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Translation planes of order q^2 admitting collineation groups of order q^3u preserving a parabolic unital

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Abstract. The set of translation planes of order q^2 that admit collineation groups of order q^3u, where u is a prime p-primitive divisor of q^2-1, consists of exactly the Desarguesian plane, assuming that the group does not contain a translation subgroup of order a multiple of q^2. This applies to show that if the group preserves a parabolic unital then the plane is forced to be Desarguesian.

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1 Introduction

This article is concerned with translation planes of order q^2 admitting collineation groups preserving a parabolic unital, or more precisely with translation planes of order q^2 admitting collineation groups G of order q^3u, where u is a prime p-primitive divisor of q^2-1.

The situation we wish to consider arises when G is a subgroup of $PSU(3,q)$ acting on a projective Desarguesian plane and G preserves a classical unital. Since $PSU(3,q)$ has order $(q^3+1)q^3(q^2-1)/(3,q+1)$, acting on the Desarguesian plane, it follows that the stabilizer of a point of the associated unital has order $q^3(q^2-1)/(3,q+1)$. Taking the tangent line as the line at infinity, it follows that there is an affine group of order $q^3(q^2-1)/(3,q+1)$ acting on the affine Desarguesian plane of order q^2.

We note that since there is a subgroup E of order q^3 acting on a unital, it then follows that there the translation subgroup has order at most q. Furthermore, E is a normal subgroup in this case. We note that there is a subgroup of order

q^3u, unless $q = 8$ or q is a Mersenne prime or when $q + 1 = 3 \cdot 2^a$.

The authors have recently classified the translation planes of order q^2 with spreads in $PG(3, q)$ that admit collineation subgroups G of $GL(4, q)$, of order $q(q+1)$ (see [7], [6]). In this setting, there are various conical flock planes that admit such a group. When this occurs, there is a unique point (∞) on the line at infinity that is fixed by G. Let T denote the translation subgroup of order q^2 whose elements have center (∞). Then T is normalized by the collineation group of order $q(q+1)$. Hence, GT has order $q^3(q+1)$. When $q+1 \neq 2^a$, there is a collineation group of order q^3u, where u is a prime p-primitive divisor of $q^2 - 1$. Therefore, a complete classification of all translation planes of order q^2 admitting collineation groups of order q^3u would necessarily involve an analysis of conical flock planes. On the other hand, if it is assumed that when there is a translation subgroup of a group of order q^3u, the translation subgroup with fixed center does not have order q^2, it turns out that the problem can be solved.

Hoffer [5] showed that a projective plane π of order q^2 admitting $PSU(3, q)$ is necessarily Desarguesian and, as we have noted, when q^2-1 has a p-primitive divisor u then there is an associated subgroup of order q^3u of $PGU(3, q)$ as E is normal in the stabilizer.

Hence, we are able to improve Hoffer's result when the plane is a translation plane. In particular, we are able to show the following result:

1 Theorem. *Let π be a translation plane of order q^2 that admits a collineation group G of order q^3u, where u is a prime p-primitive divisor of $q^2 - 1$, where the translation subgroup of G does not have order a multiple of q^2.*

Then π is Desarguesian.

2 Corollary. *Let π be a translation plane of order q^2 that admits a collineation group G of order q^3u that preserves a parabolic unital, where u is a prime p-primitive divisor of $q^2 - 1$.*

Then π is Desarguesian.

Furthermore, concerning the unital itself, we have the following result, showing the unital to be classical.

3 Theorem. *Let π be a translation plane of order q^2 that admits a collineation group G of order $q^3u(q-1)$ admitting a normal subgroup of order q^3, where u is a p-primitive divisor of $q^2 - 1$.*

(1) If G preserves a parabolic unital then π is Desarguesian.

(2) If the group is in $AGL(2, q^2)$ then the unital is classical.

2 $p^\alpha u^\beta$-groups

In this section, we discuss translation planes π of order q^2, for $q = p^r$, p a prime, that admit collineation groups of order u^β, where u is a p-primitive divisor of q^2-1. Furthermore, we then consider collineation groups of order $p^\alpha u^\beta$. When dealing with collineation groups U of order u^β in the translation complement of π, there is invariably a Desarguesian affine plane lurking in the background. Since T.G. Ostrom originated the idea of connecting such a Desarguesian plane with a translation plane under consideration, we call such a plane the 'Ostrom phantom'. More precisely,

4 Definition. Let π be a translation plane of order q^2, $q = p^r$, that admits a collineation in the translation complement τ_u of order u, where u is a prime p-primitive divisor of $q^2 - 1$. If τ_u fixes three line-sized $GF(p)$-subspaces then the set of all τ_u-invariant line-sized subspaces forms a Desarguesian spread. This spread shall be called the 'Ostrom phantom of τ_u'.

5 Lemma. *Let π be a translation plane of order q^2, $q = p^r$, that admits a collineation τ_u in the translation complement of order u, where u is a prime p-primitive divisor of $q^2 - 1$. Then τ_u fixes two components $x = 0, y = 0$ and one of the following occurs:*

(1) τ_u is an affine homology with axis-coaxis set $\{ x = 0, y = 0 \}$,

(2) τ_u acts irreducibly and semi-regularly on each component $x = 0, y = 0$, and either

(a) fixes no other non-zero proper subspace or

(b) τ_u fixes a third non-zero proper subspace distinct from $x = 0, y = 0$ and the set of all τ_u-invariant subspaces forms the Ostrom phantom of τ_u.

Proof. Since u divides $q^2 - 1$, and there are $q^2 + 1$ components, τ_u must fix at least two, say $x = 0, y = 0$. Assume that τ_u is not an affine homology with axis-coaxis set $\{ x = 0, y = 0 \}$. If τ_u is reducible on say $x = 0$, let W be a subspace fixed by τ_u. Since u does not divide $p^t - 1$ for $t < 2r$, it follows that τ_u fixes some subspace on $x = 0$ pointwise, say $\mathrm{Fix}\, \tau_{u,x=0}$. Let M be a Maschke complement of $\mathrm{Fix}\, \tau_{u,x=0}$ on $x = 0$. Then similarly τ_u fixes a proper subspace of M pointwise or M is trivial. Hence, τ_u is an affine homology if it acts reducibly on a component. Assume that τ_u is not an affine homology but fixes another subspace Z distinct from $x = 0$ or $y = 0$. Then Z is disjoint from $x = 0$ and from $x = 0$ and since τ_u is irreducible on both $x = 0$ and $y = 0$, we claim that Z is a line-sized subspace. If Z is not a line-sized subspace then τ_u would fix non-zero points on Z, implying that it is planar and hence fixes non-zero points on $x = 0$ and $y = 0$. But we may now apply the fundamental theorem of Johnson [8] to show that the set of τ_u invariant subspaces form a Desarguesian affine plane, the Ostrom phantom of τ_u. $\boxed{QED}$

6 Remark. When a collineation in the translation complement τ_u of order a prime p-primitive divisor of $q^2 - 1$ acts on a translation plane π and produces an Ostrom phantom then the normalizer of $\langle \tau_u \rangle$ in the collineation group of π is a collineation group of the Ostrom phantom.

Proof. Just note that the normalizer of $\langle \tau_u \rangle$ permutes the subspaces fixed by τ_u. $\boxed{QED}$

We now generalize the previous lemma. The arguments given are quite similar.

7 Lemma. *Let π be a translation plane of order q^2 that admits a collineation in the translation complement τ of order u, where u is a prime p-primitive divisor of $q^2 - 1$. Let U be a Sylow u-subgroup of the translation complement of π. Then U fixes at least two components $x = 0, y = 0$ and one of the following holds:*

(1) U is Abelian of rank ≤ 2 and leaves invariant $x = 0, y = 0$ and contains proper affine homologies with axis-coaxis $\{ x = 0, y = 0 \}$,

(2) U is cyclic and leaves invariant $x = 0, y = 0$ and fixes no non-trivial proper subspace distinct from $x = 0, y = 0$, or

(3) U is cyclic and semiregular on the non-zero affine points and the irreducible U-modules form a Desarguesian spread Σ_U, the Ostrom phantom of U, and U is a group of kernel homologies of Σ_U. Furthermore, the normalizer of U is a collineation group of Σ_U.

Proof. Consider the action of U on $x = 0$. The kernel of the action defines an affine homology group. However, u must be odd and Sylow u-subgroups of affine homology groups are cyclic (see, e.g., the appendices of Biliotti, Jha and Johnson [1]). Suppose some element g in U fixes a non-zero point of $x = 0$. Then from the previous lemma, g fixes $x = 0$ pointwise. Hence, the action of U on $x = 0$ is fixed-point-free. Any fixed-point-free collineation group is a Frobenius complement by adjoining an appropriate translation subgroup. Also, an odd-order u-group of a Frobenius complement is cyclic. Hence, U is metacyclic. If U is faithful on $x = 0$, then U is cyclic. Otherwise, let $U_{x=0}$ denote the subgroup of affine homologies. If U is faithful on $y = 0$, again U is cyclic. If the group induced on $y = 0$ is $\langle x \longmapsto xA \rangle$ and the group induced on $x = 0$ is $\langle y \longmapsto yB \rangle$, then the elements of U have the general form $(x, y) \longmapsto (xA^i, yB^j)$, implying that U is Abelian. Hence, either U contains affine homologies or is cyclic. So, assume that U does not contain affine homologies and fixes a non-zero proper subspace Z distinct from $x = 0, y = 0$. If Z is not line-sized then some element of U fixes non-zero points of Z, implying that this element cannot be irreducible on $x = 0$ and $y = 0$, so it is an affine homology. Hence, Z is of line size, U is cyclic and there is a U-invariant Ostrom phantom Σ_U. $\boxed{QED}$

8 Corollary. *If a translation plane of order q^2 admits a collineation group of order $p^\alpha u^\beta$ in the translation complement and a u-group U leaves invariant a subplane then the subplane is Baer and there is an associated Ostrom phantom containing the Baer subplane as a component. In the Ostrom phantom, U is a kernel homology group.*

Proof. U fixes two components $x = 0, y = 0$ and if it fixes a subplane π_0, it must be disjoint from $x = 0, y = 0$. Hence, we have three mutually disjoint line- sized U-invariant subspaces. Applying our previous lemmas completes the proof. $\boxed{QED}$

9 Corollary. *If a translation plane of order q^2 admits a collineation group of order $p^\alpha u^\beta$ in the translation complement and a u-group U centralizes a p-group P, then there is an associated Ostrom phantom such that U is a kernel homology group of this Desarguesian plane. Furthermore, one of the following two situations occurs:*

(1) P fixes a Baer subplane pointwise and $|P| \leq q$, or

(2) P is an affine elation group.

Proof. Let P fix a component L so $\operatorname{Fix} P \cap L$ is non-trivial. If P is planar then the above theorem shows that it is a Baer subplane and there is an Ostrom phantom. The statement on the order of P then follows directly (see, e.g., Biliotti, Jha, Johnson [1]). Hence, assume that $\operatorname{Fix} P$ is contained in L. It may be that $\operatorname{Fix} P$ is an elation group. In this case, $\operatorname{Fix} P$ must move any component distinct from L and fixed by U. So, again there is an Ostrom phantom. If $\operatorname{Fix} P$ is properly contained in L then U is an affine homology group with axis L. But again, P must move the coaxis of U, implying that U fixes three components, a contradiction. $\boxed{QED}$

3 Groups of order q^3u and translation subgroups

Previously, we have considered subgroups in the translation complement. We have mentioned in the introduction that the problem of determination of the translation planes of order q^2 admitting groups of order q^3u rests on the nature of the translation subgroup. In the following, we indicate the nature of the problems involving translation subgroups.

10 Theorem. *Let π be a translation plane of order q^2 admitting a collineation group of order q^3u, where u is a prime p-primitive divisor of q^2-1. Then one of the following occurs:*

(1) The translation subgroup has order q^2,

(2) the translation group is trivial, or

(3) there is an affine homology of order u.

PROOF. Let F denote the translation subgroup of G. If F is not trivial, let τ be an element of order u. We claim that either F has order q^2 or τ must centralize some translation subgroup. To see this, we note that this is clear if F has order $< q^2$. If T has order q^2p^a, then u must divide $(p^{2r+a}-1, p^{2r}-1)$, where $q = p^r$. Furthermore, $(p^{2r+a}-1, p^{2r}-1) = p^{(2r+a,2r)}-1 = p^{(a,2r)}-1$ so that if a is not 0, we must have $(a, 2r) = 2r$, so that the translation group has order q^4, a contradiction. Hence, τ centralizes some translation subgroup T or F has order q^2. Since τ of order u fixes at least one affine point (as there are q^4 affine points), then T permutes the fixed points of τ. In particular, τ fixes at least 2 points of some component L of π. Since τ is p-primitive, it follows that τ is an affine homology.

Therefore, if τ is not an affine homology, it can only be that F is trivial or has order q^2. $\boxed{QED}$

We also improve on the previous theorem as follows.

11 Theorem. *Assume a translation plane of order q^2 admits a collineation group G of order q^3u, where u is a p-primitive divisor of q^2-1, and admits an affine homology τ of order u. Then there is always a translation subgroup of order q^2p^b, normalized by τ, for some divisor p^b of q^2.*

Conversely, if there is a proper translation subgroup of order $> q^2$, normalized by an element τ of order u then τ is an affine homology and there exists a translation subgroup of order q^2 with fixed center the center of τ.

PROOF. If there is a proper translation subgroup of order not q^2, then we have a non-trivial translation group of order not q^2, and this group, since normalized by τ, admits a non-trivial subgroup centralized by τ. This implies that τ is an affine homology. Suppose that we have a translation group of order $> q^2$. Then it follows that there is a translation σ with center the cocenter of τ. If τ is of the form $(x,y) \longmapsto (x, y\circ c)$ and σ of the form $(x,y) \longrightarrow (x, y+b)$, then

$$\tau\sigma\tau^{-1} : (x,y) \longmapsto (x, y + b\circ c^{-1}).$$

Hence, τ normalizes the translation subgroup with center (∞) and cannot centralize any subgroup. Hence, this translation subgroup then must have order q^2. We note any subgroup with elements of the form $(x, y) \to (x + a, y)$, i.e., with center (0), will commute with τ, so any subgroup of order p^b together with the translation subgroup of order q^2 with center (∞) will generate a translation group of order q^2p^b. $\boxed{QED}$

12 Remark. Hence, in the following sections, we first consider groups with trivial translation subgroups and then when there are affine homologies of order u, noting that we must avoid the situation that there is a translation subgroup of order a multiple of q^2. In general, let F denote the full translation group of order q^4 then form GF. If $G \cap F = T$ and $|T| = p^\alpha$, then $(GF)_0$ has order q^3u/p^α.

4 The translation subgroup is trivial

When the translation subgroup is trivial, we have a solvable group of order q^3u of $(GF)_0$ in the translation complement (i.e., we may assume that there is an affine fixed point). Hence, we have a normal elementary Abelian subgroup N. We consider the various possible situations for N.

4.1 Case N has order u

We first consider the case when N has order u. Let τ be an element of order u. Since u divides $q^2 - 1$ and is prime, it follows that τ fixes an affine point 0 and furthermore, must fix at least two components L and M. Applying Lemma 5, we either have an Ostrom phantom or τ fixes exactly two components. In this case, there is an associated Desarguesian plane Σ admitting τ as a kernel homology group and G acts as a collineation group of Σ. Furthermore, since the translation subgroup is trivial, we have a group of order q^3u fixing an affine point 0. Every Sylow p-subgroup S_p of order q^3 in $GL(2, q^2)$ must be an elation group of order dividing q^2. In general the group is in $GL(4r, p)$, where $p^r = q$. Furthermore, the automorphism group of $\Gamma L(2, q^2)$ has order $2r$, so $q = p^r \mid 2r$, implying that $q = 2, 4$. Hence, we have a translation plane of order 16 admitting a group of order 4^3 that fixes a component L. So, there is a planar group of order $2^6/2 = 2^5$. The p -planar bound (see Biliotti, Jha, Johnson [1]) says that the maximal order of a p-planar group for a quasifield of order 2^4 is 2^3.

Hence, τ_u must fix exactly two components. If τ_u fixes exactly the zero vector 0, we have a p-planar group of order $q^3/(2, p)$, acting on a plane of order p^{2r}, so the bound is then $p^{2r-1} < p^{3r}/(2, p)$, a contradiction.

So, τ must fix additional points and hence be planar or there are exactly two fixed components. But since u is p-primitive, this latter situation cannot occur.

4.2 Case N has order $p^{\alpha} < q^2$, $q = p^r$

We now turn to the case when N has order $p^{\alpha} < q^2$, $q = p^r$. In this case, since u is a p-primitive divisor, it follows that u must divide $(q^2 - 1, p^{\alpha} - 1)$. If $p^{\alpha} - 1 \leq q^2 - 1$, it follows that either N has order q^2 or any collineation τ_u of order u must commute with N. Since N must fix a non-trivial set of points on a component L, it follows that N either fixes L pointwise or τ_u fixes L pointwise. We note that we may apply Corollary 9. In the latter case, τ_u is an affine homology that commutes with N. Hence, N must fix the co-axis of τ, since τ commutes with N. This says that N is a planar group of order p^{α}. Since N is normal then we have a group of order $q^7 u/j$ that leaves invariant a subplane π_0 of order p^k. Hence, there is a subgroup of order divisible by $q^7 u/p^{2k} j$ that fixes π_0 and fixes a point 0 of π_0. So, there is a subgroup of order $q^7 u/p^{3k} j$ that fixes a component of π_0 pointwise. Furthermore, there is a subgroup of order $q^7 u/p^{4k} j$ that fixes π_0 pointwise. Since $p^k \leq q$, we obtain a collineation group of order at least $q^3 u/j$ that fixes π_0 pointwise. But τ is p -primitive, implying a contradiction.

Hence, assume that N fixes L pointwise. So, τ normalizes an elation group of order p^{β}. Since no non-identity element of N can commute with τ, hence we have a contradiction, unless τ is not an affine homology.

So, assume that N fixes L pointwise and τ fixes at $1 + p^{\alpha}$ components of π. Choose one of the τ fixed components distinct from L, say M. We must have a have a p-planar group of order at least q, fixing M and L. Let the order of the stabilizer of M and L be $qp^{\beta}u$. Let B be a p-planar group of order qp^{β} fixing L and M. We note by the p-planar bound that B has order $\leq p^{2r-1} = q^2/p$. Suppose that τ normalizes B. Then τ will commute with some element of B, since u is p-primitive. But then, τ would fix some subplane and fix two components of that subplane, a contradiction to a previous lemma. Hence, there are exactly u Sylow p -subgroups. Suppose that $g \in B$ fixes a non-zero point on M that is fixed by $B^{\tau^j} \neq B$. Then $\left\langle g, B^{\iota^j} \right\rangle$ has order strictly larger than qp^{β}, implying that τ will fix a non-zero point on M, forcing τ be to an affine homology, a contradiction.

If B fixes a subplane of order p^{δ} then there are exactly $p^{\delta} - 1$ non-zero points on M fixed pointwise by p-elements in Sylow p-subgroups in $\langle B, \tau \rangle$, and no two Sylow p-subgroups can fix a common point. Thus, there are

$$(p^{\delta} - 1)u + 1$$

points fixed on M by Sylow p-subgroups. Take out of this set the p^β pointwise fixed by B. Suppose that qp^β does not divide $(p^\delta-1)(u-1)$. Then there would be an element of B that fixes one of these points, a contradiction, as noted above. Hence, it follows that

$$qp^\beta \text{ divides } (p^\delta-1)(u-1)=p^\delta(u-1)-u+1.$$

Note that $p^\delta \leq q$, so

$$(u-1)=tp^\delta,$$

implying that

$$qp^\beta \text{ divides } (p^\delta-1)(u-1)=tp^{2\delta}-tp^\delta.$$

Thus,

$$qp^{\beta-\delta} \text{ divides } tp^\delta-t=t(p^\delta-1),$$

implying that

$$qp^{\beta-\delta} \text{ divides } t, \text{ and } (u-q)=sqp^{\beta-\delta}p^\delta=sqp^\beta.$$

Since u divides $q+1$, this implies that $sp^\beta=1$. Thus, we have an elation group N of order q^2 and $u=1+q$. However, this implies that N has order q^2, a contradiction.

4.3 Case N has order q^2

When N is an elementary Abelian group of order q^2, N fixes pointwise a non-zero set S left invariant by G. If S is a subplane then either S is a Baer subplane or τ fixes it pointwise, a contradiction to the fact that τ is a p-primitive collineation. But a Baer subplane cannot be pointwise fixed by a group of order q^2 in a translation plane of order q^2. Hence, S is a subspace of a component L. But still τ leaves S invariant, implying that S is L or τ fixes L pointwise. If τ fixes L pointwise, N cannot leave the coaxis of τ invariant, implying that there are non-trivial elations in $\langle N,\tau\rangle$, by André's theorem. Now the elation group E is normal within $\langle N,\tau\rangle$ and if an elation commutes with τ, then this elation must move the coaxis of τ. Hence, E has order q^2, implying that the plane is a semifield plane. So, assume that S is L. Hence, again we have a semifield plane.

Thus, the plane is a semifield plane. However, we still have a group of order q^3 acting on semifield plane π. So, there is a group B of order q, which is planar. First assume that q is odd. Then, by Foulser [3], since elation and Baer p-elements cannot coexist, it follows that the group of order q fixes pointwise a subplane π_0 of order $p^a<q$. Since q cannot divide q^2-p^a, it follows that B cannot act semi-regularly on $L-\pi_0\cap L$, for any component L of π_0. Let σ be in the center of B. Then B leaves $\operatorname{Fix}\sigma$ invariant. We note that $\operatorname{Fix}\sigma$ is a semifield plane that cannot be Baer.

4.4 The semifield of order q^2 and planar groups of order q

When the plane is a semifield plane of order q^2, the elation subgroup E of order q^2 is the normal elementary Abelian subgroup of the group G of order q^3u. We know that u cannot divide (q^3-1) since it divides q^2-1 and $(q^3-1, q^2-1) = q^{(3,2)}-1 = q-1$. If there is a unique Sylow p-subgroup then τ commutes with some p-element g. If g is planar then τ fixes a proper subplane, a contradiction. If g fixes a unique component and fixes a subspace S on L pointwise, then τ is an affine homology or $S = L$. In the former case, the semifield plane is Desarguesian. Hence, τ commutes with some elation, implying that τ fixes at least three components. Actually, if there is one Sylow p-subgroup then τ permutes the q^3-q^2 elements and hence centralizes at least one of these—a contradiction. Therefore, there are exactly u Sylow p-subgroups of order q^3, each sharing an elation subgroup of order q^2. It follows that there are exactly $q^2(u-1)$ elements of order u, one group fixing each of the q^2 non-elation axis components. Since the stabilizer of $y=0$ has order qu, and no planar p-group is normal, it follows that the group of order u is normal. Hence, there are exactly $qu-u = u(q-1)$ p-elements. Hence, it follows that there are exactly u Sylow p-subgroups and they are all disjoint. Since $\langle B, B^{\tau}\rangle$ is the full group stabilizing $y = 0$, then the fixed-point subspaces are mutually disjoint. Hence, if $\operatorname{Fix} B$ has order p^a, then there are $(p^a-1)u+1$ points corresponding to these fixed-point spaces and on $y = 0$. If any element g of B fixes any of these points then g and some B^{τ_j} will fix the same point, a contradiction. Now since q does not divide q^2-p^a, as $p^a < q$ (as no p-element is Baer if q is odd), some element g in B must fix additional points outside of $\operatorname{Fix} B$. So, q must divide $(p^a-1)u+1-p^a = p^a(u-1)-u+1$. Hence, u is congruent to 1 mod p^a. Let $1+kp^a = u$. Then, q must divide $p^akp^a - kp^a = kp^a(p^a-1)$. Thus, it follows that q divides kp^a. Hence, $kp^a = zq$ and $u = 1+zq$, a contradiction unless $z = 1$ and q is even.

Now, if q is even, there is a planar group of order q so there is a Baer collineation σ of order 2 in the center $Z(B)$ of B. By Ganley [4], it follows that there is a unique Baer collineation σ of order 2 with axis $\pi_0 = \operatorname{Fix}\sigma$. Hence, $B \mid \pi_0 = B_1$ has order 2. Note that π_0 is a semifield subplane, since it contains the axis of the elation group. Let σ_1 denote a central involution of B_1. Then σ_1 fixes pointwise a subplane π_1 of order $q^{1/2}$ of π_0. The group $B_1 \mid \pi_1 = B_1$ has order 2 on π_1. Let $q = 2^r$ and $r = 2^ab$, where $(2, b) = 1$. By induction, we have a planar group of order $q/2^k$ leaving invariant a semifield subplane of order $q^{1/2^k} = 2^{r/2^k} = 2^{2^{a-k}b}$. Hence, we have a planar group of order $q/2^a = 2^{2^ab-a}$ leaving invariant a semifield subplane of order $q^{1/2^a} = 2^{2^ab/2^a} = 2^b$, that is contained in a subplane of order 2^{2b}. But this again says there is a planar group of order 2^{2^ab-a} that fixes a subplane of order 2^b pointwise that is contained in a subplane of order 2^{2b}. This says that $2^{2^a b-a} \leq 2$, so that $2^a b - a \leq 1$. This means that $a \leq 1$. If $a = 1$ then $b = 1$, implying that $2^a b = 2$. If $a = 0$ then $b = 1$. Thus, $q = 2$ or 4.

Thus, we arrive at the following two theorems.

13 Theorem. *If a semifield plane of order q^2 admits a planar group of order q and a prime p-primitive collineation then $q^2 = 16$.*

14 Theorem. *If a semifield plane of even order q^2 admits a planar group of order q then $q = 2$ or 4. Note also there is a bound on the size of a planar group: If $q = 2^r$ and $r = 2^a b$ and there is a planar group of order 2^t, then*

$$t \leq a + 1.$$

Hence, we assume that N has order $> q^2$. So, G of order $q^3\tau$ normalizes N.

4.5 Case N has order $q^2p^\beta > q^2$

When N has order $q^2p^\beta > q^2$, still N is an elementary Abelian group of order q^2p^β, normalized by τ. Then $(q^2p^\beta - 1, q^2 - 1) = (p^{2r+\beta} - 1, p^{2r} - 1) = p^{(2r+\beta,2r)} - 1 = p^{(\beta,2r)} - 1$. Hence, it follows that τ commutes with a proper subgroup T of N. If T is planar, then τ fixes a proper subplane, implying a contradiction. Hence, T is an elation subgroup. So, τ must fix at least three components of π. Therefore, there is a Desarguesian affine plane Σ, the Ostrom phantom, admitting the normalizer of τ as a collineation group. Let T^+ denote the full elation subgroup of N. Then $N = T^+N_{(0)}$, where $N_{(0)}$ fixes a second component in an elation orbit of T^+. Hence, $N_{(0)}$ is planar and elementary Abelian. Since τ normalizes T^+, there is a complement C to T^+ of N, normalized by τ. If τ normalizes $N_{(0)}$ then τ fixes $N_{(0)}$ pointwise, a contradiction. Therefore, C does not fix a component. If T^+ has order q^2 then π is a semifield plane. We have considered the semifield plane case above. Hence, we may assume that τ centralizes T^+. Thus, we may assume that $N_{(0)}$ has order $> q^2$ and is planar and commutes with T^+. This means that the order of the pointwise fixed subplane is at least the order of the translation subgroup. We know that, for q odd, elations and Baer p-collineations are incompatible. If q is odd then T^+ has order $< q$, for q odd, for otherwise, we would have a Baer collineation. Hence, we have a planar group of order $> q$ and a translation group of order $< q$. Since $T^+C = T^+N_{(0)}$, if $g \in N_{(0)}$ then $g = ct$, for c in C and t in T^+. Since the group is elementary Abelian and we have the orders of the two groups as above, then g and g^* have the same corresponding t for two different elements g and g^* of $N_{(0)}$. However, this means that $(ct)(c^*t)^{-1}$ is in $C \cap N_{(0)}$. N is normalized by G so the translation group of order T^+ of order $< q$ is normalized by a group of order q^3. Hence, there is a planar group of order at least q^2, a contradiction by the p-planar bound. If q is even then T^+ has order $\leq q$ and there is a planar group of order at least q^2, again a contradiction.

5 G contains translations

If G contains translations, we have a non-trivial translation group T and we assume that the order of T is not divisible by q^2, and this group, since normalized by τ, contains a non-trivial group centralized by τ. This implies that τ is an affine homology and then further that the translation group has order $< q^2$. Furthermore, we have a group of $(GF)_0$ of order at least pqu. Let N be an elementary Abelian normal subgroup.

5.1 N has order u

If N has order u, then the axis and coaxis of τ are both fixed, implying that there is a planar p-group of order at least pq. Since τ cannot fix any subplane, it follows that pq divides $\tau - 1$. But u divides $q+1$, so that $q+1 \geq u \geq 1+pq$, a contradiction.

5.2 N has order p^α

If N has order p^α, we still know that τ is an affine homology. Suppose that p^α is not q^2. Then τ centralizes a subgroup T of N. If T is planar, we have a contradiction by a previous lemma. If T is an elation, then there cannot be an element that centralizes τ, since τ fixes exactly two components. Hence, N has order q^2. If N is planar, again τ must fix a subplane, a contradiction. So, N fixes exactly one component and fixes a subspace S on L pointwise. Since N is not planar, N cannot leave the coaxis of τ invariant. However, this implies there are elations back in G. No elation can commute with τ so we must have an elation group of order q^2; π is a semifield plane. However, semifields with u-homology groups are Desarguesian.

6 The main theorem

15 Theorem. *Let π be a translation plane of order q^2 that admits a collineation group G of order q^3u, where u is a prime p-primitive divisor of q^2-1. Assume that G does not contain a translation subgroup of order a multiple of q^2.*

Then the plane π is Desarguesian.

Proof. It remains to consider semifield planes of order 16 that admit planar groups of order 4 and collineations of order $2^2 + 1 = 5$. The only possible non-Desarguesian semifield plane then must have kernel $GF(2)$ and the full automorphism group has an automorphism group of order 3. $\boxed{QED}$

16 Corollary. *Let π be a translation plane of order q^2 that admits a collineation group G of order q^3u that preserves a parabolic unital, where u is a prime p-primitive divisor of $q^2 - 1$.*

Then π is Desarguesian.

Proof. Since a group of order q^3 acts on the q^2 lines on the parabolic point (∞) at infinity of the parabolic unital then τ stabilizes a component, say $x = 0$, and permutes q points on $x = 0$. Hence τ fixes at least two affine points on a component. Thus, it follows that τ is an affine homology of the associated plane so we have in either of the two above situations that the plane π is Desarguesian. $\boxed{QED}$

7 Final comments

We have shown that if a translation plane of order q^2 admits a collineation group of order q^3u preserving a parabolic unital, then the translation plane is Desarguesian. What we have not determined is the type of unital that is preserved. However, by the collineation group, the unital cannot be of Buekenhout-Metz type, unless the unital is the classical unital.

17 Problem. Show that the parabolic unital preserved by a collineation group of order q^3u is classical.

However, we note the following result:

18 Theorem. *Let π be a translation plane of order q^2 that admits a collineation group G of order $q^3u(q-1)$, where u is a p-primitive divisor of q^2-1, admitting a normal subgroup of order q^3.*

If G preserves a parabolic unital then π is Desarguesian. If the group then is in $AGL(2, q^2)$, the unital is classical.

Proof. From our main theorem and corollary, π is Desarguesian, since there is a subgroup of order q^3u, as there is a normal subgroup of order q^3 and the group preserves a unital. Hence, there is a subgroup H of order $u(q-1)$ in $\Gamma L(2, q^2)$. However, our assumptions will show that H is in $GL(2, q^2)$. Moreover, even though the q^2 part may not preserve the unital, the subgroup H will preserve the unital. Hence, by Ebert and Wantz [2], the unital is Buekenhout-Metz. However, the non-classical BM unitals have linear automorphism groups of order dividing $2q^3(q-1)$. Hence, the unital is classical. $\boxed{QED}$

19 Corollary. *Assume that π is a translation plane of order q^2, $p^r = q$, such that $(2r, q-1) = (2, q-1)$. Let π admit a collineation group of order $q^3 2u(q-1)$, containing a normal subgroup of order q^3, where u is a prime p-primitive divisor of $q^2 - 1$.*

Then π is Desarguesian and the unital is classical.

Proof. We need only show that a subgroup of order $2u(q-1)$ in $\Gamma L(2, q^2)$ contains a linear subgroup of order $(q-1)$. However, it is clear by our assumptions that we have a linear subgroup of order $2(q-1)$ or $(q-1)$. We claim that any linear subgroup of order $2(q-1)$ has a subgroup of order $q-1$. To see this, we note that the group is a subgroup of the direct product of two cyclic subgroups of order $q^2 - 1$ and hence is Abelian. $\boxed{QED}$

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