and M are components such that there is an affine homology group $H_{[L],M}$ with axis L and coaxis M as well as an affine homology group $H_{[M],L}$ with axis M and coaxis L then the only way that the spread could be symplectic of any type is if $H_{[L],M}$ is isomorphic to $H_{[M],L}$.

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